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RESEARCH PAPER

Two-dimensional Cattaneo-Hristov heat diffusion in the half-plane

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Abstract

In this paper, Cattaneo-Hristov heat diffusion is discussed in the half plane for the first time, and solved under two different boundary conditions. For the solution purpose, the Laplace, and the sine- and exponential- Fourier transforms with respect to time and space variables are applied, respectively. Since the fractional term in the problem is the Caputo-Fabrizio derivative with the exponential kernel, the solutions are in terms of time-dependent exponential and spatial-dependent Bessel functions. Behaviors of the temperature functions due to the change of different parameters of the problem are interpreted by giving 2D and 3D graphics.

Keywords: Two-dimensional Cattaneo-Hristov equation; Laplace transform; sine-Fourier transform; exponential Fourier transform; Caputo-Fabrizio derivative

AMS 2020 Classification: 35R11; 35A22; 74A10

1 Introduction

Heat conduction, also called diffusion, is the exchange of thermal energy between different physical systems. The classical theory of heat conduction is based on Fourier's law, that is, almost 200 years ago. Fourier's law implies that infinitesimal heat changes propagate at an infinite speed. This result makes the law a paradox that cannot specifically represent microscopic heat distribution. The physical validity of this law is for heat transfer models in low dimensions and also in macroscopic scales.

To remove the inconsistency of Fourier's law for heat transfer occurring in non-homogeneous mediums or for microscopic scales, different non-local dependencies between heat flux and temperature gradient have been proposed. As a result, different types of heat conduction equations have emerged and this has led to the development of non-classical theories on heat conduction. In this sense, fractional operators with singular or non-singular kernels have played a significant

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role in various types of real-world problems [1–6]. For instance, in a thin rectangular plate the non-local relation between the heat flux $q(t)$ and the temperature gradient $gradT = \left[\frac{\partial T}{\partial x}\ \frac{\partial T}{\partial y}\right]$ can be given by [7]

$$q(t) = -k\int_0^t K(t-\tau)\, gradT(x,y,\tau)\, d\tau, \tag{1}$$

where k is the coefficient of thermal conductivity. When this relation is combined with the laws of conservation of energy, it leads to the following generalized heat conduction equation [8]:

$$\frac{\partial T}{\partial t} = a\int_0^t K(t-\tau)\, \Delta T(x,y,\tau)\, d\tau. \tag{2}$$

in which a is the thermal diffusivity coefficient. The decisive factor here is the type of kernel function K which physically corresponds to the memory effects in heating systems. Some leading non-local laws with various types of kernel functions can be summarized as follows:

$$q(t) = -k\int_0^t gradT(x,y,\tau)\, d\tau \quad \text{(Full memory/without fading memory [9])}, \tag{3}$$

$$q(t) = \begin{cases} \frac{k}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}\, gradT(x,y,\tau)\, d\tau, & 0<\alpha\leq 1, \\ -\frac{k}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}\, gradT(x,y,\tau)\, d\tau, & 1<\alpha\leq 2, \end{cases} \quad \text{(long-tail memory [10])}, \tag{4}$$

$$q(t) = -\frac{k}{\xi}\int_0^t \exp\left(-\frac{t-\tau}{\xi}\right) gradT(x,y,\tau)\, d\tau \quad \text{(short-tail memory [11, 12])}, \tag{5}$$

where ξ denotes the finite relaxation time of the heating process. In Eq. (3), the kernel is constant $K=1$ so there is no fading in memory. In Eq. (4), the relations between heat flux and temperature gradient have long-term memory power kernels $K=(t-\tau)^{\alpha-1}$ and $K=(t-\tau)^{\alpha-2}$. Thus, constitutive relations given in Eq. (4) led to the emergence of the heat equation with Caputo fractional derivative. Analytical solutions of these equations with different initial and boundary conditions and in different coordinate systems have been studied in detail by Povstenko [13–17]. Furthermore, the thermal stresses due to fractional heat conduction were researched [18–22], and even the optimal control problem of these thermal stresses was investigated later [23, 24].

The integro-differential equation with Jeffrey kernel $K = \exp\left(-\frac{t-\tau}{\xi}\right)$ based on the constitutive law stated in Eq. (5) was proposed for the damped heat diffusion in rigid conductors. A few years ago, Hristov conceived of relating the Jeffrey kernel in the Cattaneo model to the Caputo-Fabrizio fractional derivative that has a non-singular kernel [25]. The obtained model is called the Cattaneo-Hristov heat diffusion equation in the literature. This development shows that different constitutive equations can be reconstructed with non-singular fractional derivatives [26–28] which was detailed studied by researchers [29, 30]. In fact, this is a wise answer to understanding the physical background of fractional derivatives.

There are limited but undoubtedly valuable studies in the literature to find the analytical and numerical solutions of the Cattaneo-Hristov heat equation [31–37] in one dimensional space. Although the Cattaneo-Hristov heat equation was constructed on the half-real line, it should be enlarged to the other coordinates according to the geometry of the medium heat conduction acting. In this manner, Avci [38] investigated the solution process of the Cattaneo-Hristov heat diffusion on an axial symmetrical finite cylinder and also analyzed the thermal stresses due to the heat sources applied from the boundaries of the cylinder. Motivated by this fact, the current study aims to represent the elastic heat diffusion in the half-plane. Therefore, this work focuses on solving the two-dimensional Cattaneo-Hristov equation with the Dirichlet boundary conditions by Laplace, sine- and exponential- Fourier integral transforms. To our knowledge, this is the first study on the two-dimensional Cattaneo-Hristov diffusion equation, and therefore it is possible that the work will contribute to the technological development of thermally elastic film materials.

