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Interpolative contractions and discontinuity at fixed point

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Abstract

In this paper, we investigate new solutions to the Rhoades' discontinuity problem on the existence of a self-mapping which has a fixed point but is not continuous at the fixed point on metric spaces. To do this, we use the number defined as

$$n(\mathfrak{x},\mathfrak{y}) = [d(\mathfrak{x},\mathfrak{y})]^{\beta} [d(\mathfrak{x},T\mathfrak{x})]^{\alpha} [d(\mathfrak{y},T\mathfrak{y})]^{\gamma} \left[\frac{d(\mathfrak{x},T\mathfrak{y}) + d(\mathfrak{y},T\mathfrak{x})}{2}\right]^{1-\alpha-\beta-\gamma},$$

where $\alpha, \beta, \gamma \in (0,1)$ with $\alpha+\beta+\gamma < 1$ and some interpolative type contractive conditions. Also, we investigate some geometric properties of $Fix(T)$ under some interpolative type contractions and prove some fixed-disc (resp. fixed-circle) results. Finally, we present a new application to the discontinuous activation functions.

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Keywords: *Rhoades' open problem; fixed-circle problem; interpolative type contractive condition.*

1. Introduction and Motivation

There are some examples of self-mappings has a fixed point but is not continuous at this fixed point as seen in the following example:

Let $(\mathbb{R}, d)$ be the usual metric space with the function $d : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ defined as

$$d(\mathfrak{x},\mathfrak{y}) = |\mathfrak{x} - \mathfrak{y}|,$$

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for all $\mathfrak{x}, \mathfrak{y} \in \mathbb{R}$. If we consider the self-mappings $T : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ defined as

$$T\mathfrak{x} = \begin{cases} \mathfrak{x} & ; \quad \mathfrak{x} \geq 0 \\ \mathfrak{x}+1 & ; \quad \mathfrak{x} < 0 \end{cases}$$

and

$$S\mathfrak{x} = \begin{cases} \mathfrak{x} & ; \quad \mathfrak{x} \leq 0 \\ 2 & ; \quad 0 < \mathfrak{x} \leq 10 \\ 11 & ; \quad \mathfrak{x} > 10 \end{cases},$$

for all $\mathfrak{x} \in \mathbb{R}$, then we have $Fix(T) = \{\mathfrak{x} \in \mathbb{R} : \mathfrak{x} = T\mathfrak{x}\} = \mathbb{R}^+ \cup \{0\}$ and T is discontinuous at the fixed point $\mathfrak{x} = 0$. Also, we obtain $Fix(S) = \{\mathfrak{x} \in \mathbb{R} : \mathfrak{x} = S\mathfrak{x}\} = \mathbb{R}^- \cup \{0, 2, 11\}$ and S is discontinuous at the fixed point $\mathfrak{x}_1 = 0$ and $\mathfrak{x}_2 = 2$.

In this context, in [19], Rhoades introduced the following open problem related to the discontinuity at fixed point:

"What are the contractive conditions which are strong enough to generate a fixed point but which do not force the map to be continuous at fixed point?"

After then, in [14], Pant obtained a first solution using the following number:

$$m(\mathfrak{x}, \mathfrak{y}) = \max\left\{d(\mathfrak{x}, T\mathfrak{x}), d(\mathfrak{y}, T\mathfrak{y})\right\}.$$

As an another solution, in [2, 3], Bisht and Pant studied on this open problem using the numbers

$$M(\mathfrak{x}, \mathfrak{y}) = \max\left\{d(\mathfrak{x}, \mathfrak{y}), d(\mathfrak{x}, T\mathfrak{x}), d(\mathfrak{y}, T\mathfrak{y}), \frac{d(\mathfrak{x}, T\mathfrak{y}) + d(\mathfrak{y}, T\mathfrak{x})}{2}\right\}$$

and

$$M^*(\mathfrak{x}, \mathfrak{y}) = \max\left\{d(\mathfrak{x}, \mathfrak{y}), d(\mathfrak{x}, T\mathfrak{x}), d(\mathfrak{y}, T\mathfrak{y}), \frac{\alpha\left[d(\mathfrak{x}, T\mathfrak{y}) + d(\mathfrak{y}, T\mathfrak{x})\right]}{2}\right\}, \ \alpha \in [0, 1).$$

Also, using different approaches, new solutions to the this problem were obtained by various authors (for example, see [4], [5], [15], [16], [17], [18] and the references therein).

On the other hand, a self-mapping can have more than one fixed point. For example, the self-mappings T and S have more than one fixed point. For this reason, the fixed-point theory has been studied by geometric thinking. Recently, the following open problem has been investigated for this purpose:

"What are the geometric properties of fixed points in which case a self-mapping has more than one fixed point?"

