Research Article

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Continued fractions related to a group of linear fractional transformations

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Abstract: There are strong relations between the theory of continued fractions and groups of linear fractional transformations. We consider the group $G_{3,3}$ generated by the linear fractional transformations $a = 1 - 1/z$ and $b = z + 2$. This group is the unique subgroup of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ with index 2. We calculate the cusp point of an element given as a word in generators. Conversely, we use the continued fraction expansion of a given rational number p/q, to obtain an element in $G_{3,3}$ with cusp point p/q. As a result, we say that the action of $G_{3,3}$ on rational numbers is transitive.

Keywords: continued fractions, modular group, cusp points

MSC 2020: 11A55, 20H10

1 Introduction

The modular group is the projective special linear group $\mathrm{PSL}(2,\mathbb{Z})$, which is isomorphic to the quotient group $\mathrm{SL}(2,\mathbb{Z})/\pm I$, so that an element can be represented by a matrix

$$M = \begin{pmatrix} x & u \\ y & v \end{pmatrix}, \tag{1}$$

where $x, y, u, v \in \mathbb{Z}$ with $xv - yu = 1$. It should be noted that both the matrix M in (1) and its negative $-M$ both represent the same element in $\mathrm{PSL}(2,\mathbb{Z})$. The modular group acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ via fractional linear transformations [1]:

$$\left(\begin{pmatrix} x & u \\ y & v \end{pmatrix}, z\right) \to \frac{xz + u}{yz + v}. \tag{2}$$

Elements of the modular group are orientation preserving isometries of $\mathbb{H}$. The modular group is generated by the transformations:

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + 1,$$

and the matrix representations of these generators are

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

As T of order 2 and TU of order 3, $\mathrm{PSL}(2,\mathbb{Z})$ has the presentation

$$\mathrm{PSL}(2,\mathbb{Z}) = \langle T, U : T^2 = (TU)^3 = 1 \rangle, \tag{3}$$

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which can be thought as $\mathbb{Z}_2 * \mathbb{Z}_3$, the free product of the cyclic group of order 2 and the cyclic group of order 3 [2].

There is a natural relation between continued fractions and the modular group. Consider the element

$$W = \begin{pmatrix} x & u \\ y & v \end{pmatrix} = U^{r_0}.\ T.\ U^{r_1}.\ .....U^{r_n}.\ T^i,$$

where $r_j \in \mathbb{Z}$ and $i = 0, 1$. The fractional linear transformation that corresponds to W is

$$\begin{aligned} W(z) &= U^{r_0}.\ T.\ U^{r_1}.\ ....T.\ U^{r_n}.\ T(z) \\ &= U^{r_0}.\ T.\ U^{r_1}.\ ....T.\ U^{r_n}.\ \left(\frac{-1}{z}\right) \\ &= U^{r_0}.\ T.\ U^{r_1}.\ ....T.\ \left(r_n - \frac{1}{z}\right) \\ &= ... \\ &= r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots r_{n-1} - \cfrac{1}{r_n - \frac{1}{z}}}}}. \end{aligned}$$

We can represent the element W with the continued fraction expansion $[r_0, r_1, ..., r_n; z]$. We can adjoin to $\mathbb{H}$ a point at infinity, with the usual convention that $\frac{1}{\infty} = 0$. Then we call the image of infinity under W, the cusp point of W. This can also be written as the finite negative integer continued fraction

$$W(\infty) = \frac{x}{y} = [r_0, r_1, ..., r_n] = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots r_{n-1} - \cfrac{1}{r_n}}}}.$$

We interested in the group $G_{3,3}$, which is the only subgroup of the modular group with index 2 [3]. This group is generated by the transformations:

$$a = U.\ T = 1 - \frac{1}{z} \quad b = U^2 = z + 2$$

and has presentation

$$G_{3,3} = \langle a, b : a^3 = (a^2 b)^3 = 1 \rangle \simeq \mathbb{Z}_3 * \mathbb{Z}_3. \tag{4}$$

In this study, we obtain relations between integer continued fractions and elements of the group $G_{3,3}$. Then we calculate the cusp point of the element given as a word in terms of generators. Conversely, we obtain an element whose cusp point is a given rational. In addition, it is stated that the cusp point set of $G_{3,3}$ is $\mathbb{Q}_\infty$. As a result, we say that the action of $G_{3,3}$ on $\mathbb{Q}_\infty$ is transitive.

