Araştırma Makalesi **BAUN Fen Bil. Enst. Dergisi, 24(2), 636-644, (2022)**
DOI:10.25092/baunfbed. 1078377 **J. BAUN Inst. Sci. Technol., 24(2), 636-644, (2022)**

Convolutions and approximations in the variable exponent spaces

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Geliş Tarihi (Received Date): 24.02.2022
Kabul Tarihi (Accepted Date): 23.05.2022

Abstract

A convolution in the variable exponent Lebesgue spaces is defined and the possibility its approximation by finite linear combinations of Steklov means is proved. Moreover, the convergence of the special convolutions sequence constructed via approximate identity to the original function is showed.

Keywords: *Convolution, variable exponent Lebesgue spaces, approximate identity.*

Değişken üslü Lebesgue uzaylarında konvolüsyonların bazı özellikleri

Öz

Bu çalışmada değişken üslü Lebesgue uzaylarında konvolüsyon tanımlandı ve Steklov ortalamalarının sonlu lineer birleşimleri ile yaklaşımının mümkün olduğu kanıtlandı. Ayrıca yaklaşım birimi ile oluşturulan özel konvolüsyonlar dizisinin başlangıç fonksiyonuna yakınsadığı gösterildi.

Anahtar kelimeler: *Konvolüsyon, değişken üslü Lebesgue uzayları, yaklaşım birimi*

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1. Introduction

Let $p(\cdot):[0,2\pi]\to[1,\infty)$ be a Lebesgue measurable 2π-periodic function. We define the modular functional $\rho_{p(\cdot)}(f):=\int_0^{2\pi}|f(x)|^{p(x)}dx$ on the Lebesgue measurable functions f on $[0,2\pi]$. By $L_{2\pi}^{p(\cdot)}$ we denote the class of 2π periodic Lebesgue measurable functions f, such that for a constant $\lambda=\lambda(f)>0$ the inequality $\rho_{p(\cdot)}(f/\lambda)<\infty$ holds.

Let $\beta_{2\pi}$ be the class of Lebesgue measurable functions $p(\cdot):[0,2\pi]\to[1,\infty)$ such that

$$1\le p_-:=\underset{x\in[0,2\pi]}{ess\inf}\, p(x)\le p_+:=\underset{x\in[0,2\pi]}{ess\sup}\, p(x)<\infty, \tag{1}$$

$$|p(x)-p(y)|\le\frac{c_{p(\cdot)}}{-\log(|x-y|)} \tag{2}$$

for all $x,y\in[0,2\pi]$, $|x-y|\le 1/2$, and for some constant $c_{p(\cdot)}>0$. Then with the norm

$$\|f\|_{p(\cdot)}:=\inf\left\{\lambda>0:\ \rho_{p(\cdot)}(f/\lambda)\le 1\right\} \tag{3}$$

$L_{2\pi}^{p(\cdot)}$ creates a Banach space (detailed information about the researches carried out in these spaces can be found in the monographs [1, 2]).

Let $f,g\in L^1$. We define a convolution type operator

$$(f*g)(x,h):=\int_0^{2\pi}(\sigma_h f)(x,u)g(u)du, \tag{4}$$

where $\sigma_h f(x,u):=\left(\frac{1}{h}\int\limits_0^h f(x+ut)dt\right)$, $0<h<\pi$, $x\in[0,2\pi]$, $-\infty<u<\infty$ is the Steklov means constructed via f.

The convolution operators play an important role in approximation theory, especially for construction of approximation polynomials and modulus of smoothness, which used for estimation the speed of convergence in the different function spaces.

In the classical Lebesgue spaces, for this goal were used the convolutions constructed via classical shift operator. But this shift operator is not invariant in the some spaces, for example in the variable exponent Lebesgue spaces. In the last spaces for construction of modulus of smoothness used the Steklov means $\sigma_h f$, which are invariant in the variable cases. Using this modulus were investigated some fundamental problems of approximation theory on the intervals of real line and also on the domains of complex plane (see, for example: [2-21]. Moreover, in the variable cases were investigated (see, for example: [22-25]) basicity problems of well known systems of functions which play important role in the different areas of applied mathematics. Therefore, the problems of studying convolutions in these spaces are actual.

In this work we prove the possibility of approximation to convolutions $(f * g)$ by linear combinations of the means $\sigma_h f$ and also approximation of convolutions $(f * K_n)$, constructed by approximate identity $\{K_n\}_{n\in\mathbb{N}}$ to f in the variable exponent Lebesgue spaces $L_{2\pi}^{p(\cdot)}$, $p(\cdot)\in\beta_{2\pi}$. In the case of $p(\cdot)>1$ these results were obtained in [13].