This problem was introduced as fixed-circle problem and first discussed in [11]. A first solution was obtained in[11] as follows:

Let (X, d) be a metric space and $C_{\mathfrak{x}_0,\rho}$ a circle on X. Let us define the mapping $\varphi : X \to [0, \infty)$ as

$$\varphi(\mathfrak{x}) = d(\mathfrak{x}, \mathfrak{x}_0),$$

for all $\mathfrak{x} \in X$. If there exists a self-mapping $T : X \to X$ satisfying

$(C1)$ $d(\mathfrak{x}, T\mathfrak{x}) \leq \varphi(\mathfrak{x}) - \varphi(T\mathfrak{x})$,

$(C2)$ $d(T\mathfrak{x}, x_0) \geq \rho$,
for each $\mathfrak{x} \in C_{\mathfrak{x}_0,\rho}$, then the circle $C_{\mathfrak{x}_0,\rho}$ is a fixed circle of T.

After this study, various solutions were proved using different techniques (for example, see [10], [12], [13], [16], [17], [18], [20] and the references therein).

From the above motivation, our aim is to obtain new solutions to the above two open problems. To do this, we modify the notions of interpolative Boyd-Wong type contraction and interpolative Matkowski type contraction. Also, we give some necessary examples to show the validity of the obtained results and present a new application to the discontinuous activation functions.

2. Main Results

In this section, we present some solutions to the Rhoades' open problem on the existence of a self-mapping which has a fixed point but is not continuous at the fixed point on metric spaces. Also, we investigate some geometric properties of a fixed point set $Fix(T)$ of a self-mapping $T : X \to X$ satisfying the used contractive conditions.

At first, we recall the followings:

Let Ψ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ such that
(ψ_1) $\psi(0) = 0$,
(ψ_2) $\psi(\mathfrak{t}) < \mathfrak{t}$ for each $\mathfrak{t} > 0$,
(ψ_3) ψ is upper semi-continuous from the right.

Definition 2.1 ([9], Interpolative Boyd-Wong type contraction)**.** Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an interpolative Boyd-Wong type contraction, if there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and a nondecreasing function $\psi \in \Psi$ such that

$$d(T\mathfrak{x}, T\mathfrak{y}) \leq \psi\left([d(\mathfrak{x}, \mathfrak{y})]^{\beta} [d(\mathfrak{x}, T\mathfrak{x})]^{\alpha} [d(\mathfrak{y}, T\mathfrak{y})]^{\gamma} \left[\frac{d(\mathfrak{x}, T\mathfrak{y}) + d(\mathfrak{y}, T\mathfrak{x})}{2}\right]^{1-\alpha-\beta-\gamma}\right),$$

for all $\mathfrak{x}, \mathfrak{y} \in X - Fix(T)$.

Let Φ be the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that
(ψ_1) ϕ is nondecreasing,
(ψ_2) $\lim_{n\to\infty} \phi^n(\mathfrak{t}) = 0$ for each $\mathfrak{t} > 0$.

Lemma 2.2 ([1], [8])**.** *Let $\phi \in \Phi$. Then $\phi(\mathfrak{t}) < \mathfrak{t}$ for all $\mathfrak{t} > 0$ and $\phi(0) = 0$.*

Definition 2.3 ([9], Interpolative Matkowski type contraction)**.** Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an interpolative Matkowski type contraction, if there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and $\phi \in \Phi$ such that

$$d(T\mathfrak{x}, T\mathfrak{y}) \leq \phi\left([d(\mathfrak{x}, \mathfrak{y})]^{\beta} [d(\mathfrak{x}, T\mathfrak{x})]^{\alpha} [d(\mathfrak{y}, T\mathfrak{y})]^{\gamma} \left[\frac{d(\mathfrak{x}, T\mathfrak{y}) + d(\mathfrak{y}, T\mathfrak{x})}{2}\right]^{1-\alpha-\beta-\gamma}\right),$$

for all $\mathfrak{x}, \mathfrak{y} \in X - Fix(T)$.

2.1. **Some Discontinuity Results.** In this subsection, to obtain new results related to the discontinuity at the fixed point, we use the following number:

$$n(\mathfrak{x},\mathfrak{y}) = [d(\mathfrak{x},\mathfrak{y})]^{\beta}\,[d(\mathfrak{x},T\mathfrak{x})]^{\alpha}\,[d(\mathfrak{y},T\mathfrak{y})]^{\gamma}\left[\frac{d(\mathfrak{x},T\mathfrak{y})+d(\mathfrak{y},T\mathfrak{x})}{2}\right]^{1-\alpha-\beta-\gamma},$$

where $\alpha,\beta,\gamma \in (0,1)$ with $\alpha+\beta+\gamma<1$.

