DOI 10.1007/s11253-023-02253-z
Ukrainian Mathematical Journal, Vol. 75, No. 8, January, 2024 (Ukrainian Original Vol. 75, No. 8, August, 2023)

A MODULUS OF SMOOTHNESS FOR SOME BANACH FUNCTION SPACES

Ramazan Akgün[1]

UDC 517.5

Based on the Steklov operator, we consider a modulus of smoothness for functions in some Banach function spaces, which can be not translation invariant, and establish its main properties. A constructive characterization of the Lipschitz class is obtained with the help of the Jackson-type direct theorem and the inverse theorem on trigonometric approximation. As an application, we present several examples of related (weighted) function spaces.

1. Introduction and Main Results

The celebrated theorem of Jackson and Bernstein–Stechkin on the constructive characterization of the Lipschitz classes, states that[2] (see, e.g., [13, Chapter 7, Theorem 3.3])

a necessary and sufficient condition for $f \in L^p,\ 1 \le p \le \infty$, to belong to the Lipschitz class of order $\alpha > 0$,

$$\mathrm{Lip}(\alpha, p) := \left\{ f \in L^p \colon \omega_{\lfloor \alpha \rfloor + 1}(f, \delta)_p \lesssim \delta^\alpha,\ \delta > 0 \right\},$$

is that

$$\inf_{T_n \in \mathbb{T}_n} \|f - T_n\|_{L^p} =: E_n(f)_{L^p} \lesssim n^{-\alpha} \quad \text{for all} \quad n \in \mathbb{N} := \{1, 2, 3, \ldots\},$$

where $\lfloor x \rfloor := \max\{n \in \mathbb{N} \colon n \le x\}$ and $\mathbb{T}_n$ is a class of trigonometric polynomials

$$T_n(x) = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx), \quad a_k, b_k \in \mathbb{R},$$

of degree at most $n \in \mathbb{N}$,

$$\omega_r(f, \delta)_{L^p} := \sup_{0 \le h \le \delta} \|(I - T_h)^r f\|_{L^p}$$

is the modulus of smoothness of order $r \in \mathbb{N}$, and $T_h f(\cdot) := f(\cdot + h),\ h \in \mathbb{R}$, is a translation operator.

In view of this equivalence, functions from the Lipschitz classes are characterized only by the orders of their best approximation. To obtain this equivalence, it is necessary to relate the best approximation order $E_n(f)_p$ with the modulus of smoothness $\omega_r\left(f, \dfrac{1}{n}\right)_{L^p}$.

[1] Department of Mathematics, Faculty of Arts and Sciences, Balikesir University, Balikesir, Turkey; e-mail: rakgun@balikesir.edu.tr.
[2] Here and in what follows, $A \lesssim B$ means that there exists a positive constant C independent of essential parameters such that the inequality $A \le CB$ is true.

Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 75, No. 8, pp. 1015–1031, August, 2023. Ukrainian DOI: 10.37863/umzh.v75i8.970.
Original article submitted June 11, 2019.

The direct and inverse inequalities of trigonometric approximation give the relations

$$E_n(f)_{L^p} \lesssim \omega_r\left(f, \frac{1}{n}\right)_{L^p}, \quad r \in \mathbb{N}, \tag{1}$$

$$\omega_r\left(f, \frac{1}{n}\right)_{L^p} \lesssim \frac{1}{n^r} \sum_{j=0}^{n} (j+1)^{r-1} E_j(f)_{L^p} \tag{2}$$

for any $n \in \mathbb{N}$ with constants depending only on r. We note that inequalities (1) and (2) are true (see [15]) for more general homogeneous Banach spaces (HBS) X, i.e., the class of measurable functions defined on $T := [0, 2\pi]$ such that the translation operator T_h is a continuous isometry and the relation $\|f(-\cdot)\|_X = \|f(\cdot)\|_X$ holds. See also the results in [20, 21, 27].

Here, the definition of the modulus of smoothness $\omega_r(f, \cdot)_X$ strongly depends on the translation invariance of the analyzed space X. If the space X is not translation invariant (e.g., for the Lebesgue spaces with a weight), then the modulus of smoothness $\omega_r(f, \cdot)_X$ may be not well defined.

The main purpose of the present paper is to define a modulus of smoothness $\Omega_r(\cdot, \delta)_X$ that can be also used for the spaces X that can be not invariant under the action of the translation operator T_h. Moreover, the role of X can be played by certain weighted spaces.

We suppose that:

(I) X is a Banach function space (BFS; see [9]) on T;

(II) $\mathbb{T}_n$ is a dense subset of X;

(III) the Steklov operator

$$f(x) \mapsto \sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt, \quad x \in T, \quad f \in X,$$

is uniformly bounded (in h) on X.

Let $S_n(\cdot, f)$ be the nth partial sum of the Fourier series of $f \in X \subset L^1$.

The modulus of smoothness in X satisfying property (III) is defined as follows:

$$\Omega_r(f, \delta)_X := \sup_{0 \le h \le \delta} \|(I - \sigma_h)^r f\|_X, \quad r \in \mathbb{N},$$

where I is the identity operator on T.

The following theorem is the main result of the present paper; it gives an estimate of the best approximation error

$$E_n(f)_X := \inf_{T_n \in \mathbb{T}_n} \|f - T_n\|_X$$

from above by the modulus of smoothness $\Omega_r\left(f, \dfrac{1}{n}\right)_X$:

Theorem 1. *Suppose that X satisfy the conditions (I)–(III) and that $f \in X$. If*

(IV) the operator $f \mapsto S_n$ is uniformly bounded (in n) on X

and

(V) $E_n(g)_X \lesssim n^{-2} \left\| \dfrac{d}{dx^2}\, g(x) \right\|_X$ *for any* $g \in X'' := \left\{ g \in X : \dfrac{d}{dx^2} g(x) \in X \right\},$

then the Jackson–Stechkin-type estimate

$$E_n(f)_X \lesssim \Omega_r \left(f, \frac{1}{n}\right)_X, \quad r \in \mathbb{N}, \tag{3}$$

is true for $n \in \mathbb{N}$ *with some constant depending only on* r *and* X.

In the approximation theory, inequalities of type (3) are known as the direct theorem of trigonometric approximation. For $X = L^2$, inequality (3) was proved in [1]. If X is an HBS, then (3) can be obtained from Theorem 10.7 in [16]. In the case where X is a Lebesgue space with a weight ω satisfying the Muckenhoupt condition A_p, $1 < p < \infty$, inequality (3) of the form

$$E_n(f)_{p,\omega} \lesssim \widetilde{\Omega}_r \left(f, \frac{1}{n}\right)_{p,\omega} := \sup_{0 \le h_i \le 1/n} \left\| \prod_{i=1}^{r} (I - \sigma_{h_i}) f \right\|_{p,\omega}, \quad r \in \mathbb{N}, \tag{4}$$

was proved in [17] (see also, e.g., Theorem 2 in [19]). Considering Example 5 in § 2, we see that (3) clearly improves inequality (4) for $r \ge 2$. Similarly, (3) also improves the direct theorems obtained in [3, 4, 6, 18, 19] for $r \ge 2$.

