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# ANALYSIS OF A PRODUCTION INVENTORY SYSTEM WITH MAP ARRIVALS, PHASE-TYPE SERVICES AND VACATION TO PRODUCTION FACILITY

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Abstract. In this paper, we discuss a production inventory system with service times. Customers arrive in the system according to a Markovian arrival process. The service times follow a phase-type distribution. We assume that there is an infinite waiting space for customers. Arriving customers demand only one unit of item from the inventory. The production facility produces items according to an $(s, S)$-policy. Once the inventory level becomes the maximum level $S$, the production facility goes on a vacation of random duration. When the production facility returns from the vacation, if the inventory level depletes to the fixed level $s$, it is immediately switched on and starts production until the inventory level becomes $S$. Otherwise, if the inventory level is greater than $s$ on return from the vacation, it takes another vacation. The vacation times are exponentially distributed. The production inventory system in the steady-state is analyzed by using the matrix-geometric method. A numerical study is performed on the system performance measures. Besides, an optimization study is discussed for the inventory policy.

## 1. Introduction

In classical inventory systems, demanded items are directly delivered from stock and the amount of time required to service is negligible. Demand occurred during stockout periods either result in lost sales or is satisfied only after the arrival of the replenishments. In contrast, in most real-life situations, a positive amount of time is needed for procedures such as preparation, packing, and loading of items in the inventory. Inventory systems have positive service times are denominated queueing-inventory systems. A detailed survey of the literature for queueing-inventory systems can be found in [7] and [4].

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In classical queueing systems (the absence of vacation), the server will be idle whenever there is no customer to service. On the other hand, the vacation of a server facilitates improved utilization of server idle time. That is, in this vacation period, the server can be utilized for some other ancillary work that will improve the productivity of an organization. The queueing systems with vacation have been extensively studied. We refer to [17] and [5] for more details on this topic.

Considering the server vacation, the queueing-inventory systems have been studied very little in the literature. The literature can be divided into two main groups: (i) *in the queueing-inventory systems*, the server goes on vacation when there is no customer in the system and/or there is no item in the inventory and (ii) *in the production inventory systems with a service facility*, the server goes on vacation when there is no customer in the system and/or there is no item in the inventory or the production goes on vacation when the inventory level becomes $S$.

*Queueing-inventory systems where the server goes on vacation when there is no customer in the system and/or there is no item in the inventory.* [11] is the first study considering a vacation to server in the inventory systems with positive service times. In this study, at a service completion epoch if no customer is in the system and/or no item in the inventory, the server goes on vacation. When, on return from this vacation, if the system is again found to have either no customer waiting or no item on stock or both, the server goes on another vacation. A perishable queueing-inventory system with early and delayed vacations of the server was studied in [10]. The server is in the operational state only if the level of inventory in the system and the number of the claims in the queue are positive. If at least one of the values is zero, the server takes a vacation. When the inventory level is zero, the server enters an early vacation. If during this period the inventory replenishes, and the any claim in the queue, the server starts service; otherwise it goes to a delayed vacation. [2] investigated a retrial queueing-inventory system with two heterogeneous servers in which the first server is unreliable server and the second server permits for vacation. The second server leaves for a vacation when the server finds either the inventory level is zero and/or the number of customers in the queue is zero. At completion of the vacation, there is at least two commodities and at least two customers in the queue, then the second server starts the service immediately. Otherwise, the server takes another vacation. [16] discussed a finite source queueing-inventory system with two heterogeneous servers. Both servers can take a vacation whenever the inventory level reaches zero and/or the customer level reaches zero. At the end of a vacation period, the service starts if there is a positive inventory and at least one customer in the system. Otherwise, the server takes another vacation.

*Queueing-inventory systems where the server goes on vacation when there is no item in the inventory.* An inventory system with retrial demands and server vacation was studied by [15]. The server takes a vacation whenever the inventory level becomes zero. When the returns from vacation, if the inventory level is zero, the server starts another vacation. Otherwise, it is ready to serve any arriving demands. [14] extended the paper in [15] by adding a new feature, called idle time for server, in addition to vacation. At the time of the stockout, the server idles for random time, so that if the replenishment is received during the idle time, the server is immediately available to service. At the end of idle time, if the replenishment is not received, the server takes a vacation. On return from any vacation, if the stock is already replenished, the server becomes available, otherwise, it's idle time starts which may be followed by another vacation. [13] discussed a finite-source inventory system with postponed demands and modified $M$ vacation policy. As distinct from the vacation policy introduced in [14] the server can take at most $M$ inactive periods repeatedly until replenishment takes place. [20] considered a queueing-inventory system with ROS policy and server vacations. Once completion of the serving, if the server finds the inventory is empty, the server leaves for a vacation. At the end of the vacation, if the server finds that the inventory is not empty, the server is available to serve, otherwise, the server takes another vacation.

*Queueing-inventory systems where the server goes on vacation when there is no customer in the system.* [3] studied a perishable queueing-inventory system with delayed vacation and negative customers. If the server finds queue is empty at service completion epoch, the server goes on a delay time. If the delay time is completed before the arrival of a customer, the server takes a vacation. At the end of the vacation period, service commences if there is a customer in the queue. Otherwise, the server starts another vacation. A perishable queueing-inventory system with server vacation was discussed by [6]. Upon service completion, server takes a vacation if there are no customers in the queue and it starts service at the end of the vacation if the number of customers in the system exceeds some threshold; otherwise, it takes another vacation. Up to this point all papers mentioned are related to the server's vacation. In these papers, the server stops servicing because of no items in the inventory and/or no customer in the system. [9] studied a queueing-inventory system with working vacations. The server takes a vacation only in the absence of customers in the system at a service completion epoch. The server continues to provide service at a lower rate than in normal mode of service during working vacations. After a service completion during the working vacation period, if there are customers in the system, the server comes back to the normal mode. Otherwise, if there are no customers in the system, the server continues the vacation.

*Production queueing-inventory systems where the server goes on vacation when there is no customer in the system and/or there is no item in the inventory.* [8] studied a production inventory system with service time and server vacation. The items for the inventory are produced according to an $(s, S)$ policy. Production starts whenever the inventory level falls to $s$ and continues until the inventory level reaches $S$. If the server finds either the inventory level is zero and/or the number of customers in the system is zero, the server takes a vacation. At the completion of the vacation period if there is no customers or no inventory or both, the server goes on another vacation.

*Production queueing-inventory systems where the production goes on vacation when the inventory level becomes $S$.* [19] considered a production-inventory system with service time, perishable item and production interruptions. The production is interrupted for a vacation once the inventory level becomes $S$. On return from a vacation, if the inventory level depletes to $s$, the production is switched on. It starts production and is kept in the on mode until the inventory level becomes $S$. A production inventory system with service time and production vacations was also studied by [18]. Customers arrive in the system according to a Poisson process. The service times of the customers follow an exponential distribution. A production facility produces items according to an $(s, S)$ policy and the production time for each item is exponentially distributed. The production takes a vacation whose length has exponential distribution once the inventory level becomes $S$. At the end of the vacation if the inventory level depletes to $s$, the production is immediately switched on and it starts production until the inventory level becomes $S$.

In this paper we extend the model studied in [18] by considering a Markovian arrival process for governing the arrival of the customers and phase type distributions for service times. The paper is structured as follows. The assumptions and description of the model are elaborated in Section 2. The steady state solution of the model including the stability condition and some performance measures of the system are discussed in Section 3. In Section 4, the total expected cost function is structured and presented sensitivity analysis with numerical examples. Finally, some concluding remarks are given in Section 5.

