symmetry

MDPI

Article

Fixed Point Theorems in Symmetric Controlled *M*-Metric Type Spaces

Khaled Suwais [1,*], Nihal Taş [2], Nihal Özgür [3] and Nabil Mlaiki [4]

[1] Faculty of Computer Studies, Arab Open University, Riyadh 11681, Saudi Arabia
[2] Department of Mathematics, Balıkesir University, Balıkesir 10145, Türkiye; nihaltas@balikesir.edu.tr
[3] Department of Mathematics, Izmir Democracy University, Karabağlar, Izmir 35140, Türkiye; nihal.ozgur@idu.edu.tr
[4] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; nmlaiki@psu.edu.sa
* Correspondence: khaled.suwais@arabou.edu.sa

Abstract: One of the frequently studied approaches in metric fixed-point theory is the generalization of the used metric space. Under this approach, in this study, we introduce a new extension of M-metric spaces, called controlled M-metric spaces, achieved by modifying the triangle inequality and keeping the symmetric condition of the space. The investigation focuses on exploring fundamental properties of this newly defined space, incorporating topological aspects. Several fixed-point theorems and fixed-circle results are established within these spaces complemented by illustrative examples to demonstrate the implications of our findings. Moreover, we present an application involving high-degree polynomial equations.

Keywords: controlled M-metric-type space; fixed point; fixed circle; fixed disc

MSC: 47H10; 54H25

check for updates

Citation: Suwais, K.; Taş, N.; Özgür, N.; Mlaiki, N. Fixed Point Theorems in Symmetric Controlled *M*-Metric Type Spaces. *Symmetry* **2023**, *15*, 1665. https://doi.org/10.3390/sym15091665

Academic Editor: Alexander Zaslavski

Received: 29 July 2023
Revised: 25 August 2023
Accepted: 27 August 2023
Published: 29 August 2023

Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction and Preliminaries

Fixed-point theory has been attracting significant attention from researchers in recent times due to its powerful applications in various fields, including integral equations, differential equations and engineering. The larger the class of metric spaces, the more fields fixed-point theory can be applied on [1–3]. In this paper, we introduce the concept of a "controlled M-metric space", which serves as a generalization of both an M-metric space and a controlled metric-type space. We explore the interconnections between this novel metric space and some known generalized metrics where the space is symmetric, accompanied by illustrative examples to elucidate our findings.

Asadi et al. were the first to introduce M-metric spaces [4]. Therefore, we recall the definition of an M-metric space and present some additional notations.

Notation 1 ([4]). *Consider a map $\mathbb{D} : \mathbb{X}^2 \to [0, \infty)$ where $\mathbb{X}$ is a nonempty set, for every $\Omega, \Phi \in \mathbb{X}$*

- $\mathbb{D}_{\Omega,\Phi} := \min\{\mathbb{D}(\Omega, \Omega), \mathbb{D}(\Phi, \Phi)\}$.
- $M_{\Omega,\Phi} := \max\{\mathbb{D}(\Omega, \Omega), \mathbb{D}(\Phi, \Phi)\}$.

Definition 1 ([4]). *Let $\mathbb{X}$ be a nonempty set and let $\mathbb{D} : \mathbb{X}^2 \to \mathbb{R}^+$ be a function satisfying the following properties for all $\Omega, \Phi, \Psi \in \mathbb{X}$. Then, the pair $(\mathbb{X}, \mathbb{D})$ is called an M-metric space:*

(1) $\mathbb{D}(\Omega, \Omega) = \mathbb{D}(\Phi, \Phi) = \mathbb{D}(\Omega, \Phi)$ *if and only if* $\Omega = \Phi$,
(2) $\mathbb{D}_{\Omega,\Phi} \leq \mathbb{D}(\Omega, \Phi)$,
(3) $\mathbb{D}(\Omega, \Phi) = \mathbb{D}(\Phi, \Omega)$,
(4) $(\mathbb{D}(\Omega, \Phi) - \mathbb{D}_{\Omega,\Phi}) \leq (\mathbb{D}(\Omega, \Psi) - \mathbb{D}_{\Omega,\Psi}) + (\mathbb{D}(\Psi, \Phi) - \mathbb{D}_{\Psi,\Phi})$.

The notion of M_b-metric spaces, an extension of M-spaces, was introduced in [5], accompanied by the establishment of certain fixed-point theorems.

Let us now revisit the definitions and notations pertinent to M_b-metric spaces.

Notation 2. *Consider a map $\mathbb{D}_b : \mathbb{X}^2 \to [0, \infty)$ where $\mathbb{X}$ is a nonempty set*

- $\mathbb{D}_{b\Omega,\Phi} := \min\{\mathbb{D}_b(\Omega, \Omega), \mathbb{D}_b(\Phi, \Phi)\}$.
- $M_{b\Omega,\Phi} := \max\{\mathbb{D}_b(\Omega, \Omega), \mathbb{D}_b(\Phi, \Phi)\}$.

Definition 2 ([5])**.** *An M_b-metric space on a nonempty set $\mathbb{X}$ is a function $\mathbb{D}_b : \mathbb{X}^2 \to \mathbb{R}^+$ that satisfies the following conditions for all $\Omega, \Phi, \Psi \in \mathbb{X}$:*

(1) $\mathbb{D}_b(\Omega, \Omega) = \mathbb{D}_b(\Phi, \Phi) = \mathbb{D}_b(\Omega, \Phi)$ *if and only if* $\Omega = \Phi$,
(2) $\mathbb{D}_{b\Omega,\Phi} \leq \mathbb{D}_b(\Omega, \Phi)$,
(3) $\mathbb{D}_b(\Omega, \Phi) = \mathbb{D}_b(\Phi, \Omega)$,
(4) *There exists a real number $s \geq 1$ such that for all $\Omega, \Phi, \Psi \in \mathbb{X}$, we have*

$$(\mathbb{D}_b(\Omega, \Phi) - \mathbb{D}_{b\Omega,\Phi}) \leq s[(\mathbb{D}_b(\Omega, \Psi) - \mathbb{D}_{b\Omega,\Psi}) + (\mathbb{D}_b(\Psi, \Phi) - \mathbb{D}_{b\Psi,\Phi})] - \mathbb{D}_b(\Psi, \Psi).$$

The parameter s is referred to as the coefficient of the M_b-metric space $(\mathbb{X}, \mathbb{D}_b)$.

In [6], it was demonstrated that the condition (4) given in Definition 2 is equivalent to the following condition:

(4)′ There exists a real number $s \geq 1$ such that for all $\Omega, \Phi, \Psi \in \mathbb{X}$, the inequality holds:

$$(\mathbb{D}_b(\Omega, \Phi) - \mathbb{D}_{b\Omega,\Phi}) \leq s[(\mathbb{D}_b(\Omega, \Psi) - \mathbb{D}_{b\Omega,\Psi}) + (\mathbb{D}_b(\Psi, \Phi) - \mathbb{D}_{b\Psi,\Phi})]$$

for all $\Omega, \Phi, \Psi \in \mathbb{X}$.