The weak inverse of the Jackson-type estimate (3) is given in the following theorem:

Theorem 2. *Let* X *satisfy properties (I)–(III) and let* $f \in X$. *If*

(VI) *the inequality* $\|T_n'\|_X \lesssim n \|T_n\|_X$ *holds for any* $T_n \in \mathbb{T}_n$,

then

$$\Omega_r \left(f, \frac{1}{n}\right)_X \lesssim \frac{1}{n^{2r}} \sum_{j=0}^{n} (j+1)^{2r-1} E_j(f)_X, \quad r \in \mathbb{N},$$

for $n \in \mathbb{N}$ *with some constant that depends only on* r *and* X.

Theorems 1 and 2 give the following Marchaud-type inequality:

Corollary 1. *Under the conditions of Theorems 1 and 2,*

$$\Omega_r(f, \delta)_X \lesssim \delta^{2r} \int_{\delta}^{1} u^{-2r-1} \Omega_k(f, u)_X du, \quad 0 < \delta < 1,$$

for $r, k \in \mathbb{N}$ *with* $r < k$.

Theorem 3. *Under the conditions of Theorems 1 and 2, if*

$$E_n(f)_X \lesssim n^{-\beta}, \quad n \in \mathbb{N},$$

for some $\beta > 0$, *then, for a given* $r \in \mathbb{N}$,

$$\Omega_r(f,\delta)_X \lesssim \begin{cases} \delta^{\beta}, & r > \beta/2, \\ \delta^{\beta} \log \dfrac{1}{\delta}, & r = \beta/2, \\ \delta^{2r}, & r < \beta/2. \end{cases}$$

Definition 1. *Let* $\beta > 0$, *let* $r := \lfloor \beta/2 \rfloor + 1$, *and let* X *be a BFS satisfying condition (III). We define*

$$\mathrm{Lip}(\beta, X) := \left\{ f \in X \colon \Omega_r(f,\delta)_X \lesssim \delta^{\beta},\ \delta > 0 \right\}.$$

The following result gives a constructive characterization of the Lipschitz classes $\mathrm{Lip}(\beta, X)$. As a corollary of Theorems 1, 2, and 3 and Definition 1, we get the following corollary:

Corollary 2. *Let* $\beta > 0$. *Under the conditions of Theorems 1 and 2, the following conditions are equivalent:*

(i) $f \in \mathrm{Lip}(\beta, X)$,

(ii) $E_n(f)_X \lesssim n^{-\beta},\ n \in \mathbb{N}$.

Some examples of the space X are given in Sec. 2. In Sec. 3, we present the proofs of our results.

2. Applications

In this section, we collect some definitions of the function classes suitable for the method proposed in the previous section.

Nonweighted Setting. Let $\mathcal{M}$ be the set of all measurable and scalar-valued functions on T and let $\mathcal{M}^+$ be a subset of functions from $\mathcal{M}$ whose values lie in $[0,\infty]$. By χ_E we denote the characteristic function of a measurable set $E \subset T$.

A mapping $\rho\colon \mathcal{M}^+ \to [0,\infty]$ is called a function norm if, for all constants $a \geq 0$, for all functions f, g, f_n, $n \in \mathbb{N}$, and for all measurable subsets E of T, the following properties hold:

(i) $\rho(f) = 0$ iff $f = 0$ a.e.; $\rho(af) = a\rho(f)$ and $\rho(f+g) \leq \rho(f) + \rho(g)$;

(ii) if $0 \leq g \leq f$ a.e., then $\rho(g) \leq \rho(f)$;

(iii) if $0 \leq f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$;

(iv) if a set E in T has a finite Lebesgue measure $|E|$, then $\rho(\chi_E) < \infty$;

(v) if a set E in T satisfies $|E| < \infty$, then there exists a positive constant C depending only on E and ρ and such that

$$\int\limits_E f(x)\, dx \leq C\rho(f).$$

For a function norm ρ, the class of functions

$$X := X(\rho) = \{ f \in \mathcal{M} \colon \rho(|f|) < \infty \}$$

is called a BFS. For each $f \in X,$ we define the norm

$$\|f\|_X := \rho(|f|), \quad f \in X.$$

A BFS X equipped with the norm $\|\cdot\|_X$ is a Banach space (see [9, p. 3–5, Theorems 1.4 and 1.6]). If ρ is a function norm, then its associate norm ρ^a is defined on $\mathcal{M}^+$ by

$$\rho^a(g) := \sup\left\{\int\limits_T f(x)g(x)\,dx\colon\ f \in \mathcal{M}^+,\ \rho(f) \le 1\right\}, \quad g \in \mathcal{M}^+.$$

If ρ is a function norm, then ρ^a is itself a function norm [9, p. 8, Theorem 2.2]. The BFS $X\big(\rho^a\big)$ determined by the function norm ρ^a is called the associate space of $X = X(\rho)$ and denoted by X^a. It is well known (see, e.g., [9, p. 9]) that

$$\|f\|_X = \sup\left\{\int\limits_T |f(x)g(x)|dx\colon\ g \in X^a,\ \|g\|_{X^a} \le 1\right\}. \tag{5}$$

The distribution function μ_f of a measurable function f is defined as the Lebesgue measure of the set $\{x \in T : |f(x)| > \lambda\}$ for $\lambda \ge 0$. A Banach function norm is said to be rearrangement invariant (r.i.) if $\rho(f) = \rho(g)$ for every pair of functions $f,\ g,$ which are equimeasurable, i.e., $\mu_f(\lambda) = \mu_g(\lambda)$. If ρ is an r.i. function norm, then the BFS $X(\rho)$ is called an r.i. BFS. Let X be a BFS. We say that a function $f \in X$ has an absolutely continuous norm if

$$\lim_{n\to\infty} \|f\chi_{A_n}\|_X = 0$$

for any decreasing sequence of measurable sets $\{A_n\}_{n\ge1}$ with $\chi_{A_n} \to 0$ a.e. If every $f \in X$ has this property, then we say that X has an absolutely continuous norm.

Remark 1 [9, Chapter 3, Lemma 6.3, Theorem 6.10]**.** *Let X be an r.i. BFS. The following conditions are equivalent:*

(i) the set of trigonometric polynomials $\mathbb{T}_n$ is a dense subset of X;

(ii) the translation operator T_h is uniformly bounded (in h) on X;

(iii) X has an absolutely continuous norm;

(iv) the Fourier series of $f \in X$ converges in norm in the space X;

(v) the operator of partial sum $S_n(\cdot, f)$ is uniformly bounded (in n) on X.

We also note that if X is a separable r.i. BFS, then condition (i) in Remark 1 is equivalent to the following condition:

(vi) X has nontrivial Boyd indices α_X and β_X (i.e., $0 < \alpha_X, \beta_X < 1$; see Chapter 3, Corollary 6.11 in [9]).

We now present some examples of BFS.