We begin the following proposition:

Proposition 2.4. *Let T be a self-mapping of a complete metric space (X,d) such that for all $\mathfrak{x},\mathfrak{y}\in X$, we have*

(a) Given $\varepsilon>0$, there exists a $\delta>0$ such that

$$\varepsilon \le n(\mathfrak{x},\mathfrak{y}) < \varepsilon+\delta \Longrightarrow d(T\mathfrak{x},T\mathfrak{y})<\varepsilon.$$

Then the sequence $\{T^n\mathfrak{x}\}$ is a Cauchy sequence and

$$\lim_{n\to\infty} T^n\mathfrak{x} = \mathfrak{z},$$

for some $\mathfrak{z}\in X$ and given $\mathfrak{x}\in X$.

Proof. By the condition (a), if $n(\mathfrak{x},\mathfrak{y})>0$ then we get

$$d(T\mathfrak{x},T\mathfrak{y}) < n(\mathfrak{x},\mathfrak{y}). \tag{2.1}$$

Let $\mathfrak{x}_0\in X$ and let us define a sequence $\{\mathfrak{x}_n\}$ in X by

$$\mathfrak{x}_{n+1} = T\mathfrak{x}_n = T^n\mathfrak{x}_0$$

and

$$\mu_n = d(\mathfrak{x}_n,\mathfrak{x}_{n+1}),$$

$n\in\mathbb{N}\cup\{0\}$. Assume that $\mathfrak{x}_n\neq\mathfrak{x}_{n+1}$ for each n. Using the inequality (2.1), we obtain

$$\begin{aligned}
\mu_n &= d(\mathfrak{x}_n,\mathfrak{x}_{n+1}) = d(T\mathfrak{x}_{n-1},T\mathfrak{x}_n) < n(\mathfrak{x}_{n-1},\mathfrak{x}_n)\\
&= [d(\mathfrak{x}_{n-1},\mathfrak{x}_n)]^{\beta}\,[d(\mathfrak{x}_{n-1},T\mathfrak{x}_{n-1})]^{\alpha}\,[d(\mathfrak{x}_n,T\mathfrak{x}_n)]^{\gamma}\left[\frac{d(\mathfrak{x}_{n-1},T\mathfrak{x}_n)+d(\mathfrak{x}_n,T\mathfrak{x}_{n-1})}{2}\right]^{1-\alpha-\beta-\gamma}\\
&= [d(\mathfrak{x}_{n-1},\mathfrak{x}_n)]^{\beta}\,[d(\mathfrak{x}_{n-1},\mathfrak{x}_n)]^{\alpha}\,[d(\mathfrak{x}_n,\mathfrak{x}_{n+1})]^{\gamma}\left[\frac{d(\mathfrak{x}_{n-1},\mathfrak{x}_{n+1})+d(\mathfrak{x}_n,\mathfrak{x}_n)}{2}\right]^{1-\alpha-\beta-\gamma}\\
&= [d(\mathfrak{x}_{n-1},\mathfrak{x}_n)]^{\alpha+\beta}\,[d(\mathfrak{x}_n,\mathfrak{x}_{n+1})]^{\gamma}\left[\frac{d(\mathfrak{x}_{n-1},\mathfrak{x}_{n+1})}{2}\right]^{1-\alpha-\beta-\gamma}\\
&\le [d(\mathfrak{x}_{n-1},\mathfrak{x}_n)]^{\alpha+\beta}\,[d(\mathfrak{x}_n,\mathfrak{x}_{n+1})]^{\gamma}\left[\frac{d(\mathfrak{x}_{n-1},\mathfrak{x}_n)+d(\mathfrak{x}_n,\mathfrak{x}_{n+1})}{2}\right]^{1-\alpha-\beta-\gamma}\\
&= \mu_{n-1}^{\alpha+\beta}\mu_n^{\gamma}\left[\frac{\mu_{n-1}+\mu_n}{2}\right]^{1-\alpha-\beta-\gamma}.
\end{aligned} \tag{2.2}$$