In contrast, the concept of a controlled metric-type space was introduced as a generalization of a metric space and a b-metric space, defined as follows:

Definition 3 ([7])**.** *Consider a nonempty set $\mathbb{X}$ and a function $\alpha : \mathbb{X}^2 \to [1, \infty)$. The function $\mathbb{D} : \mathbb{X}^2 \to [0, \infty)$ is defined as a controlled metric type if the following conditions hold:*

$(\mathbb{D}_1)$ $\mathbb{D}(\Omega, \Phi) = 0$ *if and only if* $\Omega = \Phi$,
$(\mathbb{D}_2)$ $\mathbb{D}(\Omega, \Phi) = \mathbb{D}(\Phi, \Omega)$,
$(\mathbb{D}_3)$ $\mathbb{D}(\Omega, \Phi) \leq \alpha(\Omega, \Psi)\mathbb{D}(\Omega, \Psi) + \alpha(\Psi, \Phi)\mathbb{D}(\Psi, \Phi)$,

for all $\Omega, \Phi, \Psi \in \mathbb{X}$. The pair $(\mathbb{X}, d)$ is called a controlled metric-type space.

The paper is organized as follows. In Section 2, we introduce the definition and examples of a controlled M-metric space, a novel form of generalized metric space. We then explore the fundamental topological properties of these controlled M-metric spaces. Moving on to Section 3, we establish various fixed-point results for self-mappings of controlled M-metric spaces. Given that non-unique fixed points are of interest, we delve into studying the geometric properties of fixed points, specifically in the context of the fixed-circle and fixed-disc problems. Previous works on metric and generalized metric spaces, such as [8], have been influential in this area. Section 4 presents an application of our findings to high-degree polynomial equations, highlighting the practical utility of our results.

2. Controlled *M*-Metric Spaces

In this section, we present a novel generalization of the M-metric spaces. Then, we explore the fundamental properties of this newly defined concept, including various topological aspects.

2.1. The Notion of a Controlled M-Metric Space

Initially, we introduce the following notations.

Notation 3. *Consider a map $\nu : \mathbb{X}^2 \to [0, \infty)$ where $\mathbb{X}$ is a nonempty set*

- $\nu_{\Omega,\Phi} = \min\{\nu(\Omega, \Omega), \nu(\Phi, \Phi)\}$.
- $\mu_{\Omega,\Phi} = \max\{\nu(\Omega, \Omega), \nu(\Phi, \Phi)\}$.

Definition 4. *Suppose $\mathbb{X}$ is a nonempty set and $\alpha : \mathbb{X}^2 \to [1, \infty)$ and $\nu : \mathbb{X}^2 \to [0, \infty)$ are two functions. We define $(\mathbb{X}, \nu)$ as a controlled M-metric space if the following conditions are met for all distinct $\Omega, \Phi, \Psi \in \mathbb{X}$:*

(ν_1) $\nu(\Omega, \Phi) = \nu_{\Omega,\Phi} = \mu_{\Omega,\Phi}$ *if and only if* $\Omega = \Phi$;
(ν_2) $\nu_{\Omega,\Phi} \leq \nu(\Omega, \Phi)$;
(ν_3) $\nu(\Omega, \Phi) = \nu(\Phi, \Omega)$;
(ν_4) $\nu(\Omega, \Phi) - \nu_{\Omega,\Phi} \leq \alpha(\Omega, \Psi)[\nu(\Omega, \Psi) - \nu_{\Omega,\Psi}] + \alpha(\Psi, \Phi)[\nu(\Psi, \Phi) - \nu_{\Psi,\Phi}]$.

For the remainder of this paper, we refer to controlled M-metric spaces as $CMMS$. Now, let us provide an example of $CMMS$.

Example 1. *Suppose $\mathbb{C}$ is the set of all complex numbers and $A = \{\Psi \in \mathbb{C} : |\Psi| = 1\}$, $B = \{\Psi \in \mathbb{C} : |\Psi| = 2\} \subset \mathbb{C}$. Consider the set $\mathbb{X} = A \cup B \cup \{0\}$ and the functions $\alpha : \mathbb{X}^2 \to [1, \infty)$ and $\nu : \mathbb{X}^2 \to [0, \infty)$ defined by*

$$\alpha(\Psi_1, \Psi_2) = |\Psi_1||\Psi_2| + 1$$

and

$$\nu(\Psi_1, \Psi_2) = |\Psi_1 - \Psi_2|,$$

respectively, for all $\Psi_1, \Psi_2 \in \mathbb{X}$. We can easily verify that $(\mathbb{X}, \nu)$ is a controlled M-metric space. Indeed, conditions $(\nu_1) - (\nu_3)$ are trivial by the definition of the absolute value function. For the condition (ν_4), we have

$$\nu(\Omega, \Phi) - \nu_{\Omega,\Phi} = |\Omega - \Phi| - \min\{\nu(\Omega, \Omega), \nu(\Phi, \Phi)\} = |\Omega - \Phi|$$

and

$$\begin{aligned} & \alpha(\Omega, \Psi)[\nu(\Omega, \Psi) - \nu_{\Omega,\Psi}] + \alpha(\Psi, \Phi)[\nu(\Psi, \Phi) - \nu_{\Psi,\Phi}] \\ = \ & (|\Omega||\Psi| + 1)[|\Omega - \Psi| - 0] + (|\Psi||\Phi| + 1)[|\Psi - \Phi| - 0] \\ = \ & (|\Omega||\Psi| + 1)|\Omega - \Psi| + (|\Psi||\Phi| + 1)|\Psi - \Phi|. \end{aligned}$$

Then, by the triangle inequality and considering the facts that $\alpha(\Omega, \Psi) \geq 1, \alpha(\Psi, \Phi) \geq 1$ for all distinct $\Omega, \Phi, \Psi \in \mathbb{X}$, we obtain

$$\begin{aligned} |\Omega - \Phi| & \leq |\Omega - \Psi| + |\Psi - \Phi| \\ & \leq (|\Omega||\Psi| + 1)|\Omega - \Psi| + (|\Psi||\Phi| + 1)|\Psi - \Phi|. \end{aligned}$$

Thus, condition (ν_4) is a consequence of the triangle inequality for the absolute value function.

Remark 1. *(1) If we set the function $\alpha : \mathbb{X}^2 \to [1, \infty)$ as $\alpha(\Omega, \Phi) = 1$ for all $\Omega, \Phi \in \mathbb{X}$, then $(\mathbb{X}, v)$ becomes an M-metric space. Consequently, every M-metric space is a CMMS. However, it is crucial to note that a CMMS does not necessarily qualify as an M-metric space, as demonstrated in the following example.*

(2) If $\nu(\Omega, \Omega) = 0$ for all $\Omega \in \mathbb{X}$, then $(\mathbb{X}, v)$ is a controlled metric-type space. Despite this, not all CMMS instances are controlled metric-type spaces, as illustrated in the subsequent example.