(1) Lebesgue Spaces. We define Lebesgue functionals as follows:

$$\rho_p(f) := \left(\int\limits_T f(x)^p \, dx \right)^{1/p}, \quad 0 < p < \infty, \qquad \text{and} \qquad \rho_\infty(f) := \operatorname*{essup}_{x \in T} f(x).$$

Then $\rho_p(|f|)$ is a Banach function norm for $1 \le p \le \infty$. We set $L^p := X(\rho_p)$ and $\|f\|_p := \rho_p(|f|)$. In this case, property (I) was proved in Theorem 1.2 of [9]. Property (II) is well known and can be found in any monograph on the approximation theory (see, e.g., Chapter 1, Part 1.4.1 in [29]). Property (III) is a consequence of the integral Minkowski inequality [29, p. 592, (12)] and the translation invariance of L^p, $1 \le p \le \infty$. Property (V) is known from, e.g., [13, p. 206, (2.17)]. Property (VI) was proved in [8] for $p = \infty$; in [30], for $1 \le p \le \infty$, and in [7], for $0 \le p \le \infty$. For (IV), one can see [28, §3, Theorem 1].

(2a) Lorentz Spaces L^{pq}. Let $0 < p, q \le \infty$ and let $\mathcal{M}_0$ be a subset of functions from $\mathcal{M}$ that are finite a.e. on T. For $f \in \mathcal{M}_0$, we set

$$\|f\|_{p,q} := \left(\int\limits_0^\infty \left[t^{1/p} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q}, \quad 0 < p < \infty,$$

$$\|f\|_{p,\infty} := \sup_{x \in (0,\infty)} t^{1/p} f^*(t), \qquad \|f\|_{(p,\infty)} := \sup_{x \in (0,\infty)} t^{1/p} f^{**}(t),$$

$$\|f\|_{(p,q)} := \left(\int\limits_0^\infty \left[t^{1/p} f^{**}(t) \right]^q \frac{dt}{t} \right)^{1/q}, \quad 0 < p < \infty,$$

where f^* is a decreasing rearrangement of the function f [9, Chapter 2, Section 1] and

$$f^{**}(t) := \frac{1}{t} \int\limits_0^t f^*(s) \, ds, \quad t > 0.$$

The class of functions $\{f \in \mathcal{M}_0 \colon \|f\|_{p,q} < \infty\}$ is denoted by L^{pq}. It is known that L^{pq} coincides with L^p for $0 < p \le \infty$ and $\|f\|_{p,p} = \|f\|_p$, where $f \in L^p$. On the other hand, if $1 \le q \le p < \infty$ or $q = p = \infty$, then $\|\cdot\|_{p,q}$ is an r.i. Banach function norm. If $1 < p < \infty$ and $1 \le q \le \infty$ or $q = p = \infty$, then $\|\cdot\|_{(p,q)}$ is an r.i. Banach function norm (see Chapter 4, Theorem 4.3 and Lemma 4.5 in [9]).

(2b) Lorentz Spaces Λ and M. Let X be an r.i. BFS on $(\mathbb{R}^+, dx)$. Suppose that X is renormed so that its fundamental function φ_X is concave. The Lorentz space $\Lambda(X)$ consists of all functions f in $\mathcal{M}_0^+(\mathbb{R}^+, dx)$ for which

$$\|f\|_{\Lambda(X)} := \int\limits_0^\infty f^*(s) \, d\varphi_X(s) < \infty.$$

The Lorentz space $M(X)$ consists of all functions f in $\mathcal{M}_0^+(\mathbb{R}^+, dx)$ for which

$$\|f\|_{M(X)} := \sup_{t \in (0,\infty)} f^{**}(t) \varphi_X(t) < \infty.$$

The Lorentz spaces $\Lambda(X)$ and $M(X)$ are r.i. BFS (see Chapter 2, Theorem 5.13 in [9]).

(c) Zygmund Spaces. The spaces $L(\log L)$ and $L_{\exp}$ are r.i. BFS (see Chapter 4, Part 6 in [9]). If X is an r.i. BFS and has an absolutely continuous norm, then properties (I), (II) and (IV) can be obtained from Remark 1, while properties (III), (V), and (VI) were obtained in [18, Lemmas 2.2 and 2.5 and Theorem 1.2].

(3) Orlicz Spaces. A function φ is called a Young function if φ is even, continuous, and nonnegative in $\mathbb{R}$, increasing on $(0,\infty)$, and such that $\varphi(0)=0$ and $\lim_{x\to\infty}\varphi(x)=\infty$. We say that a Young function φ satisfies condition Δ_2 (and write $\varphi\in\Delta_2$) if there exists a constant $C>0$ such that $\varphi(2x)\le C\varphi(x)$ for all $x\in\mathbb{R}$. Two Young functions φ and φ_1 are called equivalent (and we write $\varphi\sim\varphi_1$) if there are constants $C,\ C'>0$ such that

$$\varphi_1(Cx)\le\varphi(x)\le\varphi_1\big(C'x\big)$$

holds for any $x>0$. A nonnegative function $M\colon\ [0,\infty)\to[0,\infty)$ is said to be quasiconvex if there exist a convex Young function Φ and a constant $C\ge 1$ such that

$$\Phi(x)\le M(x)\le\Phi(Cx)$$

holds for any $x\ge 0$. Let φ be a quasiconvex Young function. By $\tilde{L}_\varphi(T)$ we denote the class of Lebesgue measurable functions $f\colon T\to\mathbb{R}$, satisfying the condition

$$\int_T\varphi(|f(x)|)dx<\infty.$$

The linear span of the Orlicz class $\tilde{L}_\varphi(T)$, denoted by $\varphi(L)$, becomes a normed space with the Orlicz norm

$$\|f\|_\varphi:=\sup\left\{\int_T|f(x)g(x)|\,dx\colon\ \int_T\varphi^a(|g|)\,dx\le 1\right\},\tag{6}$$

where

$$\varphi^a(y):=\sup_{x\ge 0}\{xy-\varphi(x)\},\quad y\ge 0,$$

is the complementary function of φ. It can be easily seen that $\varphi(L)\subset L^1(T)$ and $\varphi(L)$ becomes a Banach space with the Orlicz norm. The Banach space $\varphi(L)$ is called an Orlicz space. In this case, condition (I) can be replaced by condition (I$'$), namely, X is a Banach space whose norm satisfies the integral Minkowski inequality. Hence, the Orlicz norm (6) has this property. Under the conditions that φ^α is a quasiconvex function for some $\alpha\in(0,1)$ and $\varphi\in\Delta_2$, property (II) is a consequence of [23, Lemma 3] and properties (III)–(VI) were proved in [6, Lemmas 2, 3, and 5 and Theorem 1].

The examples considered in items 1–3 above are HBS and, in these cases, inequalities (1) and (2) can be also obtained by the method developed in [16, §10]. On the other hand, the examples presented below are, in general, not translation invariant and, in these cases, the method proposed in [16] is not applicable. The aim of the present work arises from this fact.

The following example demonstrates a function class that is not rearrangement invariant.