2. Auxiliary results

As immediately follows from the definition of convolution type operator given in Introduction, for $\forall f, g, j \in L_{2\pi}^{p(\cdot)}$ (·)} and for $\forall\alpha\in\mathbb{R}$ the relations

a. $((\alpha f)*g)(x,h)=(f*(\alpha g)(x,h))=\alpha(f*g)(x,h)$,
b. $((f\pm g)*j)(x,h)=(f*j)(x,h)\pm(g*j)(x,h)$

hold.

But the operator $(f * g)$ isn't commutative. For example: If $f(x):=1$ and $g(x):=x$, then

$$(f*g)(x,h)=\int_0^{2\pi}\left(\frac{1}{h}\int_0^h f(x+tu)\,dt\right)g(u)\,du$$
$$=\int_0^{2\pi}\left(\frac{1}{h}\int_0^h 1dt\right)u\,du=2\pi^2. \tag{5}$$

On the other hand,

$$(g*f)(x,h)=\int_0^{2\pi}\left(\frac{1}{h}\int_0^h g(x+tu)\,dt\right)f(u)\,du$$
$$=\int_0^{2\pi}\left(\frac{1}{h}\int_0^h (x+tu)\,dt\right)du$$
$$=\int_0^{2\pi}\left(x+\frac{hu}{2}\right)du$$
$$=\pi(2x+\pi h). \tag{6}$$

Theorem 1 [1; page:34] Let $p(\cdot):[0,2\pi]\to[1,\infty)$ and let $f:[0,2\pi]\times[0,2\pi]\to\mathbb{R}$ be the measurable functions and $f(\cdot,y)\in L_{2\pi}^{p(\cdot)}$ for every $y\in[0,2\pi]$. Then

$$\left\|\int_0^{2\pi} f(\cdot, y)dy\right\|_{p(\cdot)} \leq c_{p(\cdot)}\int_0^{2\pi}\|f(\cdot, y)\|_{p(\cdot)}dy \tag{7}$$

with some positive constant $c_{p(\cdot)}$.

Let $\lambda, \gamma > 0$ and $|\tau| \leq \pi / \lambda^{\gamma}$. We consider the Steklov operator $S_{\lambda,\tau} f$, defined as $\left(S_{\lambda,\tau} f\right)(x) := \lambda\int_{x+\tau-1/2\lambda}^{x+\tau+1/2\lambda} f(t)\,dt$.

The following result was proved in [3].

Theorem 2 Let $p(\cdot) \in \beta_{2\pi}$ and $0 < \gamma \leq 1$. Then the family of the Steklov operators $S_{\lambda,\tau} f$ is uniformly bounded in $L^{p(\cdot)}(T)$ for $1 \leq \lambda < \infty$ and $|\tau| \leq \pi / \lambda^{\gamma}$, i.e. there exists a positive constant $c(p)$ such that

$$\left\|S_{\lambda,\tau}(f)\right\|_{p(\cdot)} \leq c(p)\|f\|_{p(\cdot)},\ 1 \leq \lambda < \infty,\ |\tau| \leq \pi / \lambda^{\gamma}. \tag{8}$$

3. Main results

Our new results arre following:

Therorem 3 If $f \in L_{2\pi}^{p(\cdot)}$, $p(\cdot) \in \beta_{2\pi}$ and $g \in L^1$, then there exists a positive constant $c_{p(\cdot)}$ such that

$$\|f * g\|_{p(\cdot)} \leq c_{p(\cdot)}\|f\|_{p(\cdot)}\|g\|_1. \tag{9}$$

Proof Let $f \in L_{2\pi}^{p(\cdot)}$. Since for any positive integer u with $0 < uh < 1$

$$\left\|\frac{1}{h}\int_0^h f(x+ut)\,dt\right\|_{p(\cdot)} = \left\|\frac{1}{uh}\int_x^{x+uh} f(s)\,ds\right\|_{p(\cdot)} = \left\|\frac{1}{uh}\int_{x+uh/2-uh/2}^{x+uh/2+uh/2} f(s)\,ds\right\|_{p(\cdot)}$$
$$= \left\|\left(S_{\frac{1}{uh},\frac{uh}{2}} f\right)(\cdot)\right\|_{p(\cdot)},$$

denoting $\lambda := 1/uh$, $\tau := uh/2$ and applying Theorem 1 and Theorem 2 we have

$$\|f * g\|_{p(\cdot)} = \left\|\int_0^{2\pi} \sigma_h f(\cdot, u)\,g(u)\,du\right\|_{p(\cdot)}$$
$$\leq c_{p(\cdot)}\int_0^{2\pi}\left\|\sigma_h f(\cdot, u)\right\|_{p(\cdot)}|g(u)||du|$$
$$\leq c_{p(\cdot)}\|f\|_{p(\cdot)}\|g\|_{L^1}.$$