(3) If $\nu_{\Omega,\Phi} = \mu_{\Omega,\Phi}$ for each $\Omega, \Phi \in \mathbb{X}$, then taking $\alpha = \beta$ in Definition 3.1 introduced in [3], we see that our Definition 4 coincides with this definition of a double controlled M-metric space (see Example 1).

Example 2. *Let $\mathbb{X} = \{1,2,3\}$ and the functions $\alpha : \mathbb{X}^2 \to [1,\infty)$ and $\nu : \mathbb{X}^2 \to [0,\infty)$ be defined as*

$$\alpha(\Omega,\Phi) = (\Omega+\Phi)^2$$

and

$$\begin{array}{l} v(1,1) = v(2,2) = v(3,3) = 1, \\ v(1,2) = v(2,1) = 7, \\ v(1,3) = v(3,1) = 5, \\ v(2,3) = v(3,2) = 2, \end{array}$$

for all $\Omega,\Phi \in \mathbb{X}$, respectively, then $(\mathbb{X},\nu)$ is a CMMS. However, the function v is not both an M-metric and a controlled metric type. Indeed, if we take $\Omega = 1$, $\Phi = 2$, $\Psi = 3$, then we have

$$v(1,2) - v_{1,2} = 6 \leq [v(1,3) - v_{1,3}] + [v(3,2) - v_{3,2}] = 5,$$

a contradiction. Hence, the condition (4) of Definition 1 is not satisfied, that is, v is not an M-metric. However, we observe that $v(1,1) = v(2,2) = v(3,3) = 1 \neq 0$. As a result, the condition $(\mathbb{D}_1)$ is not fulfilled, and thus, v does not qualify as a controlled metric-type space.

Next, we proceed to introduce the following intriguing proposition.

Proposition 1. *Consider a CMMS, denoted by $(\mathbb{X}, v)$, and let $\Omega,\Phi,\Psi \in \mathbb{X}$. The following hold:*
(1) $\mu_{\Omega,\Phi} + v_{\Omega,\Phi} = v(\Omega,\Omega) + v(\Phi,\Phi) \geq 0$.
(2) $\mu_{\Omega,\Phi} - v_{\Omega,\Phi} = |v(\Omega,\Omega) - v(\Phi,\Phi)| \geq 0$.
(3) $\mu_{\Omega,\Phi} - v_{\Omega,\Phi} \leq \alpha(\Omega,\Psi)[\mu_{\Omega,\Psi} - v_{\Omega,\Psi}] + \alpha(\Psi,\Phi)[\mu_{\Psi,\Phi} - v_{\Psi,\Phi}]$.

Proof. (1) First, we may assume $v(\Omega,\Omega) \geq v(\Phi,\Phi)$. Then, we obtain

$$\mu_{\Omega,\Phi} = v(\Omega,\Omega) \text{ and } v_{\Omega,\Phi} = v(\Phi,\Phi).$$

Hence, using the definition of v, we obtain

$$\mu_{\Omega,\Phi} + v_{\Omega,\Phi} = v(\Omega,\Omega) + v(\Phi,\Phi) \geq 0.$$

(2) Similar to the reasoning used as in the proof of (1), we can easily demonstrate this.
(3) Suppose that

$$v(\Phi,\Phi) < v(\Psi,\Psi) < v(\Omega,\Omega).$$

Then, we obtain

$$\begin{array}{rcl} \mu_{\Omega,\Phi} - v_{\Omega,\Phi} & = & v(\Omega,\Omega) - v(\Phi,\Phi) \\ & = & [v(\Omega,\Omega) - v(\Psi,\Psi)] + [v(\Psi,\Psi) - v(\Phi,\Phi)] \\ & \leq & \alpha(\Omega,\Psi)[v(\Omega,\Omega) - v(\Psi,\Psi)] + \alpha(\Psi,\Phi)[v(\Psi,\Psi) - v(\Phi,\Phi)] \\ & = & \alpha(\Omega,\Psi)[\mu_{\Omega,\Psi} - v_{\Omega,\Psi}] + \alpha(\Psi,\Phi)[\mu_{\Psi,\Phi} - v_{\Psi,\Phi}]. \end{array}$$

Given that $\alpha(\Omega,\Psi) \geq 1$ and $\alpha(\Psi,\Phi) \geq 1$, we find that the condition (3) is fulfilled. For the remaining cases, similar arguments are used. □

2.2. Basic Topological Properties

We define the following, which are used in the theoretical results.

Definition 5. *Consider a CMMS denoted by $(\mathbb{X}, \nu)$. The following hold:*

1. *A sequence $\{\Omega_n\}$ in $\mathbb{X}$ converges to a point Ω if and only if*

$$\lim_{n\to\infty} (\nu(\Omega_n,\Omega) - \nu_{\Omega_n,\Omega}) = 0.$$

2. *A sequence $\{\Omega_n\}$ in $\mathbb{X}$ is said to be a ν-Cauchy sequence if and only if*

$$\lim_{n,m\to\infty}(\nu(\Omega_n,\Omega_m)-\nu_{\Omega_n,\Omega_m}) \text{ and } \lim_{n\to\infty}(\mu_{\Omega_n,\Omega_m}-\nu_{\Omega_n,\Omega_m})$$

exist and are finite.

3. *A CMMS is said to be ν-complete if every ν-Cauchy sequence $\{\Omega_n\}$ converges to a point Ω such that*

$$\lim_{n\to\infty}(\nu(\Omega_n,\Omega_m)-\nu_{\Omega_n,\Omega_m})=0 \text{ and } \lim_{n\to\infty}(\mu_{\Omega_n,\Omega_m}-\nu_{\Omega_n,\Omega_m})=0.$$

Definition 6. *Consider a CMMS denoted by $(\mathbb{X},\nu)$, $\Omega\in\mathbb{X}$ and $\varepsilon\geq 0$.*

(1) The open ball $B(\Omega,\varepsilon)$ is

$$B(\Omega,\varepsilon)=\{\Phi\in\mathbb{X},\nu(\Omega,\Phi)-\nu_{\Omega,\Phi}<\varepsilon\}.$$

(2) The closed ball $B[\Omega,\varepsilon]$ is

$$B[\Omega,\varepsilon]=\{\Phi\in\mathbb{X},\nu(\Omega,\Phi)-\nu_{\Omega,\Phi}\leq\varepsilon\}.$$

(3) The circle $C^v_{\Omega,\varepsilon}$ is

$$C^v_{\Omega,\varepsilon}=\{\Phi\in\mathbb{X},\nu(\Omega,\Phi)-\nu_{\Omega,\Phi}=\varepsilon\}.$$

Definition 7. *Consider a CMMS denoted by $(\mathbb{X},\nu)$ and let $A\subset\mathbb{X}$. If there exists $\varepsilon>0$ such that $B(a,\varepsilon)\subset A$, then A is referred to as an open subset of $\mathbb{X}$.*

Definition 8. *Consider a CMMS denoted by $(\mathbb{X},\nu)$.*

(1) The self-mapping $\mathbb{T}:\mathbb{X}\to\mathbb{X}$ is considered continuous at $\Omega\in\mathbb{X}$ if, for all $\varepsilon>0$, there exists $\delta>0$ such that $\mathbb{T}(B(\Omega,\delta))\subseteq B(\mathbb{T}\Omega,\varepsilon)$.