## 2. Model Description

We analyze a production queueing-inventory system with production vacations as demonstrated in Figure 1. Customers arrive in the system according to a Markovian

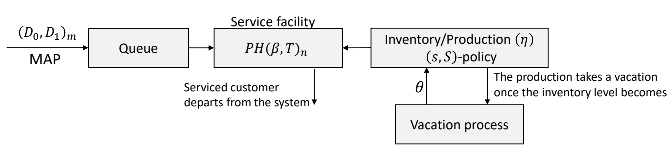


Figure 1. A production inventory system with production vacations.

arrival process ($MAP$) with representation $(\boldsymbol{D}_0, \boldsymbol{D}_1)$ of order $m$. The underlying Markov chain of the $MAP$ is governed by the matrix $\boldsymbol{D}$ $(= \boldsymbol{D}_0 + \boldsymbol{D}_1)$. Such that, the matrix $\boldsymbol{D}_0$ denotes the transition rates without arrival while the matrix $\boldsymbol{D}_1$ denotes the transition rates with arrival. So, the arrival rate is given by $\lambda = \boldsymbol{\delta} \boldsymbol{D}_1 \boldsymbol{e}$ where $\boldsymbol{\delta}$ is the stationary probability vector of the generator matrix $\boldsymbol{D}$ and it is satisfied

$$\boldsymbol{\delta} \boldsymbol{D} = \boldsymbol{0}, \ \boldsymbol{\delta} \boldsymbol{e} = 1. \tag{1}$$

For detailed information about $MAP$s, we refer to the study in [1].

When the inventory level is positive, an arriving customer finding the server idle gets into service immediately. Otherwise, the customer enters into a waiting space (queue) with infinite size to be served under the first-come first-served ($FCFS$) discipline. On the other hand, when the inventory is empty, no customer is allowed to join the queue. That is, all arriving customers are lost during the stochout case.

Each arriving customer demands a single item from the inventory. A served customer leaves immediately the system and the on-hand inventory is decreased by one at service completion epoch. The service time follows a phase-type distribution with representation $(\boldsymbol{\beta}, \boldsymbol{T})$ of order $n$ where $\boldsymbol{\beta}$ is the initial probability vector, $\boldsymbol{\beta e} = 1$, $\boldsymbol{T}$ is an infinitesimal generator matrix holding the transition rates among the $n$ transient states; $\boldsymbol{T}^0$ is a column vector contains the absorption rates into state 0 from the transient states. It is clear that $\boldsymbol{Te} + \boldsymbol{T}^0 = \mathbf{0}$. The phase-type distribution has the service rate $\mu = 1/[\boldsymbol{\beta}(-\boldsymbol{T})^{-1}\boldsymbol{e}]$. The properties in detail of phase-type distributions are given in [12].

The production inventory system studied has a single production facility that produces one type of item. The production time is exponentially distributed with parameter $\eta$. The inventory level in the system is governed by the $(s, S)$-policy. Once the inventory level becomes the maximum level $S$, the production facility takes a vacation whose duration follows an exponential distribution with parameter $\theta$. When the production facility returns from the vacation, if the inventory level depletes to the fixed level $s$, it is immediately switched on and starts production until the inventory level becomes $S$. Otherwise, if the inventory level is greater than $s$ on return from the vacation, it takes another vacation.

## 3. The Steady-State Analysis

The steady-state analysis of the production inventory system described is performed in this section. Let $N(t), I(t), K(t)$, $J_1(t)$ and $J_2(t)$ denote, respectively, the number of customers in the system, the inventory level, the state of the production process, the phase of the service and the phase of the arrival, at time $t$. The state of the production process is given by

$$K(t) = \begin{cases} 0 & \text{, if the production is taking a vacation,} \\ 1 & \text{, if the production is in ON mode.} \end{cases}$$

The process $\{(N(t), I(t), K(t), J_1(t), J_2(t)) : t \geq 0\}$ is a continuous-time Markov chain and the state space is given by

$$\Omega = \{i_0\} \cup \{i_1, i \geq 1\},$$

where

$$i_0 = \{(0, j, k, j_2) : 0 \leq j \leq S-1, k = 0, 1, 1 \leq j_2 \leq m\} \cup \\ \{(0, S, 0, j_2) : 1 \leq j_2 \leq m\}$$

and

$$i_1 = \{(i, j, k, j_1, j_2) : 0 \leq j \leq S-1, k = 0, 1, 1 \leq j_1 \leq n, 1 \leq j_2 \leq m\} \cup \\ \{(i, S, 0, j_1, j_2) : 1 \leq j_1 \leq n, 1 \leq j_2 \leq m\}.$$

The level $(0, j, 0, j_2)$ of dimension $m$ corresponds to the case when the system is idle, the inventory level $j$, $0 \le j \le S$, the production process is on vacation and the arrival process is in one of $m$ phases. The level $(0, j, 1, j_2)$ of dimension $m$ corresponds to the case when the system is idle, the inventory level $j$, $0 \le j \le S-1$, the production process is in ON mode and the arrival process is in one of $m$ phases.

The level $(i, j, 0, j_1, j_2)$of dimension $mn$ corresponds to the case when there is $i$ customers in the system, the inventory level $j$, $0 \le j \le S$, the production process is on vacation, the service process is in one of $n$ phases and the arrival process is in one of $m$ phases. The level $(i, j, 1, j_1, j_2)$ of dimension $mn$ corresponds to the case when there is $i$ customers in the system, the inventory level $j$, $0 \le j \le S-1$, the production process is in ON mode, the service process is in one of $n$ phases and the arrival process is in one of $m$ phases.

The infinitesimal generator matrix of the quasi-birth-and-death $(QBD)$ process has a block-tridiagonal matrix structure and is given by

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{C}_0 & & & \\ \boldsymbol{B}_0 & \boldsymbol{A} & \boldsymbol{C} & & \\ & \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{C} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2}$$

At this point, we need to set up some notation for use in sequel. $\boldsymbol{e}$ is a unit column vector; $\boldsymbol{e}_i$ is a column vector with 1 in the $i^{th}$ position and 0 elsewhere; $\boldsymbol{e}(j)$ is a unit column vector is of dimension $j$ and $\boldsymbol{I}_k$ is an identity matrix of order $k$. The symbols $\otimes$ and $\oplus$ represent the Kronecker product and the Kronecker sum, respectively. If $\boldsymbol{A}$ is a matrix of order $m \times n$ and if $\boldsymbol{B}$ is a matrix of order $p \times q$, then the Kronecker product of the two matrices is given by $\boldsymbol{A} \otimes \boldsymbol{B}$, a matrix of order $mp \times nq$; the Kronecker sum of two square matrices, say, $\boldsymbol{G}$ of order $g$ and $\boldsymbol{H}$ of $h$, is given by $\boldsymbol{G} \oplus \boldsymbol{H} = \boldsymbol{G} \otimes \boldsymbol{I}_h + \boldsymbol{I}_g \otimes \boldsymbol{H}$, a square matrix of order $gh$. Finally, for the dimensions of the matrices we define $d_1 = (2S+1)m$ and $d_2 = (2S+1)mn$.

The matrices $\boldsymbol{A}_0$ and $\boldsymbol{A}$ in the main diagonal of the matrix $\boldsymbol{Q}$ have dimensions $(d_1 \times d_1)$ and $(d_2 \times d_2)$, respectively.