2 Motivation and history

The modular group has been studied extensively. The abstract group structure of the modular group and its subgroups are studied in [2,4–13]. Some of the popular number sequences, like Fibonacci, Pell, Lucas, etc., are related to the modular group and Hecke groups, which are a generalization of the modular group [14–20].

In recent years, many studies have related the theory of continued fractions to the action of some subgroups of fractional linear transformations of the complex plane. Parabolic and elliptic elements of the modular group are studied in [21] from the view of continued fractions and graph theory. Demir and Koruoğlu obtained the word form of such elements using paths in the Farey tree and continued fractions. Also the transitive action of the modular group on the set of rational numbers is studied in [22].

As stated in Section 1, the cusp point (or sometimes it is called parabolic point) of an element V is the image of infinity under its action, i.e., $V(\infty)$. In [23], Koruoğlu used the modular group blocks

$$U(z) = z + 1 \quad \text{and} \quad UTU(z) = \frac{z}{z+1}$$

to calculate the cusp point of a given element in terms of these blocks. Powers of these blocks are associated with simple continued fractions.

The relations between integer continued fractions and Fibonacci numbers with cusp points of the modular group are studied in [16]. In [24], integer continued fraction expansions and geodesic expansions are studied from the perspective of graph theory. Short and Walker represented Rosen continued fractions by paths in a class of graphs in hyperbolic geometry [25]. One of the interesting studies about continued fractions with even partial quotients is [26]. Kraaikamp and Lopes considered the Theta group Θ, generated by

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U^2(z) = z + 2.$$

They obtained important results about even integer continued fractions and closed geodesic analogous to the one related to the modular group. Moreover, Short and Walker studied the geometric representation of even-integer continued fractions and some subgroups of the modular group [27].

The natural connection between the modular group and integer continued fractions is due to the generator T of order 2. In this study, we try not to use the generator T of order 2, and we focus on the subgroup $G_{3,3}$, which is the only subgroup of the modular group with index 2. By considering the presentation given in equation 4, it can be seen that every element in $G_{3,3}$ has the word form

$$b^{m_0}a^{n_0}b^{m_1}a^{n_1}. ..b^{m_r}a^{n_r},$$

where $m_i \in \mathbb{Z}\backslash\{0\}$, $n_i = 1, 2$, m_0, and n_r can be zero.

Since we do not have the generator T of order 2, it is not straightforward to see the relation between integer continued fraction expansion and word form of the element. For example, consider the word

$$a^2b^4ab,$$

then the corresponding linear fractional transformation is

$$\begin{aligned} a^2b^4ab(z) &= a^2b^4a(z+2) \\ &= a^2b^4\left(1 - \frac{1}{z+2}\right) \\ &= a^2\left(8 + 1 - \frac{1}{z+2}\right) \\ &= -\frac{1}{-1 + 8 + 1 - \frac{1}{z+2}}. \end{aligned}$$

We can represent this element with the integer continued fraction expansion $[0, 8, 2 + z]$. As an another example, one can consider the word $b^2a^2b^3a$ by the corresponding fractional linear transformation associated the integer continued fraction expansion $[4, 6, z]$.

3 Results

3.1 Cusp points of $G_{3,3}$

In this section, we calculate the cusp point of an element in $G_{3,3}$ given as a word in generators. First, we need a basic lemma about elements that have same cusp point.

Lemma 1. *Let W be an element in the group $G_{3,3}$, then the elements W and $W.\,b^m$ have same cusp point for all integers m.*

Proof. Suppose the element W has matrix representation

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix}.$$

The corresponding fractional linear transformation is

$$W(z) = \frac{xz + u}{yz + v},$$

which has the cusp point $W(\infty) = \frac{x}{y}$. Now we calculate

$$W.\, b^m = \begin{pmatrix} x & u \\ y & v \end{pmatrix} . \begin{pmatrix} 1 & 2m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 2mx + u \\ y & 2my + v \end{pmatrix}.$$