(4) Variable Exponent Lebesgue Spaces. Let $\mathcal{P}$ be a class of 2π-periodic Lebesgue measurable functions $p=p(x)\colon\ T\to(1,\infty)$ such that $\operatorname{essup}_{x\in T}p(x)<\infty$. We consider a class $L_{2\pi}^{p(\cdot)}$ of 2π-periodic measurable functions f defined on T and such that

$$\int\limits_T |f(x)|^{p(x)} dx < \infty.$$

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space [24, Theorem 2.5] with the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\alpha > 0\colon \int\limits_T \left|\frac{f(x)}{\alpha}\right|^{p(x)} dx \le 1\right\}.$$

We say that the variable exponent $p(\cdot)$ defined on T satisfies the Dini–Lipschitz property DL_γ of order γ if

$$\sup_{x_1,x_2\in T}\{|p(x_1)-p(x_2)|\colon |x_1-x_2|\le\delta\}\left(\ln\frac{1}{\delta}\right)^{\gamma} \le C, \quad 0<\delta<1. \tag{7}$$

If $p(\cdot)$ satisfies the properties

$$1 < \operatorname{essinf}_{x\in T} p(x),\ \operatorname{essup}_{x\in T} p(x) < \infty$$

and the Dini–Lipschitz condition (7) of order ≥ 1, then property (I) follows from Theorem 3.2.13 in [14]; properties (II)–(IV) follow from [26, Theorems 6.1 and 6.2 and Lemma 3.1], and properties (V) and (VI) follow from [3, Theorem 1 and Lemma 1].

Weighted Case. A function $\omega\colon T\to[0,\infty]$ is called a weight if ω is measurable and positive a.e. A 2π-periodic weight function ω belongs to the Muckenhoupt class $A_p,\ p>1,$ if

$$\sup_J \left(\frac{1}{|J|}\int\limits_J \omega(x)\,dx\right)\left(\frac{1}{|J|}\int\limits_J \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \le C$$

with a finite positive constant C independent of $J,$ where J is any subinterval of $T.$

(5) Weighted Lebesgue Spaces. For a weight $\omega,$ by $L^p(T,\omega)$ we denote a class of measurable functions defined on T and such that $\omega f \in L^p(T).$ We set

$$\|f\|_{p,\omega} := \|\omega f\|_p$$

for $f\in L^p(T,\omega).$ If $\omega^p\in A_p$ and $1<p<\infty,$ then properties (II)–(VI) are known from [19] and property (I) is a consequence of [22, Lemma 2.5 (b)].

(6a) Weighted Orlicz Spaces $\boldsymbol{\varphi_\omega(L)}$. Let φ be a quasiconvex Young function. By $\tilde{L}_{\varphi,\omega}(T)$ we denote the class of Lebesgue measurable functions $f\colon T\to\mathbb{R}$ satisfying the condition

$$\int\limits_T \varphi(|f(x)|)\omega(x)\,dx < \infty.$$

The linear span of the weighted Orlicz class $\tilde{L}_{\varphi,\omega}$, denoted by $\varphi_\omega(L)$, turns into a normed space with the Orlicz norm

$$\|f\|_{\varphi,\omega} := \sup\left\{\int\limits_T |f(x)g(x)|\omega(x)\,dx:\ \int\limits_T \varphi^a(|g|)\omega(x)\,dx \le 1\right\}. \tag{8}$$

For a quasiconvex function φ, we define the index $p(\varphi)$ of φ as follows:

$$\frac{1}{p(\varphi)} := \inf\big\{p:\ p>0,\ \varphi^p \text{ is quasiconvex}\big\}.$$

If $\omega \in A_{p(\varphi)}$, then it is easy to see that $\varphi_\omega(L) \subset L^1(T)$ and $\varphi_\omega(L)$ becomes a Banach space with the Orlicz norm. The Banach space $\varphi_\omega(L)$ is called a weighted Orlicz space. In this case, condition (I) can be also replaced by condition (I$'$). Hence, the Orlicz norm (8) satisfies property (I$'$). If the conditions that φ^α is quasiconvex for some $\alpha \in (0,1)$, $\varphi \in \Delta_2$, and $\omega \in A_{p(\varphi)}$, are satisfied, then properties (II)–(VI) were proved in [6].

(6b) Weighted Orlicz Spaces $L_{M,\omega}$. A convex and continuous function $M:\ [0,\infty) \to [0\infty)$ such that

$$M(0)=0, \qquad M(x)>0 \quad \text{for} \quad x>0,$$

$$\lim_{x\to 0}\frac{M(x)}{x}=0, \qquad \text{and} \qquad \lim_{x\to\infty}\frac{M(x)}{x}=\infty,$$

is called an N-function. The complementary Young function N of M is defined by

$$N(y) := \max\{xy - M(x):\ x \ge 0\}$$

for $y \ge 0$.

Let M be an N-function. We denote by L_M a linear space of 2π-periodic measurable functions $f:\ T \to \mathbb{R}$ such that

$$\int\limits_T M(\lambda|f(x)|)\,dx < \infty$$

holds for some $\lambda > 0$. Equipped with the norm

$$\|f\|_M := \sup\left\{\int\limits_T |f(x)g(x)|\,dx:\ \int\limits_T N(|g(x)|)\,dx \le 1\right\},$$

where N is the complementary function, L_M becomes a Banach space. This space is called the Orlicz space generated by M.

Let $M^{-1}:\ [0,\infty) \to [0,\infty)$ be the inverse of the Young function M and let

$$h(t) := \limsup_{x\to\infty}\frac{M^{-1}(x)}{M^{-1}(x/t)}, \quad t>0.$$

The numbers β_M and α_M defined by

$$\beta_M := \lim_{t\to\infty} \frac{\log h(t)}{\log t} \qquad \text{and} \qquad \alpha_M := \lim_{t\to 0^+} \frac{\log h(t)}{\log t}$$

are called the upper and lower Boyd indices of the Orlicz space L_M, respectively.

Let ω be a weight function. We denote by $L_{M,\omega}$ the space of all measurable functions $f\colon T \to \mathbb{R}$ such that $f\omega \in L_M$. The norm in $L_{M,\omega}$ is defined by $\|f\|_{M,\omega} := \|f\omega\|_M$. The normed space $L_{M,\omega}$ is called a weighted Orlicz space. In the case where M is an N-function, $L_{M,\omega}$ has nontrivial Boyd indices α_M and β_M, and $\omega^{1/\alpha_M} \in A_{1/\alpha_M}$, $\omega^{1/\beta_M} \in A_{1/\alpha_M}$, properties (I)–(VI) were proved in [19]. We note also that the spaces $L_{M,\omega}$ and $\varphi_\omega(L)$ are, in general, different (see [10]).