$$\boldsymbol{A}_0 = \begin{pmatrix} \hat{\boldsymbol{A}}_1 & \hat{\boldsymbol{A}}_4 & & & & & & & \\ & \hat{\boldsymbol{A}}_2 & \hat{\boldsymbol{A}}_4 & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & \hat{\boldsymbol{A}}_2 & \hat{\boldsymbol{A}}_4 & & & & \\ & & & & \hat{\boldsymbol{A}}_3 & \hat{\boldsymbol{A}}_4 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \hat{\boldsymbol{A}}_3 & \hat{\boldsymbol{A}}_4 & \\ & & & & & & & \hat{\boldsymbol{A}}_3 & \hat{\boldsymbol{A}}_5 \\ & & & & & & & & \boldsymbol{D}_0 \end{pmatrix},$$

with

$$\hat{\boldsymbol{A}}_1 = \begin{pmatrix} -\theta \boldsymbol{I}_m & \theta \boldsymbol{I}_m \\ \boldsymbol{0} & -\eta \boldsymbol{I}_m \end{pmatrix}, \ \hat{\boldsymbol{A}}_2 = \begin{pmatrix} \boldsymbol{D}_0 - \theta \boldsymbol{I}_m & \theta \boldsymbol{I}_m \\ \boldsymbol{0} & \boldsymbol{D}_0 - \eta \boldsymbol{I}_m \end{pmatrix},$$

$$\hat{\boldsymbol{A}}_3 = \begin{pmatrix} \boldsymbol{D}_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_0 - \eta \boldsymbol{I}_m \end{pmatrix}, \ \hat{\boldsymbol{A}}_4 = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \eta \boldsymbol{I}_m \end{pmatrix}, \ \hat{\boldsymbol{A}}_5 = \begin{pmatrix} \boldsymbol{0} \\ \eta \boldsymbol{I}_m \end{pmatrix}.$$

$$\boldsymbol{A} = \begin{pmatrix} \tilde{\boldsymbol{A}}_1 & \tilde{\boldsymbol{A}}_4 & & & & & & & \\ & \tilde{\boldsymbol{A}}_2 & \tilde{\boldsymbol{A}}_4 & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & \tilde{\boldsymbol{A}}_2 & \tilde{\boldsymbol{A}}_4 & & & & \\ & & & & \tilde{\boldsymbol{A}}_3 & \tilde{\boldsymbol{A}}_4 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \tilde{\boldsymbol{A}}_3 & \tilde{\boldsymbol{A}}_4 & \\ & & & & & & & \tilde{\boldsymbol{A}}_3 & \tilde{\boldsymbol{A}}_5 \\ & & & & & & & & \boldsymbol{T} \oplus \boldsymbol{D}_0 \end{pmatrix},$$

with

$$\tilde{\boldsymbol{A}}_1 = \begin{pmatrix} -\theta \boldsymbol{I}_{mn} & \theta \boldsymbol{I}_{mn} \\ \boldsymbol{0} & -\eta \boldsymbol{I}_{mn} \end{pmatrix}, \ \tilde{\boldsymbol{A}}_2 = \begin{pmatrix} \boldsymbol{T} \oplus \boldsymbol{D}_0 - \theta \boldsymbol{I}_{mn} & \theta \boldsymbol{I}_{mn} \\ \boldsymbol{0} & \boldsymbol{T} \oplus \boldsymbol{D}_0 - \eta \boldsymbol{I}_{mn} \end{pmatrix},$$

$$\tilde{\boldsymbol{A}}_3 = \begin{pmatrix} \boldsymbol{T} \oplus \boldsymbol{D}_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T} \oplus \boldsymbol{D}_0 - \eta \boldsymbol{I}_{mn} \end{pmatrix}, \ \tilde{\boldsymbol{A}}_4 = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \eta \boldsymbol{I}_{mn} \end{pmatrix}, \ \tilde{\boldsymbol{A}}_5 = \begin{pmatrix} \boldsymbol{0} \\ \eta \boldsymbol{I}_{mn} \end{pmatrix}.$$

The matrices $\boldsymbol{B}_0$ and $\boldsymbol{B}$ in the lower diagonal of the matrix $\boldsymbol{Q}$ have dimensions $(d_2 \times d_1)$ and $(d_2 \times d_2)$, respectively.

$$\boldsymbol{B}_0 = \begin{pmatrix} \boldsymbol{0} & & & & \\ \hat{\boldsymbol{B}}_1 & \boldsymbol{0} & & & \\ & \ddots & \ddots & & \\ & & \hat{\boldsymbol{B}}_1 & \boldsymbol{0} & \\ & & & \hat{\boldsymbol{B}}_2 & \boldsymbol{0} \end{pmatrix} \text{ with}$$

$$\hat{\boldsymbol{B}}_1 = \begin{pmatrix} \boldsymbol{T}^0 \otimes \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}^0 \otimes \boldsymbol{I}_m \end{pmatrix}, \ \hat{\boldsymbol{B}}_2 = \begin{pmatrix} \boldsymbol{T}^0 \otimes \boldsymbol{I}_m & \boldsymbol{0} \end{pmatrix}.$$

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{0} & & & & \\ \tilde{\boldsymbol{B}}_1 & \boldsymbol{0} & & & \\ & \ddots & \ddots & & \\ & & \tilde{\boldsymbol{B}}_1 & \boldsymbol{0} & \\ & & & \tilde{\boldsymbol{B}}_2 & \boldsymbol{0} \end{pmatrix} \text{ with}$$

$$\tilde{\boldsymbol{B}}_1 = \begin{pmatrix} \boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m \end{pmatrix}, \ \tilde{\boldsymbol{B}}_2 = \begin{pmatrix} \boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m & \boldsymbol{0} \end{pmatrix}.$$

The matrices $\boldsymbol{C}_0$ and $\boldsymbol{C}$ in the upper diagonal of the matrix $\boldsymbol{Q}$ have dimensions $(d_1 \times d_2)$ and $(d_2 \times d_2)$, respectively.

$$\boldsymbol{C}_0 = \begin{pmatrix} \boldsymbol{0} & & & & \\ & \hat{\boldsymbol{C}}_1 & & & \\ & & \ddots & & \\ & & & \hat{\boldsymbol{C}}_1 & \\ & & & & \hat{\boldsymbol{C}}_2 \end{pmatrix} \text{ with}$$

$$\hat{\boldsymbol{C}}_1 = \begin{pmatrix} \boldsymbol{\beta} \otimes \boldsymbol{D}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\beta} \otimes \boldsymbol{D}_1 \end{pmatrix}, \ \hat{\boldsymbol{C}}_2 = \begin{pmatrix} \boldsymbol{\beta} \otimes \boldsymbol{D}_1 \end{pmatrix}.$$

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{0} & & & & \\ & \tilde{\boldsymbol{C}}_1 & & & \\ & & \ddots & & \\ & & & \tilde{\boldsymbol{C}}_1 & \\ & & & & \tilde{\boldsymbol{C}}_2 \end{pmatrix} \text{ with}$$

$$\tilde{\boldsymbol{C}}_1 = \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{D}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_n \otimes \boldsymbol{D}_1 \end{pmatrix}, \ \tilde{\boldsymbol{C}}_2 = \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{D}_1 \end{pmatrix}.$$

3.1. **The stability condition.** Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_{0,0}, \boldsymbol{\pi}_{0,1}, \boldsymbol{\pi}_{1,0}, \boldsymbol{\pi}_{1,1}, \cdots, \boldsymbol{\pi}_{s,0}, \boldsymbol{\pi}_{s,1}, \cdots, \boldsymbol{\pi}_{S-1,0}, \boldsymbol{\pi}_{S-1,1}, \boldsymbol{\pi}_{S,0})$ denote the steady-state probability vector of the generator matrix $\boldsymbol{F} = \boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$. The probability vector satisfies

$$\boldsymbol{\pi}\boldsymbol{F} = \boldsymbol{0}, \ \boldsymbol{\pi}\boldsymbol{e} = 1. \tag{3}$$

The steady-state equations in (3) can be rewritten as following.