Now suppose $\mu_{n-1}<\mu_n$ for some $n\ge 1$. Hence, we get

$$\frac{\mu_{n-1}+\mu_n}{2} \le \mu_n$$

and using the inequality (2.2), we have

$$\mu_n \leq \mu_{n-1}^{\alpha+\beta}\mu_n^{1-\alpha-\beta}. \tag{2.3}$$

By the inequality (2.3), we find

$$\mu_n^{\alpha+\beta} \leq \mu_{n-1}^{\alpha+\beta},$$

a contradiction. Therefore, it should be

$$\mu_n \leq \mu_{n-1},$$

for all $n \geq 1$. Thereby, $\{\mu_{n-1}\}$ is a nonincreasing sequence of positive real numbers. Let

$$\mu = \lim_{n\to\infty} \mu_{n-1}.$$

Then we have

$$\frac{\mu_{n-1}+\mu_n}{2} \leq \mu_{n-1},$$

for all $n \geq 1$ and using the inequality (2.2), we get

$$\mu_n < \mu_{n-1}.$$

So we obtain

$$\lim_{n\to\infty} \mu_n = \mu < \lim_{n\to\infty} \mu_{n-1} = \mu,$$

a contradiction. It should be

$$\lim_{n\to\infty} \mu_n = \mu = 0.$$

Now, we show that $\{\mathfrak{x}_n\}$ is Cauchy. On the contrary, we assume that $\{\mathfrak{x}_n\}$ is not a Cauchy sequence. Then there exist an $\varepsilon > 0$ and a subsequence $\{\mathfrak{x}_{n_i}\}$ of $\{\mathfrak{x}_n\}$ such that

$$d(\mathfrak{x}_{n_i}, \mathfrak{x}_{n_{i+1}}) > 2\varepsilon. \tag{2.4}$$

Let us select δ in (a) such that $0 < \delta \leq \varepsilon$. Since $\lim_{n\to\infty} \mu_n = 0$, there exists a positive integer N such that

$$\mu_n < \frac{\delta}{4}, \tag{2.5}$$

whenever $n \geq N$. Let $n_i > N$. Hence there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that

$$d(\mathfrak{x}_{n_i}, \mathfrak{x}_{m_i}) \geq \varepsilon + \frac{\delta}{2}. \tag{2.6}$$

If not, then

$$d(\mathfrak{x}_{n_i}, \mathfrak{x}_{n_{i+1}}) \leq d(\mathfrak{x}_{n_i}, \mathfrak{x}_{n_{i+1}-1}) + d(\mathfrak{x}_{n_{i+1}-1}, \mathfrak{x}_{n_{i+1}}) < \varepsilon + \frac{\delta}{2} + \frac{\delta}{4} < 2\varepsilon,$$

a contradiction. Let r_i be the smallest integer such that $n_i < r_i < n_{i+1}$ and

$$d(\mathfrak{x}_{n_i}, \mathfrak{x}_{r_i}) \geq \varepsilon + \frac{\delta}{2}.$$

Then

$$d(\mathfrak{x}_{n_i}, \mathfrak{x}_{r_i-1}) < \varepsilon + \frac{\delta}{2}.$$

By the condition (a), we have

$$\begin{aligned} d(\mathfrak{x}_{n_i},\mathfrak{x}_{r_i}) &= d(T\mathfrak{x}_{n_i-1},T\mathfrak{x}_{r_i-1}) < n(\mathfrak{x}_{n_i-1},\mathfrak{x}_{r_i-1}) \\ &= [d(\mathfrak{x}_{n_i-1},\mathfrak{x}_{r_i-1})]^{\beta}[d(\mathfrak{x}_{n_i-1},T\mathfrak{x}_{n_i-1})]^{\alpha}[d(\mathfrak{x}_{r_i-1},T\mathfrak{x}_{r_i-1})]^{\gamma} \\ &\quad \left[\frac{d(\mathfrak{x}_{n_i-1},T\mathfrak{x}_{r_i-1})+d(\mathfrak{x}_{r_i-1},T\mathfrak{x}_{n_i-1})}{2}\right]^{1-\alpha-\beta-\gamma} \end{aligned}$$

and so we get

$$\varepsilon \leq 0,$$

a contradiction. Therefore, $\{\mathfrak{x}_n\}$ is Cauchy in the complete metric space (X,d) and

$$\lim_{n\to\infty}\mathfrak{x}_n = \lim_{n\to\infty}T^n\mathfrak{x} = \mathfrak{z},$$

for some $\mathfrak{z}\in X$. □

To obtain our main discontinuity theorem, we recall the definition of the notion of a k-continuity and give some examples.

Definition 2.5 ([15])**.** A self-mapping T of a metric space X is called k-continuous, $k = 1,2,3,\ldots$, if $T^k\mathfrak{x}_n \to T\mathfrak{t}$ whenever $\{\mathfrak{x}_n\}$ is a sequence in X such that $T^{k-1}\mathfrak{x}_n \to \mathfrak{t}$.