7. Weighted Variable Exponent Lebesgue Spaces. By $L_\omega^{p(\cdot)}$ we denote the class of Lebesgue measurable functions $f\colon T \to \mathbb{R}$ satisfying the condition $\omega f \in L_{2\pi}^{p(\cdot)}$. A weighted variable exponent Lebesgue space $L_\omega^{p(\cdot)}$ is a Banach space with the following norm:

$$\|f\|_{p(\cdot),\omega} := \|\omega f\|_{p(\cdot)}.$$

For given $p \in \mathcal{P}$, the class of weights ω satisfying the condition (see [11])

$$\|\omega\chi_Q\|_{p(\cdot)} \left\|\omega^{-1}\chi_Q\right\|_{p'(\cdot)} \lesssim |Q|,$$

for all balls Q in T is denoted by $A_{p(\cdot)}$. Here, $p'(x) := p(x)/(p(x)-1)$ is the conjugate exponent of $p(x)$. We say that the variable exponent $p(x)$ is log-Hölder continuous on T if there exists a constant $C \geq 0$ such that

$$|p(x_1) - p(x_2)| \lesssim \frac{1}{\log(e + 1/|x_1 - x_2|)} \quad \text{for all} \quad x_1, x_2 \in T.$$

In the case where

$$1 < \operatorname{essinf}_{x\in T} p(x), \ \operatorname{essup}_{x\in T} p(x) < \infty,$$

$$1/p \quad \text{is Log-Hölder continuous on} \quad T,$$

and

$$\omega^{p_0} \in A_{(p(\cdot)/p_0)'} \quad \text{for some} \quad p_0 \in (1, \operatorname{essinf}_{x\in T} p(x)),$$

properties (I)–(VI) were established in [4].

8. Weighted r.i. BFS. For a weight ω, by $X(T,\omega)$ we denote a class of measurable functions defined on T and such that $\omega f \in X(T)$. We set $\|f\|_{X,\omega} := \|\omega f\|_X$ for $f \in X(T,\omega)$. In the case where $X(T)$ is a reflexive r.i. BFS with nontrivial Boyd indices α_X and β_X such that $\omega^{1/\alpha_X} \in A_{1/\alpha_X}$ and $\omega^{1/\beta_X} \in A_{1/\alpha_X}$, properties (I)–(VI) were obtained in [18].

3. Proofs of the Results

The following two lemmas are required to prove Theorem 1. If $A \lesssim B$ and $B \lesssim A$ simultaneously, then we write $A \approx B$.

Lemma 1. *Assume that X satisfies conditions (I)–(III), $f \in X$, and $t, l > 0$. Then*

$$\Omega_1(f, lt)_X \lesssim (1 + \lfloor l \rfloor)^2 \Omega_1(f, t)_X$$

holds with a constant that depends only on r and X.

Proof. Let $t > 0$. Then there exists $n \in \mathbb{N}$ such that $1/n < t \le 2/n$. We define an operator

$$(U_{1/n} f)(x) := 3n^3 \int_0^{1/n} \int_0^t \int_{-u}^u f(x+s)\, ds\, du\, dt, \quad x \in T, f \in X.$$

In this case (see [3, p. 14]),

$$\frac{d^2}{dx^2} U_{1/n} f(x) = Cn^2 (I - \sigma_{1/n}) f(x)$$

holds for almost all $x \in T$ with some constant $C \in \mathbb{R}$.

Hence, in view of the uniform boundedness of the operator $f \mapsto \sigma_{1/n} f$ in X (for fixed $n \in \mathbb{N}$), we conclude that

$$\frac{d^2}{dx^2} U_{1/n} f(x) \in X \qquad \text{and} \qquad U_{1/n} f \in X''.$$

On the other hand, it follows from (5) that

$$\|U_{1/n} f\|_X = \left\| 3n^3 \int_0^{1/n} \int_0^t \int_{-u}^u f(x+s)\, ds\, du\, dt \right\|_X$$

$$\lesssim n^3 \int_0^{1/n} \int_0^t 2u \|\sigma_u f\|_X\, du\, dt$$

$$\lesssim 3n^3 \|f\|_X \int_0^{1/n} \int_0^t 2u\, du\, dt = \|f\|_X$$

and, hence, $f - U_{1/n} f \in X$. Then

$$\inf_{g \in X''} \left\{ \|f - g\|_X + t^2 \|g''\|_X \right\} =: K_2\left(f, t, X, X''\right) \le 2K_2\left(f, 1/n, X, X''\right)$$

$$\lesssim \|f - U_{1/n} f\|_X + n^{-2} \left\| \frac{d^2}{dx^2} U_{1/n} f \right\|_X =: I_1 + I_2. \tag{9}$$

We estimate I_1. By using (5), we get

$$\|f - U_{1/n} f\|_X \lesssim n^3 \int_0^{1/n} \int_0^t 2u \|(I - \sigma_u) f\|_X\, du\, dt$$

$$\lesssim \sup_{0\le u\le 1/n} \|(I-\sigma_u)f\|_X 3n^3 \int_0^{1/n}\int_0^t 2u\,du\,dt$$

$$\lesssim \sup_{0\le u\le 1/n} \|(I-\sigma_u)f\|_X = \Omega_1(f,1/n)_X. \tag{10}$$

For the estimate $I_2,$ we find

$$\frac{1}{n^2}\left\|\frac{d^2}{dx^2}U_{1/n}f(x)\right\|_X = \left\|n^{-2}\frac{d^2}{dx^2}U_{1/n}f(x)\right\|_X = \|C(I-\sigma_{1/n})f\|_X$$

$$\lesssim \sup_{0\le u\le 1/n} \|(I-\sigma_u)f\|_X = \Omega_1(f,1/n)_X. \tag{11}$$

Thus, it follows from (9)–(11) that

$$K_2\left(f,t,X,X''\right) \lesssim \Omega_1(f,1/n)_X \le \Omega_1(f,t)_X.$$

On the other hand, for $g\in X''$,

$$(I-\sigma_h)g(x) = \frac{1}{2h}\int_{-h}^{h}(g(x)-g(x+t))\,dt = -\frac{1}{8h}\int_0^h\int_0^t\int_{-u}^{u} g''(x+s)\,ds\,du\,dt.$$

Therefore,

$$\|(I-\sigma_h)g\|_X = \frac{1}{8h}\sup_{v\in X^a}\left\{\int_T\left|\int_0^h\int_0^t\int_{-u}^{u} g''(x+s)\,ds\,du\,dt\right||v(x)|\,dx:\ \|v\|_{X^a}\le 1\right\}$$

$$\le \frac{1}{8h}\int_0^h\int_0^t 2u\left\|\frac{1}{2u}\int_{-u}^{u} g''(x+s)\,ds\right\|_X du\,dt$$

$$\lesssim \frac{1}{8h}\int_0^h\int_0^t 2u\|g''\|_X\,du\,dt = h^2\|g''\|_X,$$

and we find

$$\Omega_1(g,t)_X \lesssim t^2\|g''\|_X \tag{12}$$

for $g\in X''$.