$$\begin{aligned}
-\boldsymbol{\pi}_{0,0}\theta\boldsymbol{I} + \boldsymbol{\pi}_{1,0}(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) &= \boldsymbol{0}, & & \\
\boldsymbol{\pi}_{i,0}[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0) - \theta\boldsymbol{I}] + \boldsymbol{\pi}_{i+1,0}(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) &= \boldsymbol{0}, & 1 \le i \le s,& \\
\boldsymbol{\pi}_{i,0}[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0)] + \boldsymbol{\pi}_{i+1,0}(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) &= \boldsymbol{0}, & s+1 \le i \le S-1,& \\
\boldsymbol{\pi}_{S-1,1}\eta\boldsymbol{I} + \boldsymbol{\pi}_{S,0}[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0)] &= \boldsymbol{0}, & & \\
\\
\boldsymbol{\pi}_{0,0}\theta\boldsymbol{I} - \boldsymbol{\pi}_{0,1}\eta\boldsymbol{I} + \boldsymbol{\pi}_{1,1}(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) &= \boldsymbol{0}, & & \\
\boldsymbol{\pi}_{i-1,1}\eta\boldsymbol{I} + \boldsymbol{\pi}_{i,0}\theta\boldsymbol{I} + \boldsymbol{\pi}_{i,1}[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0) - \eta\boldsymbol{I}] & & & \\
+\boldsymbol{\pi}_{i+1,1}(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) &= \boldsymbol{0}, & 1 \le i \le s,& \\
\boldsymbol{\pi}_{i-1,1}\eta\boldsymbol{I} + \boldsymbol{\pi}_{i,1}[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0) - \eta\boldsymbol{I}] & & & \\
+\boldsymbol{\pi}_{i+1,1}(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) &= \boldsymbol{0}, & s+1 \le i \le S-2,& \\
\boldsymbol{\pi}_{S-2,1}\eta\boldsymbol{I} + \boldsymbol{\pi}_{S-1,1}[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0) - \eta\boldsymbol{I}] &= \boldsymbol{0}, & &
\end{aligned} \tag{4}$$

with the normalizing condition

$$\sum_{i=0}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})\boldsymbol{e}+\boldsymbol{\pi}_{S,0}\boldsymbol{e}=1. \tag{5}$$

The production inventory model with service facility under study is stable *if and only if* $\boldsymbol{\pi}\boldsymbol{C}\boldsymbol{e}<\boldsymbol{\pi}\boldsymbol{B}\boldsymbol{e}$ (see, e.g., [12]). The stability condition is given as

$$\Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big](\boldsymbol{I}_n\otimes\boldsymbol{D}_1)\boldsymbol{e} \;<\; \Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big](\boldsymbol{T}^0\boldsymbol{\beta}\otimes\boldsymbol{I}_m)\boldsymbol{e}. \tag{6}$$

Now adding the equations given in (4) we obtain

$$\Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big][(\boldsymbol{T}+\boldsymbol{T}^0\boldsymbol{\beta})\oplus\boldsymbol{D}]=\mathbf{0}. \tag{7}$$

Post-multiplying the equation in (7) by $(\boldsymbol{e}_n\otimes\boldsymbol{I}_m)$ and using the arrival rate $\lambda=\boldsymbol{\delta}\boldsymbol{D}_1\boldsymbol{e}$ we get

$$\Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big](\boldsymbol{e}_n\otimes\boldsymbol{I}_m)\boldsymbol{D}_1\boldsymbol{e}_m=\lambda\Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big]\boldsymbol{e}. \tag{8}$$

Then we obtain the left-side of the equation given in (6) by using the normalizing condition in (5) and the equation in (8) given as

$$\Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big](\boldsymbol{I}_n\otimes\boldsymbol{D}_1)\boldsymbol{e}=\lambda[1-(\boldsymbol{\pi}_{0,0}+\boldsymbol{\pi}_{0,1})\boldsymbol{e}]. \tag{9}$$

Post-multiplying the equation in (7) by $(\boldsymbol{I}_n\otimes\boldsymbol{e}_m)$ and using the service rate $\mu=1/[\boldsymbol{\beta}(-\boldsymbol{T})^{-1}\boldsymbol{e}]$ we get

$$\Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big](\boldsymbol{T}^0\boldsymbol{\beta}\otimes\boldsymbol{e}_m)\boldsymbol{e}_n=\mu\Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big]\boldsymbol{e}. \tag{10}$$

The right-side of the equation given in (6) is obtained by using the normalizing condition in (5) and the equation in (10) given as

$$\Big[\sum_{i=1}^{S-1}(\boldsymbol{\pi}_{i,0}+\boldsymbol{\pi}_{i,1})+\boldsymbol{\pi}_{S,0}\Big](\boldsymbol{T}^0\boldsymbol{\beta}\otimes\boldsymbol{I}_m)\boldsymbol{e}=\mu[1-(\boldsymbol{\pi}_{0,0}+\boldsymbol{\pi}_{0,1})\boldsymbol{e}]. \tag{11}$$

Finally the stability condition given in (6) is given by

$$\lambda[1-(\boldsymbol{\pi}_{0,0}+\boldsymbol{\pi}_{0,1})\boldsymbol{e}] \;<\; \mu[1-(\boldsymbol{\pi}_{0,0}+\boldsymbol{\pi}_{0,1})\boldsymbol{e}].$$

It is clear that $(\boldsymbol{\pi}_{0,0}+\boldsymbol{\pi}_{0,1})\boldsymbol{e}\neq 1$, so we establish the following theorem.

**Theorem 1.** *The production inventory system with service facility under study is stable if and only if the following condition is satisfied.*

$$\lambda \ < \ \mu \tag{12}$$

*where $\lambda$ and $\mu$ are the arrival rate and the service rate, respectively.*

3.2. **The steady-state probability vector.** Let $\boldsymbol{x} = (\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \cdots)$ denote the steady-state probability vector of the generator matrix $\boldsymbol{Q}$ given in (2). The probability vector satisfies

$$\boldsymbol{x}\boldsymbol{Q} = \boldsymbol{0}, \ \boldsymbol{x}\boldsymbol{e} = 1. \tag{13}$$

The vector $\boldsymbol{x}(0)$ of dimension $(2S+1)m$ is further partitioned into vectors of dimension $m$ as $\boldsymbol{x}(0) = [\boldsymbol{x}(0,0,0), \boldsymbol{x}(0,0,1), \cdots, \boldsymbol{x}(0,S-1,0), \boldsymbol{x}(0,S-1,1), \boldsymbol{x}(0,S,0)]$. The vector $\boldsymbol{x}(0,j,0)$, $0 \le j \le S$, gives the probability that the system is idle, the inventory level is $j$, the production process is on vacation and the arrival process is in one of $m$ phases. The vector $\boldsymbol{x}(0,j,1)$, $0 \le j \le S-1$, gives the probability that the system is idle, the inventory level is $j$, the production process is in ON mode and the arrival process is in one of $m$ phases.

The vector $\boldsymbol{x}(i)$, $i \ge 1$, of dimension $(2S+1)mn$ is further partitioned into vectors of dimension $mn$ as $\boldsymbol{x}(i) = [\boldsymbol{x}(i,0,0), \boldsymbol{x}(i,0,1), \cdots, \boldsymbol{x}(i,S-1,0), \boldsymbol{x}(i,S-1,1), \boldsymbol{x}(i,S,0)]$. The vector $\boldsymbol{x}(i,j,0)$, $0 \le j \le S$, gives the probability that the number of customers in the system is $i$, the inventory level is $j$, the production process is on vacation and the service process and the arrival process are in various phases. The vector $\boldsymbol{x}(i,j,1)$, $0 \le j \le S-1$, gives the probability that the number of customers in the system is $i$, the inventory level is $j$, the production process is in ON mode and the service process and the arrival process are in various phases.

Under the stability condition given in (12) the steady-state probability vector $\boldsymbol{x}$ is obtained (see [12]) as

$$\boldsymbol{x}(i) = \boldsymbol{x}(1)\boldsymbol{R}^{i-1}, \ i > 1, \tag{14}$$

where the matrix $\boldsymbol{R}$ is the minimal nonnegative solution to the following matrix quadratic equation

$$\boldsymbol{R}^2\boldsymbol{B} + \boldsymbol{R}\boldsymbol{A} + \boldsymbol{C} = \boldsymbol{0}, \tag{15}$$

and the vectors, $\boldsymbol{x}(0)$ and $\boldsymbol{x}(1)$ are obtained by solving

$$\begin{gathered} \boldsymbol{x}(0)\boldsymbol{A}_0 + \boldsymbol{x}(1)\boldsymbol{B}_0 = \boldsymbol{0}, \\ \boldsymbol{x}(0)\boldsymbol{C}_0 + \boldsymbol{x}(1)[\boldsymbol{A} + \boldsymbol{R}\boldsymbol{B}] = \boldsymbol{0}, \end{gathered} \tag{16}$$

subject to the normalizing condition

$$\boldsymbol{x}(0)\boldsymbol{e} + \boldsymbol{x}(1)(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{e} = 1. \tag{17}$$

3.3. **The performance measures.** Some performance measures of the production inventory system under study are listed in this section.