In this paper, unless otherwise stated, we use the usual metric on $\mathbb{R}$ or $X \subset \mathbb{R}$.

Example 2.6. Let $X = [0,4]$ and $T : X \to X$ be defined by

$$T\mathfrak{x} = \begin{cases} 2 & ; \quad \mathfrak{x}\in[0,2] \\ 1 & ; \quad \mathfrak{x}\in(2,4] \end{cases}.$$

Since $T\mathfrak{x}_n \to \mathfrak{t}$ implies $\mathfrak{t}=1$ or $\mathfrak{t}=2$ and $T^2\mathfrak{x}_n = 2$ for all n, then $T\mathfrak{x}_n \to \mathfrak{t} \Longrightarrow T^2\mathfrak{x}_n \to \mathfrak{t}$. Therefore, T is 2-continuous, but T is discontinuous at $\mathfrak{x}=2$. On the other hand, the point $\mathfrak{x}=2$ is a fixed point of T.

Example 2.7. Let $X = [0,10]$ and $T : X \to X$ be defined by

$$T\mathfrak{x} = \begin{cases} 3 & ; \quad \mathfrak{x}\in[0,3] \\ 2 & ; \quad \mathfrak{x}\in(3,6] \\ \frac{\mathfrak{x}}{2} & ; \quad \mathfrak{x}\in(6,10] \end{cases}.$$

Then T is 3-continuous, but not 2-continuous. If attention, T has a fixed point $\mathfrak{x}=3$, but T is not continuous at this fixed point.

Remark 2.8. It is easy proved that 1-continuity is equivalent to continuity. Also, we have the following implications

$$\text{continuity} \Longrightarrow \text{2-continuity} \Longrightarrow \text{3-continuity} \Longrightarrow \ldots$$

and the converse statements of them are not always true (see Example 2.7 and for more details [17]).

Now we prove the following discontinuity theorem:

Theorem 2.9. *Let T be a self-mapping of a complete metric space (X,d) satisfying the condition (a) for all $\mathfrak{x},\mathfrak{y}\in X$. If T is k-continuous, then T has a fixed point $\mathfrak{z}$. Also, T is continuous at $\mathfrak{z}$ iff*

$$\lim_{\mathfrak{x}\to\mathfrak{z}} n(\mathfrak{x},\mathfrak{z})=0.$$

Proof. Let $\mathfrak{x}_0\in X$ and let us define a sequence $\{\mathfrak{x}_n\}$ in X by $\mathfrak{x}_n=T\mathfrak{x}_{n-1}$. Using Proposition 2.4, we obtain that $\{\mathfrak{x}_n\}$ is Cauchy. Since (X,d) is a complete metric space, there exists a point $\mathfrak{z}\in X$ such that $\{\mathfrak{x}_n\}\to\mathfrak{z}$. Also, we have $T^n\mathfrak{x}\to\mathfrak{z}$ for each $n\geq 1$.

Let T be a k-continuous self-mapping. Then we get

$$T^k\mathfrak{x}_n\to T\mathfrak{z} \text{ since } T^{k-1}\mathfrak{x}_n\to\mathfrak{z}$$

and so we have

$$T\mathfrak{z}=\mathfrak{z} \text{ as } T^k\mathfrak{x}_n\to\mathfrak{z}.$$

Thereby, $\mathfrak{z}$ is a fixed point of T. Finally, it is easily proved that T is continuous at $\mathfrak{z}$ iff

$$\lim_{\mathfrak{x}\to\mathfrak{z}} n(\mathfrak{x},\mathfrak{z})=0.$$

□

Example 2.10. Let $X=[0,2]$ and let us consider $T:X\to X$ defined by

$$T\mathfrak{x}=\begin{cases} 1 & ; \quad \mathfrak{x}\in[0,1] \\ 0 & ; \quad \mathfrak{x}\in(1,2] \end{cases},$$

for all $\mathfrak{x}\in X$ [14]. Then T satisfies the conditions of Theorem 2.9. Indeed, let us take

$$\delta(\varepsilon)=\begin{cases} 4-\varepsilon & ; \quad \varepsilon<1 \\ 4 & ; \quad \varepsilon\geq 1 \end{cases}.$$

Then T satisfies the condition (a) and T is 2-continuous. Consequently, T has a unique fixed point $\mathfrak{x}=1$ and T is discontinuous at this point.