Then, for $g\in X''$,

$$\Omega_1(f,t)_X \lesssim \|f-g\|_X + t^2\|g''\|_X.$$

Thus, taking the infimum on $g\in X''$, we get

$$\Omega_1(f,t)_X \lesssim K_2\left(f,t;X,X''\right).$$

This yields $\Omega_1(f,t)_X \approx K_2(f,t;X,X'')$. By using the last equivalence, we obtain

$$\Omega_1(f,lt)_X \lesssim \inf_{g\in X''}\left\{\|f-g\|_X+(lt)^2\|g''\|_X\right\}$$

$$\lesssim (1+\lfloor l\rfloor)^2 \inf_{g\in X''}\left\{\|f-g\|_X+t^2\|g''\|_X\right\}$$

$$\lesssim (1+\lfloor l\rfloor)^2\Omega_1(f,t)_X.$$

The lemma is proved.

Lemma 2. *Suppose that X satisfies conditions (I)–(III), $f\in X$, and $n,m,r\in\mathbb{N}$. Then there exists a number $\delta\in(0,1)$ depending only on X and such that*

$$\Omega_r(f,t)_X \lesssim C\delta^{mr}\|f\|_X + C'\Omega_{r+1}(f,t)_X$$

holds for any $t\in(0,1/n)$, where $C>0$ is a constant that depends only on r and X and $C'>0$ is a constant that depends only on r, m, and X.

Proof. For any $h>0$, there exists a constant $\mathtt{C}>1$ such that

$$\|\sigma_h f\|_X \le \mathtt{C}\|f\|_X.$$

We set $\delta := \mathtt{C}/(1+\mathtt{C})$. Further, for any $h\in(0,1/n)$, we first prove that

$$\|(I-\sigma_h)^r f\|_X \le \delta^r\left\|(I-\sigma_h^2)^r f\right\|_X + C\Omega_{r+1}(f,h)_X. \tag{13}$$

To prove (13), we observe that

$$I-\sigma_h = 2^{-1}(I-\sigma_h)(I+\sigma_h)+2^{-1}(I-\sigma_h)^2$$

and

$$\sigma_h(I-\sigma_h) = 2^{-1}(I-\sigma_h)(I+\sigma_h)-2^{-1}(I-\sigma_h)^2.$$

Hence, for $g\in X$,

$$\|(I-\sigma_h)g\|_X+\|\sigma_h(I-\sigma_h)g\|_X \le \|(I-\sigma_h)\,(I+\sigma_h)g\|_X+\left\|(I-\sigma_h)^2 g\right\|_X. \tag{14}$$

On the other hand,

$$\|(I-\sigma_h)^r f\|_X = \delta\left((1/\mathtt{C})\,\|(I-\sigma_h)^r f\|_X+\|(I-\sigma_h)^r f\|_X\right)$$

$$\le \delta\left(\|(I-\sigma_h)^r f\|_{p,\omega}+\|(I-\sigma_h)^r f\|_X\right)$$

$$= \delta\left(\left\|(I-\sigma_h)\,(I-\sigma_h)^{r-1} f\right\|_X+\|(I-\sigma_h)^r f\|_X\right)$$

$$
\begin{aligned}
&= \delta \left(\left\| \left(\sigma_h(I-\sigma_h) + (I-\sigma_h)^2\right)\, (I-\sigma_h)^{r-1} f \right\|_X + \|(I-\sigma_h)^r f\|_X \right) \\
&\le \delta \left(\left\| \sigma_h(I-\sigma_h)\,(I-\sigma_h)^{r-1} f \right\|_X + \left\| (I-\sigma_h)^2 (I-\sigma_h)^{r-1} f \right\|_X \right) \\
&\quad + \delta \, \|(I-\sigma_h)^r f\|_X \\
&\le \delta \left(\|\sigma_h (I-\sigma_h)^r f\|_X + \left\|(I-\sigma_h)^{r+1} f\right\|_{p,\omega} + \|(I-\sigma_h)^r f\|_X \right). \qquad (15)
\end{aligned}
$$

Setting $g := (I-\sigma_h)^{r-1} f$ in (14), we get

$$
\|\sigma_h(I-\sigma_h)^r f\|_X + \|(I-\sigma_h)^r f\|_X \le \|(I-\sigma_h)^r (\sigma_h + I) f\|_{p,\omega} + \left\|(I-\sigma_h)^{r+1} f\right\|_X .
$$

By using this inequality in (15), we find

$$
\begin{aligned}
\|(I-\sigma_h)^r f\|_X &\le \delta \left(\|\sigma_h(I-\sigma_h)^r f\|_{p,\omega} + \left\|(I-\sigma_h)^{r+1} f\right\|_X + \|(I-\sigma_h)^r f\|_X \right) \\
&\le \delta \left(\|(I-\sigma_h)^r (\sigma_h + I) f\|_X + \left\|(I-\sigma_h)^{r+1} f\right\|_X \right) + \delta \left\|(I-\sigma_h)^{r+1} f\right\|_X \\
&\le \delta \, \|(I-\sigma_h)^r (\sigma_h + I) f\|_X + 2\delta \left\|(I-\sigma_h)^{r+1} f\right\|_X . \qquad (16)
\end{aligned}
$$

Repeating the last inequality r times, we obtain

$$
\begin{aligned}
\|(I-\sigma_h)^r f\|_X &\le \delta \, \|(I-\sigma_h)^r (\sigma_h + I) f\|_X + 2\delta \left\|(I-\sigma_h)^{r+1} f\right\|_X \\
&\le \delta^2 \left\|(I-\sigma_h)^r (\sigma_h + I)^2 f\right\|_X + 2\delta^2 \left\|(I-\sigma_h)^{r+1} (\sigma_h + I) f\right\|_X + 2\delta \left\|(I-\sigma_h)^{r+1} f\right\|_X \\
&\le \ldots \le \delta^r \, \|(I-\sigma_h)^r (\sigma_h + I)^r f\|_X + 2 \sum_{k=1}^{r} \delta^k \left\| (I-\sigma_h)^{r+1} (\sigma_h + I)^{k-1} f \right\|_X \\
&= \delta^r \left\|(I-\sigma_h^2)^r f\right\|_{p,\omega} + 2 \sum_{k=1}^{r} \delta^k \left\| (I-\sigma_h)^{r+1} (\sigma_h + I)^{k-1} f \right\|_X .
\end{aligned}
$$

Hence,

$$
\|(I-\sigma_h)^r f\|_X \le \delta^r \left\|(I-\sigma_h^2)^r f\right\|_X + C(r,X) \Omega_{r+1}(f,h)_X
$$

and the proof of (13) is complete. By using (13), we get

$$
\begin{aligned}
\|(I-\sigma_h)^r f\|_X &\le \delta^r \left\|(I-\sigma_h^2)^r f\right\|_X + C(r,X) \Omega_{r+1}(f,h)_X \\
&\le \delta^{2r} \left\|(I-\sigma_h^4)^r f\right\|_X + (\delta^r + 1)\, C(r,X) \Omega_{r+1}(f,h)_X \\
&\le \ldots \le \delta^{mr} \left\| \left(I-\sigma_h^{2^m}\right)^r f \right\|_X + C(r,X,m) \Omega_{r+1}(f,h)_X . \qquad (17)
\end{aligned}
$$

For $g \in C(T)$, where $C(T)$ is the class of continuous functions on T, the inequality

$$\left\| \left(I - \sigma_h^{2^m}\right)^r g \right\|_{C(T)} \leq 2^r \|g\|_{C(T)}$$

holds with a constant c independent of m. The last inequality and Theorem 1.5 in [5] imply that

$$\left\| \left(I - \sigma_h^{2^m}\right)^r f \right\|_X \leq 2^r C \|f\|_X,$$

where C is a constant independent of m.