1. *The probability that there is no customer in the system*
$$P_{idle} = \boldsymbol{x}(0)\boldsymbol{e}.$$
2. *The mean number of customers in the system*
$$E_N = \sum_{i=1}^{\infty} i\ \boldsymbol{x}(i)\boldsymbol{e} = \boldsymbol{x}(1)(\boldsymbol{I} - \boldsymbol{R})^{-2}\boldsymbol{e}.$$
3. *The mean production rate*
$$E_{PR} = \eta\Big[\sum_{j=0}^{S-1}\boldsymbol{x}(0,j,1)\boldsymbol{e} + \sum_{i=1}^{\infty}\sum_{j=0}^{S-1}\boldsymbol{x}(i,j,1)\boldsymbol{e}\Big].$$
4. *The mean loss rate of customers*
$$E_{LR} = \lambda\Big[[\boldsymbol{x}(0,0,0) + \boldsymbol{x}(0,0,1)]\boldsymbol{e} + \sum_{i=1}^{\infty}[\boldsymbol{x}(i,0,0) + \boldsymbol{x}(i,0,1)]\boldsymbol{e}\Big].$$
5. *The mean number of items in the inventory when the production is switched ON*
$$EI_R = \sum_{j=0}^{S-1} j\ x(0,j,1)\boldsymbol{e} + \sum_{i=1}^{\infty}\sum_{j=0}^{S-1} j\ x(i,j,1)\boldsymbol{e}$$
6. *The mean number of items in the inventory when the production is switched OFF for a vacation*
$$EI_V = \sum_{j=0}^{S} j\ x(0,j,0)\boldsymbol{e} + \sum_{i=1}^{\infty}\sum_{j=0}^{S} j\ x(i,j,0)\boldsymbol{e}$$
7. *The mean number of items in the inventory*
$$EI = EI_R + EI_V$$

## 4. Numerical Study

In this section, we perform the numerical examples similar to ones given in [18] to see the effects of various parameters on the system performance measures and to discuss the optimum inventory policies under various scenarios by using a constructed cost function. In other words, the examples in [18] were performed by considering an exponential distribution for both of the inter-arrival times and the service times. We expand the examples for different phase-type distributions. So, we consider the same values used in [18] for the parameters in the all examples.

For the arrival process, we consider the following five sets of values for $\boldsymbol{D}_0$ and $\boldsymbol{D}_1$. The five arrival processes have the same mean of 1 but each one of them is qualitatively different. The values of the standard deviation of the inter-arrival times of the arrival processes with respect to ERLA are, respectively, 1, 1.41421, 3.17451, 1.99336, and 1.99336. The $MAP$ processes are normalized to have a specific arrival rate $\lambda$ as given in [1]. The arrival processes labeled MNCA and MPCA have negative and positive correlation for two successive inter-arrival times with values -0.4889 and 0.4889, respectively, whereas the first three arrival processes have zero correlation for two successive inter-arrival times.

Erlang distribution (**ERLA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}, \ \boldsymbol{D}_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

Exponential distribution (**EXPA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -1 \end{pmatrix}, \ \boldsymbol{D}_1 = \begin{pmatrix} 1 \end{pmatrix}.$$

Hyperexponential distribution (**HEXA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -1.9 & 0 \\ 0 & -0.19 \end{pmatrix}, \ \boldsymbol{D}_1 = \begin{pmatrix} 1.71 & 0.19 \\ 0.171 & 0.019 \end{pmatrix}.$$

MAP with negative correlation (**MNCA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{pmatrix}, \ \boldsymbol{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.9922 \\ 223.4925 & 0 & 2.2575 \end{pmatrix}.$$

MAP with positive correlation (**MPCA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{pmatrix}, \ \boldsymbol{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.9922 & 0 & 0.01002 \\ 2.2575 & 0 & 223.4925 \end{pmatrix}.$$

For the service times, we consider three phase-type distributions with parameter $(\boldsymbol{\beta}, \boldsymbol{T})$. The three phase-type distributions have the same mean of 1 but each one of them is qualitatively different. The values of the standard deviation of the distributions are, respectively, 0.70711, 1, and 2.24472. The distributions are normalized at a specific value for the service rate $\mu$.

Erlang distribution (**ERLS**):

$$\boldsymbol{\beta} = \begin{pmatrix} 1, & 0 \end{pmatrix}, \ \boldsymbol{T} = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}.$$

Exponential distribution (**EXPS**):

$$\boldsymbol{\beta} = \begin{pmatrix} 1 \end{pmatrix}, \ \boldsymbol{T} = \begin{pmatrix} -1 \end{pmatrix}.$$

Hyperexponential distribution (**HEXS**):

$$\boldsymbol{\beta} = \begin{pmatrix} 0.9, & 0.1 \end{pmatrix}, \quad \boldsymbol{T} = \begin{pmatrix} -1.9 & 0 \\ 0 & -0.19 \end{pmatrix}.$$

4.1. **The effect of the parameters on the performance measures.** The purpose of all examples in this section is to examine how some of the performance measures are effected by the increasing values of parameters. We assume that the inventory policy is $(s, S) = (5, 45)$ and the service rate is $\mu = 4$ for all examples.

**Example 1:** The effect of the arrival rate $\lambda$ on the performance measures such as $E_{PR}$ and $E_{LR}$, is represented in Table 1. Also, the effect on the performance measures consist of $EI$, $EI_V$ and $EI_R$ is illustrated in Figure 2. For the purposes we fixed $\eta = 2.5$ and $\theta = 1.5$.

TABLE 1. The performance measures for the increasing values of $\lambda$

| | | $E_{PR}$ | | | $E_{LR}$ | | |
|---|---|---|---|---|---|---|---|
| $\lambda$ | | ERLS | EXPS | HEXS | ERLS | EXPS | HEXS |
| 1.5 | ERLA | 1.4980 | 1.4979 | 1.4957 | 0.0020 | 0.0021 | 0.0043 |
| | EXPA | 1.4960 | **1.4958** | 1.4932 | 0.0040 | **0.0042** | 0.0068 |
| | HEXA | 1.4807 | 1.4802 | 1.4759 | 0.0193 | 0.0198 | 0.0241 |
| | MNCA | 1.4959 | 1.4957 | 1.4930 | 0.0041 | 0.0043 | 0.0070 |
| | MPCA | 1.3472 | 1.3467 | 1.3424 | 0.1527 | 0.1533 | 0.1575 |
| 2.3 | ERLA | 2.2691 | 2.2682 | 2.2544 | 0.0309 | 0.0318 | 0.0456 |
| | EXPA | 2.2549 | **2.2539** | 2.2388 | 0.0451 | **0.0461** | 0.0612 |
| | HEXA | 2.1552 | 2.1539 | 2.1379 | 0.1448 | 0.1461 | 0.1621 |
| | MNCA | 2.2546 | 2.2535 | 2.2381 | 0.0454 | 0.0465 | 0.0619 |
| | MPCA | 1.9021 | 1.9013 | 1.8939 | 0.3977 | 0.3985 | 0.4058 |
| 3.1 | ERLA | 2.5000 | 2.5000 | 2.4950 | 0.6000 | 0.6000 | 0.6050 |
| | EXPA | 2.4998 | **2.4998** | 2.4936 | 0.6002 | **0.6002** | 0.6064 |
| | HEXA | 2.4644 | 2.4636 | 2.4490 | 0.6356 | 0.6364 | 0.6510 |
| | MNCA | 2.4998 | 2.4997 | 2.4935 | 0.6002 | 0.6003 | 0.6065 |
| | MPCA | 2.2945 | 2.2938 | 2.2856 | 0.8055 | 0.8062 | 0.8144 |

From Table 1 we notice the following observations.