(2) The mapping $\mathbb{T}:\mathbb{X}\to Y$ is referred to as sequentially continuous at $\Omega\in\mathbb{X}$ if and only if $\{\mathbb{T}\Omega_n\}$ converges to a point $\mathbb{T}\Omega$ whenever $\{\Omega_n\}$ converges to a point Ω.

Next, we present the following lemma.

Lemma 1. *Consider a CMMS denoted by $(\mathbb{X},\nu)$. If the sequence Ω_n in $\mathbb{X}$ converges to both Ω and Φ with $\Omega\neq\Phi$, then we have $v(\Omega,\Phi)-v_{\Omega,\Phi}=0$*

Proof. Suppose the sequence $\{\Omega_n\}$ converges to two different points, say, Ω and Φ. Using Definition 5 (1), we obtain

$$\lim_{n\to\infty}(\nu(\Omega_n,\Omega)-\nu_{\Omega_n,\Omega})=0$$

and

$$\lim_{n\to\infty}(\nu(\Omega_n,\Phi)-\nu_{\Omega_n,\Phi})=0.$$

Using the conditions (v_3) and (v_4), we obtain

$$\nu(\Omega,\Phi)-\nu_{\Omega,\Phi}\leq\alpha(\Omega,\Omega_n)[\nu(\Omega_n,\Omega)-\nu_{\Omega_n,\Omega}]+\alpha(\Omega_n,\Phi)[\nu(\Omega_n,\Phi)-\nu_{\Omega_n,\Phi}]$$

and so taking a limit for $n\to\infty$, by the condition (v_2), we have

$$\nu(\Omega,\Phi)-\nu_{\Omega,\Phi}=0.$$

□

Lemma 2. *Consider a CMMS denoted by $(\mathbb{X},\nu)$. If the function ν is sequentially continuous, then the limit of a convergent sequence is unique.*

Proof. Let Ω_n be a convergent sequence in a *CMMS* $(\mathbb{X}, \nu)$, assume that Ω_n converges to x and y in $\mathbb{X}$. Now, using the fact that the function ν is sequentially continuous, we deduce that

$$0 = \lim_{n\to\infty} (\nu(\Omega_n, \Omega_n) - \nu_{\Omega_n,\Omega_n}) = \nu(x, y) - \nu_{x,y}.$$

Similarly,

$$0 = \lim_{n\to\infty} (\nu(\Omega_n, \Omega_n) - \nu_{\Omega_n,\Omega_n}) = \mu(x, y) - \nu_{x,y}.$$

Thus, $x = y$ as required. □

3. Fixed-Point Results

First, we present this interesting and useful lemma.

Lemma 3. *Consider a CMMS denoted by $(\mathbb{X}, \nu)$ and let $\mathbb{T}$ be self-mapping on $\mathbb{X}$ such that for all $\Omega, \Phi \in \mathbb{X}$, we have,*

$$\nu(\mathbb{T}\Omega, \mathbb{T}\Phi) \leq k\nu(\Omega, \Phi), \text{ where } 0 < k < 1. \tag{1}$$

For some $\Omega_0 \in \mathbb{X}$, the sequence $\{\Omega_n\}_{n\geq 0}$ is defined by $\Omega_{n+1} = \mathbb{T}\Omega_n$. If $\Omega_n \to u$ as $n \to \infty$, then $\mathbb{T}\Omega_n \to \mathbb{T}u$ as $n \to \infty$.

Proof. Noting that if $\nu(\mathbb{T}\Omega_n, \mathbb{T}u) = 0$, then $\nu_{\mathbb{T}\Omega_n,\mathbb{T}u} = 0$. This is due to the fact that $\nu_{\mathbb{T}\Omega_n,\mathbb{T}u} \leq \nu(\mathbb{T}\Omega_n, \mathbb{T}u)$, implying that

$$\nu(\mathbb{T}\Omega_n, \mathbb{T}u) - \nu_{\mathbb{T}\Omega_n,\mathbb{T}u} \to 0 \text{ as } n \to \infty \text{ and hence } \mathbb{T}\Omega_n \to \mathbb{T}u \text{ as } n \to \infty.$$

Hence, if we assume that $\nu(\mathbb{T}\Omega_n, \mathbb{T}u) > 0$, by inequality (1), we may deduce that $\nu(\mathbb{T}\Omega_n, \mathbb{T}u) < \nu(\Omega_n, u)$.

Case 1: If $\nu(u, u) \leq \nu(\Omega_n, \Omega_n)$ and by (1), we find that $\nu(\Omega_n, \Omega_n) \to 0$, implying that $\nu(u, u) = 0$ and with $\nu(\mathbb{T}u, \mathbb{T}u) \leq k\nu(u, u) = 0$ we find that $\nu(\mathbb{T}u, \mathbb{T}u) = \nu(u, u) = 0$ and $\nu(\Omega_n, u) \to 0$. On the other hand, we have $\nu(\mathbb{T}\Omega_n, \mathbb{T}u) \leq k\nu(\Omega_n, u) \to 0$. Therefore, $\nu(\mathbb{T}\Omega_n, \mathbb{T}u) - \nu_{\mathbb{T}u,\mathbb{T}\Omega_n} \to 0$ and thus $\mathbb{T}\Omega_n \to \mathbb{T}u$.

Case 2: If $\nu(u, u) \geq \nu(\Omega_n, \Omega_n)$ and by (1), we find that $\nu(\Omega_n, \Omega_n) \to 0$, implying that $\nu_{\Omega_n,u} \to 0$. From this, $\nu(\Omega_n, u) \to 0$ and since $\nu(\mathbb{T}\Omega_n, \mathbb{T}u) < \nu(\Omega_n, u) \to 0$, then $\nu(\mathbb{T}\Omega_n, \mathbb{T}u) - \nu_{\mathbb{T}u,\mathbb{T}\Omega_n} \to 0$ and thus $\mathbb{T}\Omega_n \to \mathbb{T}u$ as desired. □

The first fixed-point result is as the following.