Using the sets Ψ, Φ and the above similar arguments, we can easily prove the following discontinuity corollaries:

Corollary 2.11. *Let T be a self-mapping of a complete metric space (X,d) satisfying the condition (a) and*

(b) $d(T\mathfrak{x},T\mathfrak{y})\leq\psi(n(\mathfrak{x},\mathfrak{y}))$ where $\psi\in\Psi$,

for all $\mathfrak{x},\mathfrak{y}\in X$. If T is k-continuous, then T has a fixed point $\mathfrak{z}$. Also, T is continuous at $\mathfrak{z}$ iff

$$\lim_{\mathfrak{x}\to\mathfrak{z}} n(\mathfrak{x},\mathfrak{z})=0.$$

Corollary 2.12. *Let T be a self-mapping of a complete metric space (X,d) satisfying the condition (a) and*

(c) $d(T\mathfrak{x},T\mathfrak{y})\leq\phi(n(\mathfrak{x},\mathfrak{y}))$ where $\phi\in\Phi$,

for all $\mathfrak{x},\mathfrak{y}\in X$. If T is k-continuous, then T has a fixed point $\mathfrak{z}$. Also, T is continuous at $\mathfrak{z}$ iff

$$\lim_{\mathfrak{x}\to\mathfrak{z}} n(\mathfrak{x},\mathfrak{z})=0.$$

2.2. **Some Fixed-Disc Results.** In this subsection, we investigate some geometric properties of $Fix(T)$ under some interpolative type contractions. Before, we recall the followings:

"*Fixed-circle problem*" has been occurred as a geometric approach to the fixed-point theory when the self-mapping $T : X \to X$ has more than one fixed point [11].

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. Then the circle is defined by

$$C_{\mathfrak{x}_0,\rho} = \{\mathfrak{x} \in X : d(\mathfrak{x}, \mathfrak{x}_0) = \rho\}.$$

If $T\mathfrak{x} = \mathfrak{x}$ for every $\mathfrak{x} \in C_{\mathfrak{x}_0,\rho}$ then $C_{\mathfrak{x}_0,\rho}$ is called as the fixed circle of T (see [11]).

After then, the notion of a fixed circle was generalized the notion of a fixed disc as follows:

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. Then the disc is defined by

$$D_{\mathfrak{x}_0,\rho} = \{\mathfrak{x} \in X : d(\mathfrak{x}, \mathfrak{x}_0) \leq \rho\}.$$

If $T\mathfrak{x} = \mathfrak{x}$ for every $\mathfrak{x} \in D_{\mathfrak{x}_0,\rho}$ then $D_{\mathfrak{x}_0,\rho}$ is called as the fixed disc of T (see [10] and [12]).

Now, we begin the following theorem.

Theorem 2.13. *Let T be a self-mapping of a metric space (X, d) and the number ρ defined by*

$$\rho = \inf\{d(\mathfrak{x}, T\mathfrak{x}) : \mathfrak{x} \notin Fix(T)\}. \tag{2.7}$$

If there exists $\mathfrak{x}_0 \in X$ such that

$$d(\mathfrak{x}, T\mathfrak{x}) < n(\mathfrak{x}, \mathfrak{x}_0), \tag{2.8}$$

for all $\mathfrak{x} \in X - Fix(T)$, then

(i) *$\mathfrak{x}_0 \in Fix(T)$,*

(ii) *T fixes the disc $D_{\mathfrak{x}_0,\rho}$,*

(iii) *T fixes the circle $C_{\mathfrak{x}_0,\rho}$.*

Proof. (i) Let $\mathfrak{x}_0 \in X - Fix(T)$. By (2.8), we get

$$d(\mathfrak{x}_0, T\mathfrak{x}_0) < n(\mathfrak{x}_0, \mathfrak{x}_0) = 0,$$

a contradiction. Hence it should be $\mathfrak{x}_0 \in Fix(T)$.

(ii) Let $\mathfrak{x} \in D_{\mathfrak{x}_0,\rho}$ and $\mathfrak{x} \in X - Fix(T)$. Using (2.8) and the condition (i), we obtain

$$d(\mathfrak{x}, T\mathfrak{x}) < n(\mathfrak{x}, \mathfrak{x}_0) = 0,$$

a contradiction. So, we have $\mathfrak{x} \in Fix(T)$. Consequently, T fixes the disc $D_{\mathfrak{x}_0,\rho}$.

(iii) It is clear from the condition (ii). □

Remark 2.14. If T satisfies the conditions of Theorem 2.13, then we obtain that T is an identity map. Therefore, we rewrite Theorem 2.13 as follows:

Let T be a self-mapping of a metric space (X, d). There exists $\mathfrak{x}_0 \in X$ such that

$$d(\mathfrak{x}, T\mathfrak{x}) < n(\mathfrak{x}, \mathfrak{x}_0),$$

for all $\mathfrak{x} \in X - Fix(T)$ iff T is an identity map.