- When the arrival rate $\lambda$ is increased, the mean production rate $E_{PR}$ and the mean loss rate of customers $E_{LR}$ increase.
- The increase the variability in the inter-arrival times causes the values of $E_{PR}$ to decrease. Moreover, at the values of $E_{PR}$ are looked, we should note that the $MAP$ process with positive correlation labeled MPCA is significantly separated from the other $MAP$ processes, especially for the systems where the traffic intensity is low (the cases of $\lambda = 1.5$ and $\lambda = 2.3$).

- The values of $E_{LR}$ increase as the variability in the inter-arrival times increases. The values of $E_{LR}$ are dramatically more at the processes named HEXA and MPCA.
- Compared to $MAP$ process, the distribution of the service times has less effect on the values of $E_{PR}$ and $E_{LR}$. The increase the variability in the service times induces a decrement on $E_{PR}$ and an increment on $E_{LR}$.

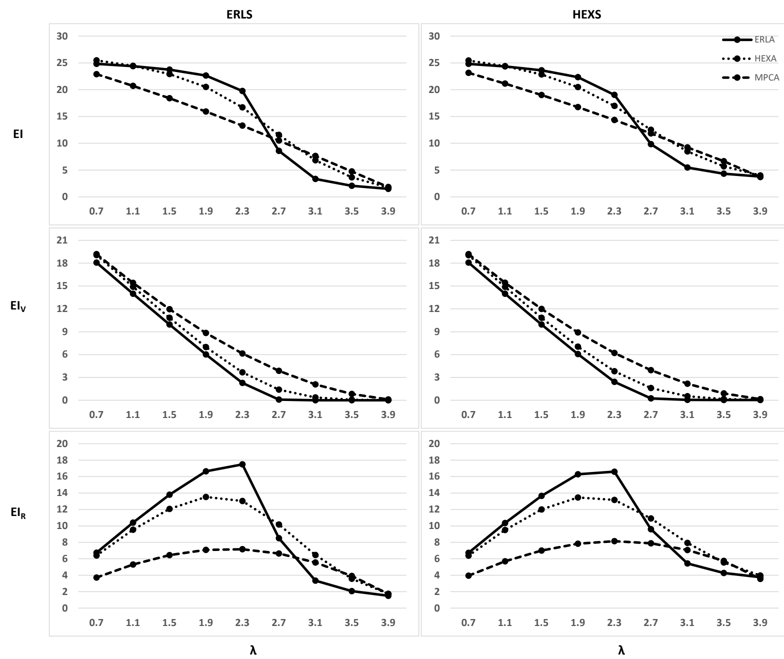


FIGURE 2. The effect of the arrival rate on the inventory level

When the arrival rate $\lambda$ is increased, the mean number of items in the inventory $EI$ decreases. The effect of variability in the arrival process on $EI$ changes differently depending on the arrival rate. That is, as the variability increases the values of $EI$ decrease for the low arrival rates while increases for the high arrival rates. The behaviour of $EI$ $(= EI_V + EI_R)$ is detailed consideringly its components in Figure 2. When the arrival rate $\lambda$ is increased, the values of $EI_V$ decrease for all arrival and service scenarios. On the other hand, the values of $EI_R$ firstly increase and then decrease after a certain point. That is, while the values of $\lambda$ increases, the values of $EI_R$ have a concave structure. The variability in the arrival process affects the concave structure, for example, ERLA with low variability has a faster decline compared to other arrival processes. The decrease in the values of $EI_R$ occurs when the arrival rate $\lambda$ is greater than the production rate $\eta$ (=2.5). We note that in the case of HEXA where the variability is high, the break starts earlier.

**Example 2:** The effect of the production rate $\eta$ on the performance measures such as $E_{PR}$ and $E_{LR}$ is represented in Table 2. Also, the effect on the performance measures consist of $EI$, $EI_V$ and $EI_R$ is illustrated in Figure 3. For this example we fixed $\lambda = 1.5$ and $\theta = 1.5$.

TABLE 2. The performance measures for the increasing values of $\eta$

| | | $E_{PR}$ | | | $E_{LR}$ | | |
|---|---|---|---|---|---|---|---|
| $\eta$ | | ERLS | EXPS | HEXS | ERLS | EXPS | HEXS |
| | ERLA | 1.3938 | 1.3937 | 1.3928 | 0.1062 | 0.1063 | 0.1072 |
| | EXPA | 1.3862 | **1.3862** | 1.3849 | 0.1138 | **0.1138** | 0.1151 |
| 1.4 | HEXA | 1.3136 | 1.3135 | 1.3121 | 0.1864 | 0.1865 | 0.1879 |
| | MNCA | 1.3860 | 1.3859 | 1.3846 | 0.1140 | 0.1141 | 0.1154 |
| | MPCA | 1.0994 | 1.0993 | 1.0985 | 0.4006 | 0.4007 | 0.4015 |
| | ERLA | 1.4779 | 1.4778 | 1.4758 | 0.0221 | 0.0222 | 0.0242 |
| | EXPA | 1.4673 | **1.4671** | 1.4649 | 0.0327 | **0.0329** | 0.0351 |
| 1.6 | HEXA | 1.3928 | 1.3926 | 1.3901 | 0.1072 | 0.1074 | 0.1099 |
| | MNCA | 1.4671 | 1.4669 | 1.4645 | 0.0329 | 0.0331 | 0.0355 |
| | MPCA | 1.1615 | 1.1614 | 1.1602 | 0.3385 | 0.3386 | 0.3398 |
| | ERLA | 1.4903 | 1.4902 | 1.4881 | 0.0097 | 0.0098 | 0.0119 |
| | EXPA | 1.4841 | **1.4839** | 1.4816 | 0.0159 | **0.0161** | 0.0184 |
| 1.8 | HEXA | 1.4339 | 1.4336 | 1.4305 | 0.0661 | 0.0664 | 0.0695 |
| | MNCA | 1.4840 | 1.4837 | 1.4813 | 0.0160 | 0.0163 | 0.0187 |
| | MPCA | 1.2139 | 1.2137 | 1.2122 | 0.2860 | 0.2862 | 0.2878 |

From Table 2 we notice the following observations.

- As is to be expected when the production rate $\eta$ is increased, the mean production rate $E_{PR}$ increases and the mean loss rate of customers $E_{LR}$ decreases due to there is more items in the inventory.
- The increase the variability in the inter-arrival times causes the values of $E_{PR}$ to decrease. On the cases of HEXA and MPCA the decrement is significantly different compared the other $MAP$ processes.
- The values of $E_{LR}$ increase as the variability in the inter-arrival times increases. It is dramatically more on MPCA.
- We do not see that the variability of the service distribution has a significant effect on the values of $E_{PR}$ and $E_{LR}$. Even so, it can be said that the HEXS with high variability is distinguished from the others.

When the production rate $\eta$ is increased, the mean number of items in the inventory $EI$ increase. It is clear in Figure 3. The variability in the arrival process causes differently effects on $EI$ depending on the production rate. In other words, as the variability increases the value of $EI$ increases for the low production rate and decreases for the high production rate. The results for its components $EI_V$ and $EI_R$ are also illustrated in Figure 3. When the production rate $\eta$ is increased, the values of $EI_V$ increase for all scenarios. As the values of $\eta$ increases, the values of $EI_R$ have a concave structure except the arrival process labeled MPCA. We can say that the variability in the arrival process affects the concave structure. That is, ERLA with low variability has a faster increment compared to HEXA with high variability.