Theorem 1. *Consider a CMMS denoted by $(\mathbb{X}, \nu)$ that is ν-complete. Let $\mathbb{T}$ be self-mapping on $\mathbb{X}$ such that for all $\Omega, \Phi \in \mathbb{X}$. Then, we have*

$$\nu(\mathbb{T}\Omega, \mathbb{T}\Phi) \leq k\nu(\Omega, \Phi), \text{ where } 0 < k < 1. \tag{2}$$

Subsequently, consider the sequence $\Omega_n = \mathbb{T}^n\Omega_0$ for some $\Omega_0 \in \mathbb{X}$ where

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\alpha(\Omega_{i+1}, \Omega_{i+2})}{\alpha(\Omega_i, \Omega_{i+1})} \alpha(\Omega_{i+1}, \Omega_m) < \frac{1}{k}. \tag{3}$$

If for each $\Omega \in \mathbb{X}$, say

$$\lim_{n\to\infty} \alpha(\Omega_n, \Omega) \text{ and } \lim_{n\to\infty} \alpha(\Omega, \Omega_n) \text{ exist and are finite,} \tag{4}$$

there is a unique fixed point for $\mathbb{T}$.

Proof. Throughout this proof, we denote by Ω_n the sequence defined in Lemma 3. In addition, let

$$\nu_n^m = \nu(\Omega_n, \Omega_m) - \nu_{\Omega_n,\Omega_m}.$$

Now, it is not difficult to see that by using (2), we obtain

$$\nu(\Omega_n, \Omega_{n+1}) \leq k^n \nu(\Omega_0, \Omega_1) \text{ for all } n \geq 0,$$

which implies that

$$\nu_n^{n+1} \leq \nu(\Omega_n, \Omega_{n+1}) \leq k^n \nu(\Omega_0, \Omega_1).$$

For all natural numbers $n < m$, we have

$$\begin{aligned}
\nu_n^m &\leq \alpha(\Omega_n, \Omega_{n+1})\nu_n^{n+1} + \alpha(\Omega_{n+1}, \Omega_m)\nu_{n+1}^m \\
&\leq \alpha(\Omega_n, \Omega_{n+1})\nu_n^{n+1} + \alpha(\Omega_{n+1}, \Omega_m)\alpha(\Omega_{n+1}, \Omega_{n+2})\nu_{n+1}^{n+2} \\
&+ \alpha(\Omega_{n+1}, \Omega_m)\alpha(\Omega_{n+2}, \Omega_m)\nu_{n+2}^m \\
&\leq \alpha(\Omega_n, \Omega_{n+1})\nu_n^{n+1} + \alpha(\Omega_{n+1}, \Omega_m)\alpha(\Omega_{n+1}, \Omega_{n+2})\nu_{n+1}^{n+2} \\
&+ \alpha(\Omega_{n+1}, \Omega_m)\alpha(\Omega_{n+2}, \Omega_m)\alpha(\Omega_{n+2}, \Omega_{n+3})\nu_{n+2}^{n+3} \\
&+ \alpha(\Omega_{n+1}, \Omega_m)\alpha(\Omega_{n+2}, \Omega_m)\alpha(\Omega_{n+3}, \Omega_m)\nu_{n+3}^m \\
&\leq \cdots \\
&\leq \alpha(\Omega_n, \Omega_{n+1})\nu_n^{n+1} + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(\Omega_j, \Omega_m)\right)\alpha(\Omega_i, \Omega_{i+1})\nu_i^{i+1} \\
&+ \prod_{k=n+1}^{m-1}\alpha(\Omega_k, \Omega_m)\nu_{m-1}^m \\
&\leq \alpha(\Omega_n, \Omega_{n+1})k^n\nu(\Omega_0, \Omega_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(\Omega_j, \Omega_m)\right)\alpha(\Omega_i, \Omega_{i+1})k^i\nu(\Omega_0, \Omega_1) \\
&+ \prod_{i=n+1}^{m-1}\alpha(\Omega_i, \Omega_m)k^{m-1}\nu(\Omega_0, \Omega_1).
\end{aligned}$$

Hence,

$$\begin{aligned}
\nu_n^m &\leq \alpha(\Omega_n, \Omega_{n+1})k^n\nu(\Omega_0, \Omega_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(\Omega_j, \Omega_m)\right)\alpha(\Omega_i, \Omega_{i+1})k^i\nu(\Omega_0, \Omega_1) \\
&+ \left(\prod_{i=n+1}^{m-1}\alpha(\Omega_i, \Omega_m)\right)k^{m-1}\alpha(\Omega_{m-1}, \Omega_m)\nu(\Omega_0, \Omega_1) \\
&= \alpha(\Omega_n, \Omega_{n+1})k^n\nu(\Omega_0, \Omega_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\alpha(\Omega_j, \Omega_m)\right)\alpha(\Omega_i, \Omega_{i+1})k^i\nu(\Omega_0, \Omega_1) \\
&\leq \alpha(\Omega_n, \Omega_{n+1})k^n\nu(\Omega_0, \Omega_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\alpha(\Omega_j, \Omega_m)\right)\alpha(\Omega_i, \Omega_{i+1})k^i\nu(\Omega_0, \Omega_1).
\end{aligned}$$

Since $\alpha(\Omega, \Phi) \geq 1$, we deduce that

$$F_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\alpha(\Omega_j, \Omega_m)\right)\alpha(\Omega_i, \Omega_{i+1})k^i.$$

Hence, we have

$$\nu_n^m \leq \nu(\Omega_0, \Omega_1)[k^n\alpha(\Omega_n, \Omega_{n+1}) + (F_{m-1} - F_n)]. \tag{5}$$

Using the condition (3), and applying the ratio test, we can establish that the sequence $\{F_n\}$ is ν-Cauchy. Now, taking the limit as $n, m \to \infty$ in the inequality (5), we conclude that

$$\lim_{n,m\to\infty} \nu_n^m = 0. \tag{6}$$

Moreover, by using (2), it is easy to see that the sequence $\{\nu(\Omega_n, \Omega_n)\}_{n\geq 0}$ is decreasing. Hence, for $n \leq m$, we have

$$\mu_{\Omega_n,\Omega_m} = \nu(\Omega_n, \Omega_n).$$

Thus,

$$\mu_{\Omega_n,\Omega_m} - \nu_{\Omega_n,\Omega_m} \leq \nu(\Omega_n, \Omega_n) \leq k\nu(\Omega_{n-1}, \Omega_{n-1}) \leq \cdots \leq k^n \nu(\Omega_0, \Omega_0). \tag{7}$$