Taking supremum in inequality (17), we arrive at the relation

$$\Omega_r(f,t)_X \lesssim \delta^{mr} \|f\|_X + \Omega_{r+1}(f,t)_X.$$

The lemma is proved.

Proof of Theorem 1. *Case* $r = 1$. Let $n \in \mathbb{N}$ and $f \in X$ be fixed. We use the operator $U_{1/n}f$. By using (IV), (10), and (11), we find

$$\begin{aligned} E_n(f)_X &= E_n(f - U_{1/n}f + U_{1/n}f)_X \\ &\leq E_n(f - U_{1/n}f)_X + E_n(U_{1/n}f)_X \\ &\lesssim \|f - U_{1/n}f\|_X + n^{-2} \left\| \frac{d^2}{dx^2} U_{1/n}f(x) \right\|_X \lesssim \Omega_1\left(f, \frac{1}{n}\right)_X \end{aligned} \tag{18}$$

for any $n \in \mathbb{N}$.

Case $r \geq 2$. Following the idea proposed in [12], we proceed by induction on r. We know that the Jackson-type estimate (3) holds for $r = 1$ [see (18)]. Suppose that inequality (3) is true for $g \in X$ and some $r = 2, 3, 4, \ldots$:

$$E_n(g)_X \lesssim \Omega_r\left(g, \frac{1}{n}\right)_X. \tag{19}$$

It is necessary to verify the validity of inequality (3) for $r + 1$. We use the mean $S_n f$ and show that

$$\|f - S_n f\|_X \lesssim \Omega_{r+1}\left(f, \frac{1}{n}\right)_X.$$

We set $u(\cdot) := f(\cdot) - S_n f(\cdot)$. In this case, $S_n(u) = 0$. Since $S_n f$ is the near best approximant for f, i.e.,

$$\|f - S_n f\|_X \lesssim E_n(f)_X,$$

by using the induction hypothesis (19), we obtain

$$\|u\|_X = \|u - S_n(u)\|_X \lesssim E_n(u)_X \leq \mathbf{C}\, \Omega_r\left(u, \frac{1}{n}\right)_X.$$

It follows from Lemma 2 that

$$\Omega_r\left(u, \frac{1}{n}\right)_X \leq C\delta^{mr}\|u\|_X + C'\Omega_{r+1}\left(u, \frac{1}{n}\right)_X .$$

If we choose m sufficiently large to guarantee that $C\mathbf{C}\delta^{mr} < 1/2,$ then we get

$$\|u\|_X \leq \mathbf{C}\Omega_r\left(u, \frac{1}{n}\right)_X \leq C\mathbf{C}\delta^{mr}\|u\|_X + C\Omega_{r+1}\left(u, \frac{1}{n}\right)_X$$

and

$$\|u\|_X \lesssim \Omega_{r+1}\left(u, \frac{1}{n}\right)_X .$$

It follows from the uniform boundedness of the operator $f \longmapsto S_n f$ in X that

$$\Omega_{r+1}\left(u, \frac{1}{n}\right)_X \lesssim \Omega_{r+1}\left(f, \frac{1}{n}\right)_X ,$$

and the required result

$$E_n(f)_X \lesssim \|f - S_n f\|_X = \|u\|_X \lesssim \Omega_{r+1}\left(u, \frac{1}{n}\right)_X \lesssim \Omega_{r+1}\left(f, \frac{1}{n}\right)_X$$

holds for $r \in \mathbb{N}$.

Theorem 1 is proved.

Proof of Theorem 2. Let $T_n \in \mathbb{T}_n,\ n \in \{0\} \cup \mathbb{N},$ be the best approximating trigonometric polynomial for $f \in X$. By using (12), we get

$$\Omega_r(g, \delta)_X \lesssim \delta^{2r}\left\|g^{(2r)}\right\|_X , \quad r \in \mathbb{N},$$

for $g^{(2r)} \in X$ and $\delta > 0$. On the other hand, for any $m \in \mathbb{N},$ we conclude that

$$\Omega^r(f, \delta)_X \leq \Omega^r\left(f - T_{2^{m+1}}, \delta\right)_X + \Omega^r\left(T_{2^{m+1}}, \delta\right)_X \tag{20}$$

and

$$\Omega^r\left(f - T_{2^{m+1}}, \delta\right)_X \lesssim \|f - T_{2^{m+1}}\|_X \lesssim E_{2^{m+1}}(f)_X. \tag{21}$$

Then

$$\begin{aligned}\Omega^r\left(T_{2^{m+1}}, \delta\right)_X &\lesssim \delta^{2r}\left\|T^{(2r)}_{2^{m+1}}\right\|_X \\ &\lesssim \delta^{2r}\left\{\left\|T^{(2r)}_1 - T^{(2r)}_0\right\|_X + \sum_{i=1}^{m}\left\|T^{(2r)}_{2^{i+1}} - T^{(2r)}_{2^i}\right\|_X\right\}\end{aligned}$$

$$\lesssim \delta^{2r}\left\{E_0(f)_X+\sum_{i=1}^{m}2^{(i+1)2r}E_{2^i}(f)_X\right\}$$

$$\lesssim \delta^{2r}\left\{E_0(f)_X+2^{2r}E_1(f)_X+\sum_{i=1}^{m}2^{(i+1)2r}E_{2^i}(f)_X\right\}.$$

Further, applying the inequality

$$2^{(i+1)2r}E_{2^i}(f)_X\lesssim\sum_{k=2^{i-1}+1}^{2^m}k^{2r-1}E_k(f)_X,\quad i\ge 1, \tag{22}$$

we get

$$\Omega^r\left(T_{2^{m+1}},\delta\right)_X\lesssim\delta^{2r}\left\{E_0(f)_X+2^{2r}E_1(f)_X+\sum_{k=2}^{2^m}k^{2r-1}E_k(f)_X\right\}$$

$$\lesssim \delta^{2r}\left\{E_0(f)_X+\sum_{k=1}^{2^m}k^{2r-1}E_k(f)_X\right\}. \tag{23}$$

Note that

$$E_{2^{m+1}}(f)_X\lesssim\frac{1}{n^{2r}}\sum_{k=2^{m-1}+1}^{2^m}k^{2r-1}E_k(f)_X.$$

Thus, if we choose m such that $2^m\le n<2^{m+1}$, then the required result follows from (20)–(23).

Theorem 2 is proved.