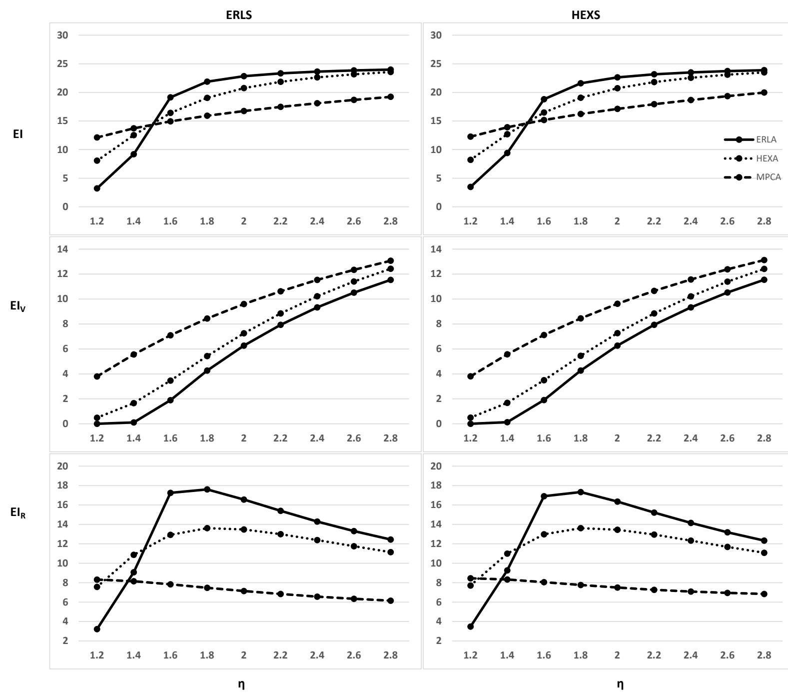


FIGURE 3. The effect of the production rate on the inventory level

**Example 3:** The effect of the vacation rate $\theta$ on the performance measures such as $E_{PR}$ and $E_{LR}$, is represented in Table 3. We fixed $\lambda = 1.5$ and $\eta = 2.5$ for this example.

Looking at the values in Table 3, it is seen that similar comments in Table 2 will be made here as well. When the vacation rate $\theta$ is increased, the values of $E_{PR}$ increase and the values of $E_{LR}$ decrease. The increase the variability in arrival processes causes the values of $E_{PR}$ to decrease and the values of $E_{LR}$ to increase. These changes are significantly different on the cases of HEXA and MPCA. When the variability of the service distribution is observed, it is seen that the HEXS with high variability is distinguished from the others. When the vacation rate $\theta$ is increased, $EI$ increase. As the variability in the arrival process increases the values of $EI$ and its components $EI_V$ and $EI_R$ increase. So, the concave structure in the values of $EI_R$ does not exist as the vacation rate $\theta$ increases.

TABLE 3. The performance measures for the increasing values of $\theta$

| | | $E_{PR}$ | | | $E_{LR}$ | | |
|---|---|---|---|---|---|---|---|
| $\theta$ | | ERLS | EXPS | HEXS | ERLS | EXPS | HEXS |
| | ERLA | 1.4893 | 1.4890 | 1.4861 | 0.0107 | 0.0110 | 0.0139 |
| | EXPA | 1.4862 | **1.4859** | 1.4826 | 0.0138 | **0.0141** | 0.0174 |
| 0.6 | HEXA | 1.4676 | 1.4671 | 1.4625 | 0.0324 | 0.0329 | 0.0375 |
| | MNCA | 1.4863 | 1.4858 | 1.4824 | 0.0137 | 0.0142 | 0.0176 |
| | MPCA | 1.3424 | 1.3418 | 1.3374 | 0.1576 | 0.1582 | 0.1626 |
| | ERLA | 1.4951 | 1.4949 | 1.4923 | 0.0049 | 0.0051 | 0.0077 |
| | EXPA | 1.4924 | **1.4922** | 1.4893 | 0.0076 | **0.0078** | 0.0107 |
| 0.9 | HEXA | 1.4753 | 1.4747 | 1.4703 | 0.0247 | 0.0253 | 0.0297 |
| | MNCA | 1.4924 | 1.4921 | 1.4890 | 0.0076 | 0.0079 | 0.0110 |
| | MPCA | 1.3453 | 1.3447 | 1.3404 | 0.1547 | 0.1553 | 0.1596 |
| | ERLA | 1.4971 | 1.4969 | 1.4946 | 0.0029 | 0.0031 | 0.0054 |
| | EXPA | 1.4948 | **1.4946** | 1.4919 | 0.0052 | **0.0054** | 0.0081 |
| 1.2 | HEXA | 1.4787 | 1.4782 | 1.4739 | 0.0213 | 0.0218 | 0.0261 |
| | MNCA | 1.4948 | 1.4945 | 1.4917 | 0.0052 | 0.0055 | 0.0083 |
| | MPCA | 1.3465 | 1.3460 | 1.3417 | 0.1535 | 0.1540 | 0.1583 |

4.2. **Optimization.** In this section we construct a objective function, $ETC$, giving the expected total cost per unit of time, and then discuss an optimization problem under various scenarios.

In the cost function, $c_{lost}$, $c_s$, $c_p$ and $c_h$ denote, respectively, cost incured due to the loss of customers, cost per unit of time of servicing per customer, cost per unit time of producing per inventory and cost per unit of time of holding per inventory.

$$ETC = c_{lost}E_{LR} + c_sEN + c_pE_{PR} + c_hEI$$

Towards finding the optimum values for the inventory policy and the total cost, we fixed the unit values of the costs by $c_{lost} = 200$, $c_h = 1$, $c_p = 10$ and $c_s = 2$. Also, we consider the other parameters as follows. The optimum values are given for various $\lambda$ and fixed $\mu = 2.5$, $\theta = 0.8$ and $\eta = 2.5$ in Table 4; for various $\eta$ and fixed $\lambda = 1.5$, $\theta = 0.8$ and $\mu = 2.5$ in Table 5; for various $\theta$ and fixed $\lambda = 2$, $\mu = 2.5$ and $\eta = 2.5$ in Table 6; and for various $\mu$ and fixed $\lambda = 1$, $\theta = 0.8$ and $\eta = 2.5$ in Table 7. We should remark that the above $ETC$ function and considered the values for the costs and parameters are the same ones given in [18].

The optimum values of the inventory policy, $(s^*, S^*)$, and the optimum total cost increase as the arrival rate $\lambda$ increases in Table 4. On the other hand, Table 5 and

Table 6 show that both the values of the optimum policy and the optimum cost decreases as the production rate $\eta$ increases and as the vacation rate $\theta$ increases, respectively. As $\mu$ increases, the optimum inventory policy remains almost the same except to the arrival process labeled MPCA in Table 7. The process requests more item on hand inventory with the increasing service rate. Also, Table 7 represents that the increase the service rate $\mu$ causes the decrement on the optimum total cost for all scenarios.

It can be seen in Tables 4-7 that the variability in the arrival process and the service process both have a significant effect on the optimum values. As the variability in the arrival process or the variability in the service process increase, both the optimum inventory policy increases and the optimum total cost increases.

TABLE 4. The optimum policy for the increasing values of $\lambda$

| | | ERLS | | EXPS | | HEXS | |
|---|---|---|---|---|---|---|---|
| | $\lambda$ | $(s^*, S^*)$ | Cost | $(s^*, S^*)$ | Cost | $(s^*, S^*)$ | Cost |
| ERLA | 1 | (7,8) | 18.4140 | (7,8) | 18.6544 | (8,9) | 21.3692 |
| | 1.3 | (9,10) | 23.4965 | (9,10) | 23.9754 | (11,12) | 29.0665 |
| | 1.6 | (11,12) | 28.8253 | (11,12) | 29.7873 | (15,16) | 38.9059 |
| EXPA | 1 | (7,8) | 19.2833 | **(7,8)** | **19.6339** | (9,10) | 22.2820 |
| | 1.3 | (9,10) | 24.5579 | **(10,11)** | **25.1954** | (12,13) | 30.1895 |
| | 1.6 | (11,12) | 30.3250 | **(12,13)** | **31.4095** | (16,17) | 40.4592 |
| HEXA | 1 | (9,10) | 21.7124 | (9,10) | 22.2780 | (10,11) | 25.7178 |
| | 1.3 | (11,12) | 28.5094 | (12,13) | 29.6037 | (14,15) | 35.6625 |
| | 1.6 | (14,15) | 37.1754 | (15,16) | 38.9603 | (19,20) | 49.2598 |
| MNCA | 1 | (8,9) | 19.6845 | (8,9) | 19.9280 | (8,9) | 22.5474 |
| | 1.3 | (9,10) | 24.9916 | (10,11) | 25.5240 | (12,13) | 30.4506 |
| | 1.6 | (11,12) | 30.7686 | (12,13) | 31.7932 | (16,17) | 40.7547 |
| MPCA | 1 | (9,14) | 93.2177 | (10,15) | 95.2172 | (14,19) | 104.7880 |
| | 1.3 | (12,17) | 140.8718 | (13,18) | 143.5009 | (20,24) | 156.8172 |
| | 1.6 | (15,20) | 216.2425 | (17,22) | 219.6613 | (26,30) | 237.9998 |