Limiting in inequalities (7), as n, m tends to ∞, we obtain

$$\lim_{n,m\to\infty} \mu_{\Omega_n,\Omega_m} - \nu_{\Omega_n,\Omega_m} = 0.$$

Therefore, $\{\Omega_n\}$ is a ν-Cauchy sequence in the ν-complete $CMMS$ $(\mathbb{X}, \nu)$, so $\{\Omega_n\}$ converges to some $a \in \mathbb{X}$. Next, we prove that a is a fixed point of $\mathbb{T}$. Using Lemma 3, we have

$$a = \lim_{n\to\infty} \Omega_{n+1} = \lim_{n\to\infty} \mathbb{T}\Omega_n = \mathbb{T}a.$$

a is therefore a fixed point of $\mathbb{T}$. Suppose that $\mathbb{T}$ has two fixed points, a and b, and

$$\nu_{a,b} = \nu(a, a) \text{ and } \mu_{a,b} = \nu(b, b).$$

Thus, we obtain

$$\begin{aligned}
\nu(a,b) - \nu_{a,b} &\leq \nu(a,b) \\
&= \nu(\mathbb{T}a, \mathbb{T}b) \\
&\leq k\nu(a,b) \\
&= k\nu(\mathbb{T}a, \mathbb{T}b) \\
&\leq k^2\nu(a,b) \\
&= k^2\nu(\mathbb{T}a, \mathbb{T}b) \\
&< \vdots \\
&\leq k^n\nu(a,b) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

implying that $\nu(a, b) = \nu_{a,b} = \nu(a, a)$. In addition,

$$\begin{aligned}
\mu_{a,b} - \nu_{a,b} &\leq \mu_{a,b} \\
&= \nu(b,b) \\
&= \nu(\mathbb{T}b, \mathbb{T}b) \\
&\leq k\nu(b,b) \\
&= k\nu(\mathbb{T}b, \mathbb{T}b) \\
&\leq k^2\nu(b,b) \\
&= k^2\nu(\mathbb{T}b, \mathbb{T}b) \\
&< \vdots \\
&\leq k^n\nu(b,b) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, $\nu(a, b) = \nu_{a,b} = \mu_{a,b}$, which implies $a = b$, as desired. $\square$

Definition 9. *Consider a self-mapping $\mathbb{T}$ on a nonempty set $\mathbb{X}$ and let $\Omega_0 \in \mathbb{X}$, and $O(\Omega_0) = \{\Omega_0, \mathbb{T}\Omega_0, \mathbb{T}^2\Omega_0, \ldots\}$ be the orbit of Ω_0. A function $H : \mathbb{X} \longrightarrow \mathbb{R}$ is said to be $\mathbb{T}$-orbitally lower semi-continuous at $v \in \mathbb{X}$ if for $\{\Omega_n\} \subset O(\Omega_0)$ such that $\Omega_n \longrightarrow v$, we have $H(v) \leq \lim_{n\to\infty} \inf H(\Omega_n)$.*

Similar to [9], using Definition 9, we can present the following corollary of Theorem 1, which is a generalization of Theorem 1 in [10].

Corollary 1. *Consider a CMMS denoted by $(\mathbb{X}, \nu)$ that is ν-complete. Let $\Omega_0 \in \mathbb{X}$ and $\mathbb{T}$ be self-mapping on $\mathbb{X}$. Suppose there exists $k \in (0,1)$ such that*

$$\nu(\mathbb{T}\Phi, \mathbb{T}^2 y) \leq k\nu(\Phi, \mathbb{T}\Phi) \text{ for each } y \in O(\Omega_0).$$

Take $\Omega_n = \mathbb{T}^n \Omega_0$. Let

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(\Omega_{i+1}, \Omega_{i+2})}{\alpha(\Omega_i, \Omega_{i+1})} \alpha(\Omega_{i+1}, \Omega_m) < \frac{1}{k}. \tag{8}$$

Then, $\Omega_n \to u \in \mathbb{X}$ (as $n \to \infty$). Moreover, $\mathbb{T}u = u$ if and only if the functional $\Omega \mapsto \nu(\Omega, \mathbb{T}\Omega)$ is $\mathbb{T}$-orbitally lower semi-continuous at u.

We recall the following lemma.

Lemma 4. *If a self-mapping $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ is surjective, then there exists a self-mapping $\mathbb{T}^* : \mathbb{X} \to \mathbb{X}$ such that $\mathbb{T} \circ \mathbb{T}^*$ is the identity map on $\mathbb{X}$.*

We have proved another fixed-point results for an expansive condition.

Theorem 2. *Consider a CMMS denoted by $(\mathbb{X}, \nu)$ that is ν-complete and $\mathbb{T}$ be a surjective self-mapping on $\mathbb{X}$ such that for all $\Omega, \Phi \in \mathbb{X}$ we have*

$$v(\mathbb{T}\Omega, \mathbb{T}\Phi) \geq kv(\Omega, \Phi), \tag{9}$$

where $k > 1$. Let us consider the sequence $\Omega_n = \mathbb{T}^n \Omega_0$ for some $\Omega_0 \in \mathbb{X}$ where

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(\Omega_{i+1}, \Omega_{i+2})}{\alpha(\Omega_i, \Omega_{i+1})} \alpha(\Omega_{i+1}, \Omega_m) < k.$$

Additionally, if for every $\Omega \in \mathbb{X}$, we have

$$\lim_{n \to \infty} \alpha(\Omega_n, \Omega) \text{ and } \lim_{n \to \infty} \alpha(\Omega, \Omega_n) \text{ exist and are finite.}$$

This implies that $\mathbb{T}$ has a unique fixed point.

Proof. Since $\mathbb{T}$ is surjective, by Lemma 4, there exists a self-mapping $\mathbb{T}^* : \mathbb{X} \to \mathbb{X}$ such that $\mathbb{T} \circ \mathbb{T}^*$ is the identity map on $\mathbb{X}$. Let $\Omega, \Phi \in \mathbb{X}$ be arbitrary points. Assume that $\mathbb{T}^*\Omega = \Psi$ and $\mathbb{T}^*\Phi = w$. Then, we have

$$\mathbb{T}\Psi = \mathbb{T}\mathbb{T}^*\Omega = \Omega \text{ and } \mathbb{T}w = \mathbb{T}\mathbb{T}^*\Phi = \Phi.$$

From inequality (9), we have

$$v(\mathbb{T}\Psi, \mathbb{T}w) \geq kv(\Psi, w),$$

this implies

$$v(\Omega, \Phi) \geq kv(\mathbb{T}^*\Omega, \mathbb{T}^*y)$$

and

$$v(\mathbb{T}^*\Omega, \mathbb{T}^*y) \leq \frac{1}{k} v(\Omega, \Phi),$$

where $\frac{1}{k} \in (0,1)$. Hence, $\mathbb{T}^*$ satisfies the inequality (2). From Theorem 1, $\mathbb{T}^*$ has a unique fixed point $a \in \mathbb{X}$. Therefore, we have

$$\mathbb{T}a = \mathbb{T}\mathbb{T}^* a = a,$$

that is, the point a is a fixed point of $\mathbb{T}$. Suppose that the point b is another fixed point of $\mathbb{T}$ such that $a \neq b$. Therefore, using inequality (9), we obtain

$$v(\mathbb{T}a, \mathbb{T}b) = v(a,b) \geq kv(a,b),$$

a contradiction. Consequently, $\mathbb{T}$ has a unique fixed point in $\mathbb{X}$. □

Recently, the fixed-circle (resp. fixed-disc) problem has been studied by different methods on a metric space and some generalized metric spaces in the cases where the fixed point is non-unique (for example, see [11] and the references therein). In this context, we define the notion of a fixed circle and obtain new fixed-circle results on a controlled M-metric space.