The cusp point of $W.\, b^m$ is equal to the cusp point W. □

Theorem 1. *Let $W = b^{m_0}a^{n_0}b^{m_1}a^{n_1}\ldots b^{m_r}a^{n_r}$ be an element in the group $G_{3,3}$, where $m_i \in \mathbb{Z}\backslash\{0\}$, $n_i = 1, 2$, m_0 can be zero. Then, the cusp point of W has integer continued fraction expansion*:

$$\left[2m_0 + \left\lfloor\frac{1}{n_0}\right\rfloor, 2m_1 + \left\lfloor\frac{1}{n_0}\right\rfloor + \left\lfloor\frac{1}{n_1}\right\rfloor - 1, 2m_2 + \left\lfloor\frac{1}{n_1}\right\rfloor + \left\lfloor\frac{1}{n_2}\right\rfloor - 1, \ldots, \right.$$
$$\left. 2m_{r-1} + \left\lfloor\frac{1}{n_{r-2}}\right\rfloor + \left\lfloor\frac{1}{n_{r-1}}\right\rfloor - 1, 2m_r + \left\lfloor\frac{1}{n_{r-1}}\right\rfloor + \left\lfloor\frac{1}{n_r}\right\rfloor - 1\right].$$

Proof. Firstly, if $W(z) = b^{m_0}(z) = z + 2m_0$, then the cusp point of W is equal to infinity. Now we prove the claim by induction on r. Let us start with $r = 0$. There are two cases of n_0

Case 1: $n_0 = 1$

$W(z) = b^{m_0}a(z) = 2m_0 + 1 - \frac{1}{z}$. The cusp point is $W(\infty) = 2m_0 + 1$, which is an integer.

Case 2: $n_0 = 2$

$W(z) = b^{m_0}a^2(z) = 2m_0 - \frac{1}{-1+z}$. Then the cusp point is $W(\infty) = 2m_0 = [2m_0]$.

We see that the claim is true for $r = 0$. Suppose the claim is true for integers $0, 1, 2, \ldots, r - 1$. Now we continue with the integer r.

$$W = b^{m_0}a^{n_0}b^{m_1}a^{n_1}.\,..b^{m_{r-1}}a^{n_{r-1}}b^{m_r}a^{n_r}.$$

Here, n_r can be equal to 1 or 2.

By the induction hypothesis, we can calculate the cusp point of the elements that consist of $r - 1$ times a and $r - 1$ times b. In other words, the cusp point of

$$b^{m_1}a^{n_1} \ldots b^{m_{r-1}}a^{n_{r-1}}b^{m_r}a^{n_r}$$

is equal to

$$\left[2m_1 + \left\lfloor\frac{1}{n_1}\right\rfloor, 2m_2 + \left\lfloor\frac{1}{n_1}\right\rfloor + \left\lfloor\frac{1}{n_2}\right\rfloor - 1, 2m_3 + \left\lfloor\frac{1}{n_2}\right\rfloor + \left\lfloor\frac{1}{n_3}\right\rfloor - 1, \ldots, 2m_{r-1} + \left\lfloor\frac{1}{n_{r-2}}\right\rfloor + \left\lfloor\frac{1}{n_{r-1}}\right\rfloor - 1, 2m_r + \left\lfloor\frac{1}{n_{r-1}}\right\rfloor + \left\lfloor\frac{1}{n_r}\right\rfloor - 1\right].$$

We denote this number by κ. Now we are ready to calculate the cusp point of the element W.

$$\begin{aligned} W(\infty) &= b^{m_0}a^{n_0}b^{m_1}a^{n_1}.\,..b^{m_{r-1}}a^{n_{r-1}}b^{m_r}a^{n_r}(\infty) \\ &= b^{m_0}a^{n_0}\{b^{m_1}a^{n_1}.\,..b^{m_{r-1}}a^{n_{r-1}}b^{m_r}a^{n_r}(\infty)\} \\ &= b^{m_0}a^{n_0}(\kappa). \end{aligned}$$