Theorem 4 If $f \in L_{2\pi}^{p(\cdot)}$, $p(\cdot) \in \beta_{2\pi}$ and $g \in L^1$, then the convolution $f * g$ in $L_{2\pi}^{p(\cdot)}$ can be approximated by the finite linear combinations of means f, i.e. for $\forall \varepsilon > 0$, there are the sets of numbers $\{\lambda_k\}_1^n \subset \mathbb{R}$ and $\{u_k\}_1^n \subset [0, 2\pi]$ such that

$$\left\| (f * g)(\cdot, h) - \sum_{k=1}^{n} \lambda_k \sigma_h f(\cdot, u_k) \right\|_{p(\cdot)} < \varepsilon . \tag{10}$$

Proof Let $S_{2\pi}$ be the set of simple functions defined on $[0, 2\pi]$. Since is dense in L^1, it is sufficient to prove this theorem in the case of $g \in S_{2\pi}$. Taking into account that every function $g \in S_{2\pi}$ can be represented as a linear combination of the characteristic functions of some subsets of $[0, 2\pi]$, it is sufficient to proof this theorem in the case of

$$g(u) := \chi_M(u) := \begin{cases} 1, & u \in M \\ 0, & u \notin M \end{cases} \tag{11}$$

where $M = [a, b]$, $0 < a < b < 2\pi$ is an arbitrary interval.

For a given $\delta > 0$ we divide M into finite subintervals I_k with the length $|I_k| < \delta$ and satisfying the conditions $I_i \cap I_j = \varnothing$, $i \neq j$ and $M = \bigcup_k I_k$. Then

$$\begin{aligned}
&(f * g)(x, h) = \int_0^{2\pi} (\sigma_h f)(x, u)\, g(u)\, du \\
&= \int_0^{2\pi} (\sigma_h f)(x, u)\, \chi_M(u)\, du \\
&= \int_M (\sigma_h f)(x, u)\, du \\
&= \sum_k \int_{I_k} (\sigma_h f)(x, u)\, du
\end{aligned}$$

Taking $u_k \in I_k$ we have

$$\begin{aligned}
&(f * g)(x, h) - \sum_k |I_k| (\sigma_h f)(x, u_k) \\
&= \sum_k \int_{I_k} (\sigma_h f)(x, u)\, du - \sum_k \int_{I_k} (\sigma_h f)(x, u_k)\, du \\
&= \sum_k \int_{I_k} \left[(\sigma_h f)(x, u) - (\sigma_h f)(x, u_k) \right] du .
\end{aligned}$$

Using the triangle property in $L_{2\pi}^{p(\cdot)}$ norm and Theorem 1, we have

$$\left\| (f * g)(\cdot, h) - \sum_k |I_k| (\sigma_h f)(\cdot, u_k) \right\|_{p(\cdot)}$$

$$=\left\|\sum_k \int_{I_k}\left[(\sigma_h f)(\cdot,u)-(\sigma_h f)(\cdot,u_k)\right]du\right\|_{p(\cdot)}$$

$$\leq c_{p(\cdot)}\sum_k \int_{I_k}\left\|(\sigma_h f)(\cdot,u)-(\sigma_h f)(\cdot,u_k)\right\|_{p(\cdot)} du . \tag{12}$$

By continuity of the mean value operators, for $\varepsilon>0$ and for every finite subintervals $I_k \subset M$ there is a $\delta>0$ such that for $|I_k|<\delta$ and $u\in I_k$ the inequality

$$\left\|(\sigma_h f)(\cdot,u)-(\sigma_h f)(\cdot,u_k)\right\|_{p(\cdot)}<\varepsilon \tag{13}$$

holds. Hence by (12) and (13) we get

$$\left\|(f*g)(\cdot,h)-\sum_k |I_k|(\sigma_h f)(\cdot,u_k)\right\|_{p(\cdot)} \leq c_{p(\cdot)}\sum_k \int_{I_k}\varepsilon du$$

$$=c_{p(\cdot)}\sum_k |I_k|\varepsilon$$

$$=c_{p(\cdot)}|M|\varepsilon \tag{14}$$

where $|M|$ is the Lebesgue measure of M .

Let us consider the approximate identities in the space $L^{p(\cdot)}$. By the approximate identity we mean a sequence $\{K_n(\cdot)\}_{n\in\mathbb{N}} \subset L_1(-\pi,\pi)$), satisfying the conditions:

a. $\sup_n \|K_n\|_{L^1}<\infty$,

b. $\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx=1$,

c. $\lim_{n\to\infty}\frac{1}{2\pi}\int_{\delta\leq|x|\leq\pi} K_n(x)dx=0$, $\forall\delta\in(0,\pi)$.