## 5. Conclusions

In this study, we considered a production inventory system with $MAP$ arrivals and phase-type service times. The production facility in the system is governed by $(s, S)$-policy and can be taken a vacation. We obtained the stability condition in closed form and then analyzed the production inventory system in the steady-state by using the matrix-geometric method. Some numerical examples were performed to see the effect of the parameters on the system performance measures and to define the optimum inventory policy. In all the examples, we observed that the variability in the inter-arrival times and the variability in the service times affect

TABLE 5. The optimum policy for the increasing values of $\eta$

| | | ERLS | | EXPS | | HEXS | |
|---|---|---|---|---|---|---|---|
| | $\eta$ | $(s^*, S^*)$ | Cost | $(s^*, S^*)$ | Cost | $(s^*, S^*)$ | Cost |
| ERLA | 2.2 | (10,12) | 27.3130 | (11,12) | 28.0789 | (15,16) | 36.1313 |
| | 2.8 | (10,11) | 26.9440 | (10,11) | 27.6879 | (13,14) | 34.7214 |
| | 3.4 | (9,11) | 27.0759 | (10,11) | 27.7608 | (12,13) | 34.0837 |
| EXPA | 2.2 | (11,12) | 28.9516 | **(11,12)** | **29.8165** | (16,17) | 37.7226 |
| | 2.8 | (10,11) | 28.1052 | **(11,12)** | **29.0030** | (13,14) | 35.9469 |
| | 3.4 | (10,11) | 28.1014 | **(10,11)** | **28.9231** | (12,13) | 35.1845 |
| HEXA | 2.2 | (15,16) | 36.3293 | (16,17) | 37.7399 | (19,20) | 46.5107 |
| | 2.8 | (12,13) | 32.9618 | (12,13) | 34.4631 | (16,17) | 42.6655 |
| | 3.4 | (11,12) | 32.4254 | (12,13) | 33.7934 | (14,15) | 41.0429 |
| MNCA | 2.2 | (12,13) | 29.3807 | (12,13) | 30.2023 | (16,17) | 38.0177 |
| | 2.8 | (10,11) | 28.4921 | (11,12) | 29.3866 | (13,14) | 36.2475 |
| | 3.4 | (10,11) | 28.4689 | (10,11) | 29.2740 | (12,13) | 35.4591 |
| MPCA | 2.2 | (15,19) | 201.1493 | (16,22) | 203.5517 | (26,31) | 218.4687 |
| | 2.8 | (13,16) | 180.9230 | (14,18) | 183.7347 | (21,24) | 198.6872 |
| | 3.4 | (11,14) | 178.3297 | (12,15) | 180.2573 | (18,20) | 191.2252 |

TABLE 6. The optimum policy for the increasing values of $\theta$

| | | ERLS | | EXPS | | HEXS | |
|---|---|---|---|---|---|---|---|
| | $\theta$ | $(s^*, S^*)$ | Cost | $(s^*, S^*)$ | Cost | $(s^*, S^*)$ | Cost |
| ERLA | 0.6 | (16,17) | 39.2035 | (17,18) | 41.6564 | (25,26) | 62.2963 |
| | 1 | (13,14) | 37.3256 | (15,16) | 39.9393 | (22,23) | 61.2640 |
| | 1.4 | (12,13) | 36.7212 | (14,15) | 39.3947 | (21,22) | 60.9467 |
| EXPA | 0.6 | (17,18) | 41.9681 | **(18,19)** | **44.5635** | (25,26) | 65.1413 |
| | 1 | (14,15) | 40.3000 | **(15,16)** | **43.0441** | (23,24) | 64.1669 |
| | 1.4 | (13,14) | 39.7809 | **(14,15)** | **42.5706** | (22,23) | 63.8602 |
| HEXA | 0.6 | (19,20) | 57.7754 | (21,22) | 61.2931 | (29,30) | 83.5578 |
| | 1 | (17,18) | 56.6264 | (19,20) | 60.2413 | (27,28) | 82.7774 |
| | 1.4 | (16,17) | 56.2707 | (18,19) | 59.9048 | (26,27) | 82.5154 |
| MNCA | 0.6 | (17,18) | 42.4377 | (18,19) | 44.9924 | (25,26) | 65.4902 |
| | 1 | (14,15) | 40.7755 | (16,17) | 43.4808 | (23,24) | 64.5153 |
| | 1.4 | (13,14) | 40.2577 | (15,16) | 43.0026 | (22,23) | 64.2096 |
| MPCA | 0.6 | (20,26) | 446.6197 | (23,29) | 451.8148 | (37,41) | 482.5901 |
| | 1 | (20,24) | 446.1006 | (22,27) | 451.3087 | (36,39) | 482.0961 |
| | 1.4 | (19,23) | 445.9109 | (22,26) | 451.1244 | (35,38) | 481.9004 |

the values of the performance measures and the optimum inventory policy. These observations are very important in the modelling of real systems. The production

TABLE 7. The optimum policy for the increasing values of $\mu$

| | | ERLS | | EXPS | | HEXS | |
|---|---|---|---|---|---|---|---|
| | $\mu$ | $(s^*, S^*)$ | Cost | $(s^*, S^*)$ | Cost | $(s^*, S^*)$ | Cost |
| ERLA | 1.9 | (7,8) | 18.9509 | (7,8) | 19.4664 | (8,9) | 24.0530 |
| | 2.2 | (7,8) | 18.6219 | (7,8) | 18.9634 | (8,9) | 22.4089 |
| | 2.5 | (7,8) | 18.4140 | (7,8) | 18.6544 | (8,9) | 21.3692 |
| EXPA | 1.9 | (7,8) | 19.8580 | **(7,8)** | **20.5228** | (9,10) | 24.9252 |
| | 2.2 | (7,8) | 19.4955 | **(7,8)** | **19.9673** | (9,10) | 23.2919 |
| | 2.5 | (7,8) | 19.2833 | **(7,8)** | **19.6339** | (9,10) | 22.2820 |
| HEXA | 1.9 | (8,9) | 22.9172 | (9,10) | 23.9924 | (10,11) | 29.3001 |
| | 2.2 | (8,9) | 22.1004 | (9,10) | 22.8603 | (10,11) | 27.0563 |
| | 2.5 | (9,10) | 21.7124 | (9,10) | 22.2780 | (10,11) | 25.7178 |
| MNCA | 1.9 | (7,8) | 20.2947 | (8,9) | 20.8661 | (9,10) | 25.2099 |
| | 2.2 | (7,9) | 19.9420 | (8,9) | 20.2841 | (9,10) | 23.5695 |
| | 2.5 | (8,9) | 19.6845 | (8,9) | 19.9280 | (9,10) | 22.5532 |
| MPCA | 1.9 | (9,12) | 132.4985 | (10,13) | 134.3205 | (14,17) | 143.5750 |
| | 2.2 | (9,13) | 106.7248 | (10,14) | 108.7262 | (15,18) | 118.3877 |
| | 2.5 | (9,14) | 93.2177 | (10,15) | 95.2172 | (14,19) | 104.7880 |

inventory model considered in this paper can be studied further in a number of ways. Some specific ones are as follows. First, one can generalize this to include $BMAP$ arrivals and/or batch services. Secondly, the production times and/or the vacation times can be considered as a phase-type distribution. Thirdly, it would be interesting to study the present model considering a hidden Markov model where allows us to talk about both observed events and hidden events that we think of as causal factors.

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