From the above reason, we ignore the third term of $n(\mathfrak{x}, \mathfrak{y})$ as follows:

$$n'(\mathfrak{x}, \mathfrak{y}) = [d(\mathfrak{x}, \mathfrak{y})]^{\beta} [d(\mathfrak{x}, T\mathfrak{x})]^{\alpha} \left[\frac{d(\mathfrak{x}, T\mathfrak{y}) + d(\mathfrak{y}, T\mathfrak{x})}{2} \right]^{1-\alpha-\beta},$$

where $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$.

Using the numbers $n'(\mathfrak{x}, \mathfrak{y})$ and ρ, we prove the following theorem:

Theorem 2.15. *Let T be a self-mapping of a metric space (X, d) and the number ρ defined as in (2.7). If there exists $\mathfrak{x}_0 \in X$ such that*

$$d(\mathfrak{x}, T\mathfrak{x}) < n'(\mathfrak{x}, \mathfrak{x}_0) \tag{2.9}$$

and

$$0 < d(\mathfrak{x}_0, T\mathfrak{x}) \leq \rho, \tag{2.10}$$

for all $\mathfrak{x} \in X - Fix(T)$, then

(i) $\mathfrak{x}_0 \in Fix(T)$,

(ii) T fixes the disc $D_{\mathfrak{x}_0,\rho}$,

(iii) T fixes the circle $C_{\mathfrak{x}_0,\rho}$.

Proof. (i) Let $\mathfrak{x}_0 \in X - Fix(T)$. By (2.9), we get

$$d(\mathfrak{x}_0, T\mathfrak{x}_0) < n'(\mathfrak{x}_0, \mathfrak{x}_0) = 0,$$

a contradiction. Hence it should be $\mathfrak{x}_0 \in Fix(T)$.

(ii) Let $\mathfrak{x} \in D_{\mathfrak{x}_0,\rho}$ and $\mathfrak{x} \in X - Fix(T)$. Using (2.9), (2.10) and the condition (i), we get

$$\begin{aligned} d(\mathfrak{x}, T\mathfrak{x}) &< n'(\mathfrak{x}, \mathfrak{x}_0) \\ &= [d(\mathfrak{x}, \mathfrak{x}_0)]^{\beta} [d(\mathfrak{x}, T\mathfrak{x})]^{\alpha} \left[\frac{d(\mathfrak{x}, T\mathfrak{x}_0) + d(\mathfrak{x}_0, T\mathfrak{x})}{2} \right]^{1-\alpha-\beta} \\ &\leq \rho^{\beta} [d(\mathfrak{x}, T\mathfrak{x})]^{\alpha} \rho^{1-\alpha-\beta} = \rho^{1-\alpha} [d(\mathfrak{x}, T\mathfrak{x})]^{\alpha} \\ &\leq [d(\mathfrak{x}, T\mathfrak{x})]^{1-\alpha} [d(\mathfrak{x}, T\mathfrak{x})]^{\alpha} = d(\mathfrak{x}, T\mathfrak{x}), \end{aligned}$$

a contradiction and so we have $\mathfrak{x} \in Fix(T)$. Consequently, T fixes the disc $D_{\mathfrak{x}_0,\rho}$.

(iii) It is a natural consequence of the condition (ii). □

We give an illustrative example:

Example 2.16. Let $X = \{-1, 0, 1, 2, 3, 4\}$ and $T : X \to X$ be a self-mapping defined by

$$T\mathfrak{x} = \begin{pmatrix} -1 & 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 1 & 4 \end{pmatrix},$$

for all $\mathfrak{x} \in X$. Then T satisfies the hypothesis of Theorem 2.15 with $\mathfrak{x}_0 = 0$. Indeed, for $x = 3$, we obtain

$$\rho = 2,$$
$$d(3, 1) = 2 < n'(3, 0) \approx 2.38,$$

with $\beta = \frac{1}{2}$, $\alpha \in \left(0, \frac{1}{2}\right)$ and

$$0 < d(0,1) = 1 \leq 2.$$

Consequently, $0 \in Fix(T) = X - \{3\}$, T fixes the disc $D_{0,2} = Fix(T) - \{4\}$ and so T fixes the circle $C_{0,2} = \{2\}$.