The last row of the above equation is equal to $2m_0 + 1 - \frac{1}{\kappa}$ if $n_0 = 1$, and $2m_0 - \frac{1}{-1+\kappa}$ if $n_0 = 2$. We combine the both subcases

$$W(\infty) = b^{m_0}a^{n_0}(\kappa) = 2m_0 + \left\lfloor \frac{1}{n_0} \right\rfloor - \frac{1}{\left\lfloor \frac{1}{n_0} \right\rfloor - 1 + \kappa},$$

which concludes the proof. □

Remark 1. We omit the case $n_r = 0$ in Theorem 1. Because, if $n_r = 0$ then we have the word

$$W = b^{m_0}a^{n_0}b^{m_1}a^{n_1}. ..b^{m_{r-1}}a^{n_{r-1}}b^{m_r}.$$

We know from Lemma 1 that W has the same cusp point with the element $W' = b^{m_0}a^{n_0}b^{m_1}a^{n_1}. ..b^{m_{r-1}}a^{n_{r-1}}$.

Example 1. Consider the word $W = a^2b^5a^2b^3aba^2$ in the group $G_{3,3}$. Here, we point out the values $r = 3$, $m_0 = 0$ and $n_3 = 2$. By Theorem 1, we have

$$\begin{aligned} W(\infty) &= \left[0, 10 + \lfloor \frac{1}{2} \rfloor + \lfloor \frac{1}{2} \rfloor - 1, 6 + \lfloor \frac{1}{2} \rfloor + \lfloor \frac{1}{1} \rfloor - 1, 2 + \lfloor \frac{1}{1} \rfloor + \lfloor \frac{1}{2} \rfloor - 1\right] \\ &= [0, 9, 6, 2] \\ &= -\frac{11}{97}. \end{aligned}$$

3.2 Obtaining an element in $G_{3,3}$, with given cusp point

Here, our aim is to construct an element in $G_{3,3}$ with a given cusp point. We know that any rational has an integer continued fraction expansion. We use the digits in the expansion to obtain the powers of the generators a and b in the word form of the element.

Theorem 2. *Let the reduced rational number $\frac{p}{q}$ has integer continued fraction expansion $[r_0, r_1, r_2, \ldots, r_n]$. Then,*

$$W = b^{k_0}a^{m_0}b^{k_1}a^{m_1}. ..b^{k_n}a^{m_n}b^{i}$$

is the element in $G_{3,3}$ with cusp point $\frac{p}{q}$, where

$$k_0 = \left\lfloor \frac{r_0}{2} \right\rfloor, \quad m_0 = \begin{cases} 1, & \text{if } r_0 \text{ is odd,} \\ 2, & \text{if } r_0 \text{ is even,} \end{cases}$$

i is an arbitrary integer and for $1 \le j \le n$

$$k_j = \begin{cases} \left\lfloor \dfrac{r_j}{2} \right\rfloor, & \text{if } m_{j-1} = 1, \\ \left\lfloor \dfrac{r_j + 1}{2} \right\rfloor, & \text{if } m_{j-1} = 2, \end{cases}$$

$$m_j = \begin{cases} 1, & \text{if } (m_{j-1} = 1 \ \wedge \ r_j \text{ is odd}) \ \vee \ (m_{j-1} = 2 \ \wedge r_j \text{ is even}), \\ 2, & \text{if } (m_{j-1} = 1 \ \wedge \ r_j \text{ is even}) \ \vee \ (m_{j-1} = 2 \ \wedge r_j \text{ is odd}). \end{cases}$$

Proof. We prove the theorem by induction on n. First, $n = 0$ means $\frac{p}{q} = [r_0] = r_0$ is an integer. Since the generator b provides a 2 unit translation, we need $a^2 = \frac{-1}{z-1}$ to map infinity to zero and than $b^{\frac{r_0}{2}}$ to map 0 to r_0 if r_0 is even. In other words, the word $b^{\frac{r_0}{2}}a^2$ has the cusp point r_0. If r_0 is odd, we first map infinity to 1 by $a = 1 - \frac{1}{z}$. Then we apply $\frac{r_0-1}{2}$ units translation by $b^{\frac{r_0-1}{2}}$. Hence, the desired element is $b^{\frac{r_0-1}{2}}a$. This means the claim is true for $n = 0$.