The following theorem is true.

Theorem 5 Let $\{K_n(\cdot)\}_{n\in\mathbb{N}}$ be an approximate identity. Then for every $f\in L^{p(\cdot)}(-\pi,\pi)$, $p(\cdot)\in\beta_{2\pi}$, the following relation holds

$$\lim_{n\to\infty}\left\|\frac{f*K_n}{2\pi}-f\right\|_{p(\cdot)}=0 . \tag{15}$$

Proof We suppose that f is continuous on $[-\pi,\pi]$, i. e., $f\in C([-\pi,\pi])$. By the triangle inequality

$$\left\|\frac{f*K_n}{2\pi}-f\right\|_{p(\cdot)}\le\left\|\frac{f*K_n}{2\pi}-\frac{1}{2\pi}\int_0^{2\pi}f(\cdot)K_n(u)\,du\right\|_{p(\cdot)}$$
$$+\left\|\frac{1}{2\pi}\int_0^{2\pi}f(\cdot)K_n(u)\,du-f\right\|_{p(\cdot)} \tag{16}$$

For the first term in the right side of (16) we have

$$\left|\frac{(f*K_n)(x)}{2\pi}-\frac{1}{2\pi}\int_0^{2\pi}f(x)K_n(u)\,du\right|$$
$$=\left|\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{h}\int_0^h f(x+ut)\,dt\right)K_n(u)\,du-\frac{1}{2\pi}\int_0^{2\pi}f(x)K_n(u)\,du\right|$$
$$=\left|\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{h}\int_0^h\left(f(x+ut)-f(x)\right)dt\right)K_n(u)\,du\right|$$
$$\le\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{h}\int_0^h\left|f(x+ut)-f(x)\right||dt|\right)|K_n(u)|\,du\,.$$

Let $\varepsilon>0$. By continuity of f, for a given ε there is a $\delta>0$ such that for $h<\delta$, the inequality $|f(x+ut)-f(x)|<\varepsilon$ holds. Hence from a. property of approximate identity we have

$$\left|\frac{(f*K_n)(x,h)}{2\pi}-\frac{1}{2\pi}\int_0^{2\pi}f(x)K_n(u)\,du\right|$$
$$\le\varepsilon\frac{1}{2\pi}\int_0^{2\pi}|K_n(u)|\,du\le c\varepsilon\,. \tag{17}$$

On the other hand by $b.$ property of approximate identity

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}f(x)K_n(u)\,du=f(x),\quad x\in[0,2\pi] \tag{18}$$

and hence for a given $\varepsilon>0$ there is a number $n_0\in\mathbb{N}$ such that for every $n\ge n_0$

$$\left\|\frac{1}{2\pi}\int_0^{2\pi}f(\cdot)K_n(u)\,du-f(\cdot)\right\|_{p(\cdot)}\le\varepsilon\,. \tag{19}$$

Using (17) and (19) in (16) we have

$$\left\|\frac{f*K_n}{2\pi}-f\right\|_{p(\cdot)}<c\varepsilon\,. \tag{20}$$

Let $f\in L^{p(\cdot)}([-\pi,\pi])$. Since $C([-\pi,\pi])$ is dense [26] in $L^{p(\cdot)}([-\pi,\pi])$, for every $\varepsilon>0$ there is a function $g\in C([-\pi,\pi])$ such that

$$\|f-g\|_{p(\cdot)}<\varepsilon \tag{21}$$

By the property $a.$ of approximate identity and Theorem 3 we have

$$\left\|\frac{f*K_n}{2\pi}-\frac{g*K_n}{2\pi}\right\|_{p(\cdot)}=\left\|\frac{(f-g)*K_n}{2\pi}\right\|_{p(\cdot)}$$

$$\leq\frac{c_{p(\cdot)}}{2\pi}\|(f-g)\|_{p(\cdot)}\|K_n\|_{L^1}\leq M\varepsilon \tag{22}$$

with some positive constant M independent of n.

Now for $\forall n\geq n_0$, using the relations (22), (20) and (21) we have

$$\left\|\frac{f*K_n}{2\pi}-f\right\|_{p(\cdot)}\leq\left\|\frac{f*K_n}{2\pi}-\frac{g*K_n}{2\pi}\right\|_{p(\cdot)}+\left\|\frac{g*K_n}{2\pi}-g\right\|_{p(\cdot)}+\|f-g\|_{p(\cdot)}$$

$$\leq M\varepsilon+c\varepsilon+\varepsilon=(M+c+1)\varepsilon.$$

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