Proof of Theorem 3. Let $f\in X$ and let

$$E_n(f)_X\lesssim n^{-\beta},\quad n=1,2,3,\ldots,$$

for some $\beta>0$. Suppose that $\delta>0$ and $n:=\lfloor 1/\delta\rfloor$. From Theorem 2, we obtain

$$\Omega_r(f,\delta)_X\le\Omega_r\left(f,\frac{1}{n}\right)_X\lesssim\frac{1}{n^{2r}}\sum_{j=0}^{n}(j+1)^{2r-1}E_j(f)_X$$

$$\lesssim\delta^{2r}\left(E_0(f)_X+\sum_{j=1}^{n}j^{2r-1}E_j(f)_X\right)$$

$$\lesssim\delta^{2r}\left(E_0(f)_X+\sum_{j=1}^{n}j^{2r-1-\beta}\right).$$

If $2r>\beta$, then we get

$$\Omega_r(f,\delta)_X\lesssim\delta^{\beta}.$$

Further, if $2r = \beta$, then

$$\sum_{j=1}^{n} j^{2r-1-\beta} = \sum_{j=1}^{n} j^{-1} \leq 1 + \log(1/\delta),$$

and, hence,

$$\Omega_r(f,\delta)_X \lesssim \delta^{\beta} \log(1/\delta).$$

If $2r < \beta$, then the series $\sum_{j=0}^{n} j^{2r-1-\beta}$ is convergent and

$$\Omega_r(f,\delta)_X \lesssim \delta^{2r} \left(E_0(f)_X + \sum_{j=1}^{n} j^{2r-1-\beta} \right) \lesssim \delta^{2r}$$

is true.

Theorem 3 is proved.

The present research was supported by the TÜBİTAK, the Scientific and Technological Research Council of Turkey, and partially by the MTM2011-27637, and the Scientific Research Project 2019/61 of the Balikesir University.

The author states that there is no conflict of interest.

REFERENCES

1. V. A. Abilov and F. V. Abilova, "Some problems of the approximation of 2π-periodic functions by Fourier sums in the space $L^2_{2\pi}$," *Math. Notes*, **76**, No. 5-6, 749–757 (2004).
2. R. Akgün, "Polynomial approximation in weighted Lebesgue spaces," *East J. Approx.*, **17**, No. 3, 253–266 (2011).
3. R. Akgün, "Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent," *Ukr. Math. Zh.*, **63**, No. 1, 3–23 (2011); ***English translation:*** *Ukr. Math. J.*, **63**, No. 1, 1–26 (2011).
4. R. Akgün, "Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth," *Georgian Math. J.*, **18**, No. 2, 203–235 (2011).
5. R. Akgün, "Exponential approximation in variable exponent Lebesgue spaces on the real line," *Constr. Math. Anal.*, **5**, No. 4, 214–237 (2022).
6. R. Akgün and D. M. Israfilov, "Approximation in weighted Orlicz spaces," *Math. Slovaca*, **61**, No. 4, 601–618 (2011).
7. V. V. Arestov, "Integral inequalities for trigonometric polynomials and their derivatives," *Izv. Akad. Nauk SSSR Ser. Mat.*, **45**, No. 1, 3–22 (1981).
8. S. N. Bernstein, *Collected works, Vol. 1: Constructive Theory of Functions [1905–1930]*, Izd. Akad. Nauk SSSR, Moscow (1952).
9. C. Bennett and R. Sharpley, "Interpolation of operators," *Pure Appl. Math.*, **129**, Academic Press, Boston, MA (1988).
10. S. Bloom and R. Kerman, "Weighted L_Φ integral inequalities for operators of Hardy type," *Studia Math.*, **110**, No. 1, 35–52 (1994).
11. D. Cruz-Uribe, L. Diening, and P. Hästö, "The maximal operator on weighted variable Lebesgue spaces," *Fract. Calc. Appl. Anal.*, **14**, No. 3, 361–374 (2011).
12. F. Dai, "Jackson-type inequality for doubling weights on the sphere," *Constr. Approx.*, **24**, No. 1, 91–112 (2006).
13. R. A. De Vore and G. G. Lorentz, "Constructive approximation," *Fundamental Principles of Mathematical Sciences*, **303**, Springer-Verlag, Berlin (1993).
14. L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, "Lebesgue and Sobolev spaces with variable exponents," *Lecture Notes Math.*, **2017**, Springer-Verlag, Heidelberg (2011).
15. Z. Ditzian, "Rearrangement invariance and relations among measures of smoothness," *Acta Math. Hungar.*, **135**, No. 3, 270–285 (2012).
16. Z. Ditzian and K. G. Ivanov, "Strong converse inequalities," *J. Anal. Math.*, **61**, 61–111 (1993).
17. E. A. Gadjieva, *Investigation of the Properties of Functions with Quasimonotone Fourier Coefficients in Generalized Nikolskii–Besov Spaces* [in Russian], Author's Abstract of the PhD Thesis, Tbilisi (1986).

18. A. Guven and D. M. Israfilov, “Approximation by trigonometric polynomials in weighted rearrangement invariant spaces,” *Glas. Mat., Ser. III*, **44 (64)**, No. 2, 423–446 (2009).
19. D. M. Israfilov and A. Guven, “Approximation by trigonometric polynomials in weighted Orlicz spaces,” *Studia Math.*, **174**, No. 2, 147–168 (2006).
20. S. Z. Jafarov, “Approximation by Fejér sums of Fourier trigonometric series in weighted Orlicz space,” *Hacet. J. Math. Stat.*, **43**, No. 3, 259–268 (2013).
21. S. Z. Jafarov, “On moduli of smoothness in Orlicz classes,” *Proc. Inst. Math. Mech. Azerb. Natl. Acad. Sci. Azerb.*, **33**, No. 51, 85–92 (2010).
22. A. Y. Karlovich, “Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces,” *J. Integral Equat. Appl.*, **15**, No. 3, 263–320 (2003).
23. M. Khabazi, “The mean convergence of trigonometric Fourier series in weighted Orlicz classes,” *Proc. Razmadze Math. Inst.*, **129**, 65–75 (2002).
24. O. Kováčik and J. Rákosník, “On spaces $L^{p(x)}$ and $W^{k,p(x)}$,” *Czechoslovak Math. J.*, **41 (116)**, No. 4, 592–618 (1991).
25. I. I. Sharapudinov, “On the uniform boundedness in L^p $(p = p(x))$ of some families of convolution operators,” *Math. Notes*, **59**, No. 1–2, 205–212 (1996).
26. I. I. Sharapudinov, “Some problems in approximation theory in the spaces $L^{p(x)}(E)$,” *Anal. Math.*, **33**, No. 2, 135–153 (2007).
27. I. I. Sharapudinov, “Approximation of functions in $L_{2\pi}^{p(\cdot)}$ by trigonometric polynomials,” *Izv. Ros. Akad. Nauk, Ser. Mat.*, **77**, No. 2, 197–224 (2013).
28. R. Taberski, “Approximation of functions possessing derivatives of positive orders,” *Ann. Polon. Math.*, **34**, No. 1, 13–23 (1977).
29. A. F. Timan, “Theory of approximation of functions of a real variable,” *Internat. Ser. Monogr., Pure Appl. Math.*, **34**, Pergamon Press Book, Macmillan Co., New York (1963).
30. A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, New York (1959).