Definition 10. *Let $(\mathbb{X}, v)$ be a CMMS and $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ be self-mapping. If $\mathbb{T}$ fixes all of the points $C^v_{\Omega_0,r}$, that is, $\mathbb{T}\Omega = \Omega$ for all $\Omega \in C^v_{\Omega_0,r}$, then $C^v_{\Omega_0,r}$ is called the fixed circle of $\mathbb{T}$.*

Definition 11. *Let $(\mathbb{X}, v)$ be a CMMS and $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ be self-mapping. $\mathbb{T}$ is called a controlled Ω_0-mapping if there exist $\Omega_0 \in \mathbb{X}$ and $k \in (0,1)$ such that for all $\Omega \in \mathbb{X}$, we have*

$$v(\Omega, \mathbb{T}\Omega) - v_{\Omega,\mathbb{T}\Omega} \leq k\big[v(\Omega, \Omega_0) - v_{\Omega,\Omega_0}\big]. \tag{10}$$

Before we prove our fixed-circle result, we define the number r as

$$r = \inf_{\Omega \in \mathbb{X}} \{(\Omega, \mathbb{T}\Omega) - v_{\Omega,\mathbb{T}\Omega} : \mathbb{T}\Omega \neq \Omega\}. \tag{11}$$

Theorem 3. *Let $(\mathbb{X}, v)$ be a CMMS, $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ be a controlled Ω_0-mapping with $\Omega_0 \in \mathbb{X}$ and r be defined as in (11). If $v_{\Omega,\mathbb{T}\Omega} = \mu_{\Omega,\mathbb{T}\Omega}$ for each $\Omega \in C^v_{\Omega_0,r}$, then $\mathbb{T}$ fixes the circle $C^v_{\Omega_0,r}$.*

Proof. Let $r = 0$. Then, we have $\Omega_0 \in C^v_{\Omega_0,r} = \{\Omega : v(\Omega, \Omega_0) = v_{\Omega,\Omega_0}\}$. Using the controlled Ω_0-mapping property, we obtain

$$v(\Omega_0, \mathbb{T}\Omega_0) - v_{\Omega_0,\mathbb{T}\Omega_0} \leq k\big[v(\Omega_0, \Omega_0) - v_{\Omega_0,\Omega_0}\big] = 0$$

when using the hypothesis $v_{\Omega,\mathbb{T}\Omega} = \mu_{\Omega,\mathbb{T}\Omega}$ and the condition (v_2), we get

$$v(\Omega_0, \mathbb{T}\Omega_0) = v_{\Omega_0,\mathbb{T}\Omega_0} = \mu_{\Omega_0,\mathbb{T}\Omega_0}.$$

By condition (v_1), we find $\mathbb{T}\Omega_0 = \Omega_0$.

Let $r > 0$ and $\Omega \in C^v_{\Omega_0,r}$. Now, we show that $\mathbb{T}\Omega = \Omega$ whenever $\Omega \in C^v_{\Omega_0,r}$. From the inequality (10), we have

$$v(\Omega, \mathbb{T}\Omega) - v_{\Omega,\mathbb{T}\Omega} \leq k\big[v(\Omega, \Omega_0) - v_{\Omega,\Omega_0}\big] = kr,$$

which implies $v(\Omega, \mathbb{T}\Omega) - v_{\Omega,\mathbb{T}\Omega} = 0$ by the definition of r and the condition (v_2). Using the hypothesis $v_{\Omega,\mathbb{T}\Omega} = \mu_{\Omega,\mathbb{T}\Omega}$, we obtain

$$v(\Omega, \mathbb{T}\Omega) = v_{\Omega,\mathbb{T}\Omega} = \mu_{\Omega,\mathbb{T}\Omega},$$

that is, by condition (v_1), $\Omega = \mathbb{T}\Omega$. Consequently, $C^v_{\Omega_0,r}$ is a fixed circle of $\mathbb{T}$. □

Example 3. *Consider the CMMS defined in Example 1. Let us define the self-mapping $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ as*

$$\mathbb{T}\Psi = \begin{cases} \Psi & ; \quad \Psi \in A \cup \{0\} \\ \frac{\Psi}{2} & ; \quad \Psi \in B \end{cases},$$

for all $\Psi \in \mathbb{X}$. Then, $\mathbb{T}$ is a controlled Ω_0-mapping with $\Omega_0 = 0$. In addition, we obtain

$$r = \inf_{\Omega \in \mathbb{X}} \{(\Omega, \mathbb{T}\Omega) - v_{\Omega,\mathbb{T}\Omega} : \mathbb{T}\Omega \neq \Omega\} = 1$$

and

$$C^{v}_{0,1} = \{\Omega : v(\Omega, 0) - v_{\Omega,0} = 1\} = A.$$

Since the self-mapping $\mathbb{T}$ satisfies the conditions of Theorem 3, $\mathbb{T}$ fixes the circle $C^{v}_{0,1}$.

If $\mathbb{T}$ fixes all of the points $B[\Omega_0, r]$, then $B[\Omega_0, r]$ is called as the fixed-closed ball (or fixed disc) of $\mathbb{T}$.

Corollary 2. *Consider a CMMS denoted by $(\mathbb{X}, v)$ and let $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ be a controlled Ω_0-mapping where $\Omega_0 \in \mathbb{X}$ and r be defined as in (11). If $v_{\Omega,\mathbb{T}\Omega} = \mu_{\Omega,\mathbb{T}\Omega}$ for each $\Omega \in B[\Omega_0, r]$, then $\mathbb{T}$ fixes the closed ball $B[\Omega_0, r]$.*

Proof. By employing similar arguments as in the proof of Theorem 3, we can readily verify this. □

4. An Application to the Determination of Polynomial Zeros

In this section, we showcase the following application utilizing our fixed-point result Theorem 4.

Theorem 4. *Let $n \geq 2$ be any even natural number. The equation*

$$\Omega^{n} + 1 = (n^{3} - 1)\Omega^{n+1} + n^{3}\Omega \tag{12}$$

has a unique positive real solution.

Proof. If $|\Omega| > 1$, then clearly Equation (12) does not have a solution. Therefore, we consider the interval $[-1, 1]$. Now, by Descartes' Rule of Signs (see [12] and the references therein), we know that the equation

$$(n^{3} - 1)\Omega^{n+1} - \Omega^{n} + n^{3}\Omega - 1 = 0 \tag{13}$$

has 3 or 1 positive real roots and has no negative real roots. Therefore, we take $\mathbb{X} = [0, 1]$. Let us define the functions

$$\nu(\Omega, \Phi) = |\Omega - \Phi|$$

and

$$\alpha(\Omega, \Phi) = |\Omega| + |\Phi| + 2,$$

for all $\Omega, \Phi \in \mathbb{X}$. It is not difficult to see that $(\mathbb{X}, \nu)$ is a ν-complete *CMMS*.