Using the sets Ψ and Φ, we obtain the following two results:

Theorem 2.17. (*Interpolative Boyd-Wong type fixed-disc result*) *Let T be a self-mapping of a metric space (X, d) and the number ρ defined as in (2.7). If there exist $\mathfrak{x}_0 \in X$ and $\psi \in \Psi$ such that*

$$d(\mathfrak{x}, T\mathfrak{x}) < \psi\left(n'(\mathfrak{x}, \mathfrak{x}_0)\right)$$

and

$$0 < d(\mathfrak{x}_0, T\mathfrak{x}) \leq \rho,$$

for all $\mathfrak{x} \in X - Fix(T)$, then
(*i*) $\mathfrak{x}_0 \in Fix(T)$,
(*ii*) *T fixes the disc $D_{\mathfrak{x}_0,\rho}$,*
(*iii*) *T fixes the circle $C_{\mathfrak{x}_0,\rho}$.*

Proof. Using the similar arguments given in the proof of Theorem 2.15 and the conditions (ψ_1), (ψ_2), we can easily proved it. □

Theorem 2.18. (*Interpolative Matkowski type fixed-disc result*) *Let T be a self-mapping of a metric space (X, d) and the number ρ defined as in (2.7). If there exist $\mathfrak{x}_0 \in X$ and $\phi \in \Phi$ such that*

$$d(\mathfrak{x}, T\mathfrak{x}) < \phi\left(n'(\mathfrak{x}, \mathfrak{x}_0)\right)$$

and

$$0 < d(\mathfrak{x}_0, T\mathfrak{x}) \leq \rho,$$

for all $\mathfrak{x} \in X - Fix(T)$, then
(*i*) $\mathfrak{x}_0 \in Fix(T)$,
(*ii*) *T fixes the disc $D_{\mathfrak{x}_0,\rho}$,*
(*iii*) *T fixes the circle $C_{\mathfrak{x}_0,\rho}$.*

Proof. By the similar arguments used in the proof of Theorem 2.15 and Lemma 2.2, we can easily proved it. □

2.3. **Discontinuous Activation Functions and an Application.** In the literature, there are some examples of activation functions which have a fixed point and are not continuous at this fixed point. Such functions are used as an activation function in neural networks. When the number of fixed points of an activation function is more than one, then it is important to study a geometry of the fixed point set of a used activation function. Therefore, in this subsection, we give an example of discontinuous activation functions and present an application to these functions using the obtained theoretical results as follows:

Discontinuous activation functions are used the ability of train networks which enables us to investigate networks of neurons (see [7] for more details).

For example, in [7], we consider any neuron activation x_i represented by the value y_i, an integer $-n \leq y_i \leq n$ such that

$$x_i = \begin{cases} 2^{y_i - n} & ; \quad y_i > 0 \\ 0 & ; \quad y_i = 0 \\ -2^{-y_i - n} & ; \quad y_i < 0 \end{cases}.$$

In terms of the y_i, the network can be mathematically described by an activation function of the form

$$y_i = F\left(\sum_j sgn(y_j)2^{|y_j|}w_{ij} + 2^n v_i\right),$$

where

$$sgn(z) = \begin{cases} 1 & ; \quad z > 0 \\ 0 & ; \quad z = 0 \\ -1 & ; \quad z < 0 \end{cases}$$

and F is any monotonic nondecreasing function returning an integer between $-n$ and n.

Now, let us consider the function sgn on $X = \{-1, 0, 1, 2, 3\}$. Then the function sgn satisfies the conditions of Theorem 2.15 with $z_0 = 0$. Indeed, for $z \in \{2, 3\}$, we get

$$\rho = \inf\{d(2,1), d(3,1)\} = 1,$$

$$d(2, T2) = d(2, 1) < n'(2, 0),\ d(3, T3) = d(3, 1) < n'(3, 0)$$

and

$$0 < d(0, T2) \leq 1,\ 0 < d(0, T3) \leq 1,$$

with $\beta = \frac{1}{2}$ and $\alpha \leq \frac{1}{3}$. Then we have $sgn(0) = 0$ and the function sgn fixes the disc $D_{0,1} = \{-1, 0, 1\}$, the circle $C_{0,1} = \{-1, 1\}$. On the other hand, the function sgn is discontinuous at the point $z_0 = 0$. Hence, it is not continuous at all points in $D_{0,1}$.

3. Conclusion

In this paper, we obtain new solutions to the Rhoades' open problem on the existence of a self-mapping which has a fixed point but is not continuous at the fixed point and fixed-disc problem on metric spaces. For this purpose, we inspire the known interpolative Boyd-Wong and Matkowski type contractions. Finally, we give an application to the discontinuous activation functions.

Also, Bisht presented an overview that aims to discuss a brief historical account of the development through the definitions and comparison of weaker forms of continuity notions in metric fixed point theory in [6]. If we consider some known continuity notions given in [6], we obtain the following open problem:

Problem 3.1. *Does Theorem 2.9 hold under other weaker continuity notions?*

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