Suppose the theorem is true for $0, 1, 2, \ldots, n-1$. We continue the induction with n. Let the rational $\frac{p}{q}$ has continued fraction expansion $[r_0, r_1, \ldots, r_{n-1}, r_n]$. Consider the rational $\frac{p'}{q'} = [r_0, r_1, \ldots, r_{n-1}]$. By the induction hypothesis, the element

$$W' = b^{k_0} a^{m_0} b^{k_1} a^{m_1}. ..b^{k_{n-1}} a^{m_{n-1}}$$

has cusp point $\frac{p'}{q'}$ where the powers of the generators are calculated as in the stated theorem. Hence, the linear fractional transformation that corresponds to W' has one of the following forms:

$$W'(z) = r_0 - \cfrac{1}{r_1 - \cfrac{1}{\ddots r_{n-1} - \frac{1}{z-1}}}$$

or

$$W'(z) = r_0 - \cfrac{1}{r_1 - \cfrac{1}{\ddots r_{n-1} - 1 + 1 - \frac{1}{z}}}.$$

It is easy to see that $m_{n-1} = 2$ in the first case and $m_{n-1} = 1$ for the latter. We investigate both.

Case 1: $m_{n-1} = 2$

In this case, let us set $k_n = \lfloor \frac{r_n+1}{2} \rfloor$. If r_n is odd, we consider the word $b^{k_n} a^2$. Then one can see by calculation that the desired word is $W = W'. b^{k_n} a^2$. Similarly, if r_n is even, we now choose the block $b^{k_n} a$. And the element with cusp point $\frac{p}{q}$ is $W = W'. b^{k_n} a$

Case 2: $m_{n-1} = 1$

As in the first case, we set $k_n = \lfloor \frac{r_n}{2} \rfloor$. For r_n is odd, the element with cusp point $\frac{p}{q}$ is $W = W'. b^{k_n} a$. And if r_n is even, then we obtain the element $W = W'. b^{k_n} a^2$.

After obtaining the word W, by Lemma 1, one can multiply this word with any integer power of b from the right. □

Example 2. Suppose the given rational is $[0, 9, 6, 2] = -\frac{11}{97}$. By using Theorem 2, we have

$$\begin{aligned}
k_0 &= 0 \quad m_0 = 2 \\
k_1 &= \left\lfloor \frac{r_1+1}{2} \right\rfloor = 5 \quad m_1 = 2 \\
k_2 &= \left\lfloor \frac{r_2+1}{2} \right\rfloor = 3 \quad m_2 = 1 \\
k_3 &= \left\lfloor \frac{r_3}{2} \right\rfloor = 1 \quad m_3 = 2.
\end{aligned}$$

Hence, we obtain the word $W = a^2 b^5 a^2 b^3 a b a^2$. Considering Lemma 1, we can also say that the elements $a^2 b^5 a^2 b^3 a b a^2 b^i$ have cusp point $-\frac{11}{97}$ for $i \in \mathbb{Z}$.

We know that a real number is rational if and only if it has finite integer continued fraction expansion. Theorem 2 says every rational number can be considered as a cusp point of an element in $G_{3,3}$. In addition to that, we can consider infinity as a fraction $\frac{1}{0}$, the cusp point of the generator $b(z) = z + 2$. Hence, the cusp point set of $G_{3,3}$ is $\mathbb{Q}_\infty$. Now let W_1 and W_2 be two elements in $G_{3,3}$ with cusp points $\frac{p}{q}, \frac{p'}{q'} \in \mathbb{Q}_\infty$, respectively. Then the element $W = W_2. W_1^{-1}$ maps $\frac{p}{q}$ to $\frac{p'}{q'}$. As a result, we obtain the following corollary.

Corollary 1. *The action of the group $G_{3,3}$ on $\mathbb{Q}_\infty$ is transitive.*

4 Conclusion

The theory of continued fractions arises from the Euclidean algorithm, one of the oldest and most basic concepts of mathematics. There is a natural relation between continued fractions and the modular group. We exhibit this relation for the subgroup $G_{3,3}$. We calculate the integer continued fraction expansion of the cusp point of an element in the group $G_{3,3}$. Then conversely, we obtain an element with a given cusp point. For further research, one can consider the elliptic generators

$$a^2 = x = -\frac{1}{z-1} \quad \text{and} \quad a^2b = y = -\frac{1}{z+1}$$

Then every element in the group $G_{3,3}$ consists of the blocks xy, x^2y, xy^2, and x^2y^2. By using the technique in [3,20], one can relate our results to Fibonacci and Pell numbers.

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