Let us consider the self-mapping

$$\mathbb{T}\Omega = \frac{\Omega^{n} + 1}{(n^{3} - 1)\Omega^{n} + n^{3}}.$$

We obtain

$$\frac{\Omega^{n} + 1}{(n^{3} - 1)\Omega^{n} + n^{3}} \leq \frac{\Omega^{n} + 1}{n^{3}}.$$

Next, since $n \geq 2$, we can deduce that $n^3 > 6$. Hence, we obtain

$$\begin{aligned}
\nu(\mathbb{T}\Omega, \mathbb{T}\Phi) &= \left| \frac{\Omega^n + 1}{(n^3 - 1)\Omega^n + n^3} - \frac{\Phi^n + 1}{(n^3 - 1)\Phi^n + n^3} \right| \\
&= \left| \frac{\Omega^n - \Phi^n}{((n^3 - 1)\Omega^n + n^3)((n^3 - 1)\Phi^n + n^3)} \right| \\
&\leq \frac{|\Omega - \Phi|}{n^3} \\
&\leq \frac{|\Omega - \Phi|}{6} = \frac{1}{6}\nu(\Omega, \Phi).
\end{aligned}$$

Therefore, we obtain

$$\nu(\mathbb{T}\Omega, \mathbb{T}\Phi) \leq k\nu(\Omega, \Phi) \text{ where } k = \frac{1}{6}.$$

On the other hand, for each $\Omega_0 \in \mathbb{X}$ we have

$$\Omega_n = \mathbb{T}^n \Omega_0 \leq \frac{2}{n^3}.$$

Thus,

$$\begin{aligned}
\sup_{n \geq 1} \lim_{i \to \infty} \frac{\alpha(\Omega_{i+1}, \Omega_{i+2})}{\alpha(\Omega_i, \Omega_{i+1})} \alpha(\Omega_{i+1}, \Omega_m) &= \sup_{m \geq 1} \lim_{i \to \infty} \frac{|\Omega_{i+1}| + |\Omega_{i+2}| + 2}{|\Omega_i| + |\Omega_{i+1}| + 2} (|\Omega_{i+1}| + |\Omega_m| + 2) \\
&\leq \sup_{n \geq 1} \lim_{i \to \infty} \frac{\frac{4}{n^3} + 2}{\frac{4}{n^3} + 2} \left(\frac{4}{n^3} + 2 \right) \\
&= \frac{4}{n^3} + 2 \\
&\leq 4 + 2 = 6 = \frac{1}{k}.
\end{aligned}$$

Similarly, we can see that

$$\lim_{n \to \infty} \alpha(\Omega_m, \Omega) \text{ and } \lim_{m \to \infty} \alpha(\Omega, \Omega_m) \text{ exist and are finite.}$$

Hence, all the conditions of Theorem 1 are satisfied. Consequently, $\mathbb{T}$ possesses a unique fixed point in $\mathbb{X}$, implying that Equation (12) has only one real solution. □

Descartes' Rule of Signs only says that the polynomial defined in (13) has 3 or 1 positive real roots, while Theorem 1 guarantees the uniqueness of the positive real roots. Thus, Theorem 1 provides complementary support to the Descartes' Rule of Signs in the study of polynomial zeros.

5. Conclusions

In this paper, we have presented the concept of a controlled M-metric space, which includes both M-metric spaces and controlled metric-type spaces. As a result, our findings extend and generalize numerous results already presented in the literature. Nevertheless, we invite the reader to explore further the applications of Meir–Keeler contraction and Suzuki contraction in the context of controlled M-metric spaces. These investigations could lead to new and intriguing outcomes. Furthermore, on this new space, our fixed-circle results can be extended to several fixed-figure results. That is, the existence of a geometric figure (e.g., a disc, an ellipse, a Cassini curve) in the fixed-point set of a self-mapping on a controlled M-metric space can be investigated (for more details see, for instance, ref. [11] and the references therein).

Author Contributions: Conceptualization, K.S., N.T. and N.Ö.; methodology, N.Ö. and N.M.; formal analysis, K.S., N.T. and N.Ö.; investigation, K.S., N.T., N.Ö. and N.M.; writing—original draft preparation,K.S., N.T., N.Ö. and N.M.; writing—review and editing, K.S., N.T., N.Ö. and N.M.; supervision, N.Ö. and N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Arab Open University, Saudi Arabia through AOU research fund No. (AOURG-2023-006).

Data Availability Statement: No data were used to support this work.

Acknowledgments: The authors extend their appreciation to the Arab Open University, Saudi Arabia for Funding this work through AOU research fund No. (AOURG-2023-006).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Borisut, P.; Kumam, P.; Gupta, V.; Mani, N. Generalized (ψ, α, β)-weak Contractions for Initial Value Problems. *Mathematics* **2019**, *7*, 266. [CrossRef]
2. Gupta, V.; Shatanawi, W.; Mani, N. Fixed point theorems for (ψ, β)-Geraghty contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations. *J. Fixed Point Theory Appl.* **2017**, *19*, 1251–1267. [CrossRef]
3. Uddin, F.; Adeel, F.; Javed, K.; Park, C.; Arshad, M. Double controlled M-metric spaces and some fixed point results. *AIMS Math.* **2022**, *7*, 15298–15312. [CrossRef]
4. Asadi, M.; Karapınar, E.; Salimi, P. New extention of p-metric spaces with some fixed point results on M-metric spaces. *J. Inequal. Appl.* **2014**, *2014*, 18. [CrossRef]
5. Mlaiki, N.; Zarrad, A.; Souayah, N.; Mukheimer, A.; Abdeljawed, T. Fixed Point Theorems in M_b-metric spaces. *J. Math. Anal.* **2016**, *7*, 1–9.
6. Mlaiki, N.; Özgür, N.Y.; Taş, A.M.N. A new extension of the M_b-metric spaces. *J. Math. Anal.* **2018**, *9*, 118–133.
7. Mlaiki, N.; Aydi, H.; Souayah, N.; Abdeljawad, T. Controlled metric type spaces and the related contraction principle. *Mathematics* **2018**, *6*, 194. [CrossRef]
8. Ahmad, H.; Younis, M.; Koksal, M.E. Double Controlled Partial Metric Type Spaces and Convergence Results. *J. Math.* **2021**, *2021*, 7008737. [CrossRef]
9. Kamran, T.; Samreen, M.; Ain, Q.U.L. A Generalization of b -metric space and some fixed point theorems. *Mathematics* **2017**, *5*, 19. [CrossRef]
10. Hicks, T.L.; Rhodes, B.E. A Banach type fixed point theorem. *Math. Japan.* **1979**, *24*, 327–330.
11. Taş, N.; Özgür, N. New fixed-figure results on metric spaces. In *Fixed Point Theory and Fractional Calculus-Recent Advances and Applications*; Forum for Interdisciplinary Mathematics; Springer: Singapore, 2022; pp. 33–62._3 [CrossRef]
12. Grabiner, D.J. Descartes' rule of signs: Another construction. *Am. Math. Mon.* **1999**, *106*, 854–856. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.