On the other hand, we aim to investigate the harmonic temperature effect, which is a remarkable concept in the classical or fractional diffusion processes, for the Cattaneo-Hristov diffusion model. The behavior of the classical diffusion under a harmonic effect was first investigated by Ångström [39]. This physical phenomenon is referred to as "'oscillatory diffusion"' or "'diffusion waves"' in the literature. The harmonic effect on diffusion can be analyzed in two ways. In the first one, a harmonic source function is stipulated [40, 41]. On the other hand, it is considered that there is a harmonic effect at the boundary [42]. In [43], all possible harmonic effects are analyzed for a one-dimensional diffusion problem. The harmonic effect on the fractional diffusion models has been studied in the recent few years [44–47]. It should be noted that these fractional diffusion equations were described by Caputo derivative with the singular kernel. As far as is known, the current study is the first to examine the Cattaneo-Hristov diffusion process modeled with the Caputo-Fabrizio derivative under a harmonic boundary effect.

The paper is organized as follows: In Section 2, we give some preliminary definitions required for the formulation of problems. In Section 3, we obtain the fundamental solutions to the Dirichlet problem for the Dirac pulse and non-moving harmonic pulse, then evaluate the behavior of temperatures according to the change of order of the fractional derivative by the graphics. Moreover, we discuss the results from both the mathematical and physical perspectives in this section. Finally, we provide the concluding remarks in Section 4.

2 Preliminaries

The birth of fractional analysis occurred per se in the solution of Abel's tautochrone problem. In fact, Abel was unaware that he had found a new theory today known as the Riemann-Liouville fractional calculus [48]. This clearly shows us that fractional operators actually arise naturally when trying to understand physical phenomena. Fractional operators are particularly effective tools for understanding memory effects, clarifying hereditary properties, and modelling transport processes in complex environments. What is important is the accurate use and interpretation of fractional operators that differ depending on their kernel functions. As is known, the leading Riemann-Liouville and Caputo operators of conventional fractional calculus include singular kernels denoting long-tail memory. On the other hand, computational difficulties arising from the nature of these derivatives and their weakness in model processes complying with the exponential decay law have led to the emergence of the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives with regular kernels.

Now, we remind the Caputo-Fabrizio fractional derivative, which also models the Cattaneo-Hristov heat diffusion discussed in the present study.

Definition 1 *[49] Let $f \in H^1(0,t)$ and $0 < \alpha < 1$, then the Caputo-Fabrizio fractional derivative is* *defined by*

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t \frac{df(s)}{ds} \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds, \tag{6} $$

where $\mathcal{N}(\alpha)$ denotes the normalization function satisfying $\mathcal{N}(0) = \mathcal{N}(1) = 1$.

The closed-form solution to the problem will be obtained using integral transforms. Since the Laplace transform is applied for the time variable, we indicate the Laplace transform property of the Caputo-Fabrizio derivative is as follows:

$$ \mathcal{L}\left\{ {}^{CF}D_t^{\alpha} f(t) \right\}(s) = \frac{sf^*(s) - f(0)}{s + \alpha(1-s)}, \; 0 < \alpha \leq 1, \tag{7} $$

in which asterisk denotes the Laplace transform of the function. For the Dirichlet problems considered in the half plane, we apply the exponential Fourier transform via y variable [50]:

$$ \mathcal{F}\{f(y)\} = \overline{f}(\eta) = \int_{-\infty}^{\infty} f(y) e^{iy\eta} dy, \; -\infty < y < \infty, \tag{8} $$

with its inverse transform:

$$ \mathcal{F}^{-1}\left\{ \overline{f}(\eta) \right\} = f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(\eta) e^{-iy\eta} d\eta. \tag{9} $$

Also, the exponential-Fourier transform of the second-order derivative is reminded as

$$ \mathcal{F}\left\{ \frac{d^2 f(y)}{dy^2} \right\} = -\eta^2 \overline{f}(\eta). \tag{10} $$

Then, we use the following sine-Fourier transform for the Dirichlet problem [50]:

$$ \mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \int_0^{\infty} f(x) \sin(x\xi)\, dx, \; 0 \leq x < \infty, \tag{11} $$

with the relevant inverse transform:

$$ \mathcal{F}^{-1}\left\{ \widetilde{f}(\xi) \right\} = f(x) = \frac{2}{\pi} \int_0^{\infty} \widetilde{f}(\xi) \sin(x\xi)\, d\xi, \tag{12} $$

Since the sine-Fourier transform is used in the domain $0 \leq x < \infty$ for a prescribed Dirichlet boundary condition, we apply the following property

$$ \mathcal{F}\left\{ \frac{d^2 f(x)}{dx^2} \right\} = -\xi^2 \widetilde{f}(\xi) + \xi f(0). \tag{13} $$

3 Statement of the problem

In this section, we aim to obtain the closed-form solutions to the Cattaneo-Hristov heat conduction problem in the half-plane. For this purpose, let us consider the initial boundary value problem defined as follows:

$$\begin{aligned} \frac{\partial T(x,y,t)}{\partial t} &= a_1 \Delta T(x,y,t) + a_2(1-\alpha)\ {}^{CF}D_t^{\alpha} \Delta T(x,y,t), \quad (14)\\ & \quad 0<x<\infty, -\infty<y<\infty,\ 0<t<\infty,\\ t &= 0:\ T(x,y,0)=0, \quad (15)\\ x &= 0:\ T(0,y,t)=f(y,t), \quad (16)\end{aligned}$$

where $\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$ is the Laplacian of temperature function. Firstly, applying the exponential-Fourier transform with respect to the spatial coordinate y and considering the property Eq. (10) lead to

$$\begin{aligned} \frac{\partial \overline{T}(x,\eta,t)}{\partial t} &= a_1\left(\frac{\partial^2 \overline{T}(x,\eta,t)}{\partial x^2} - \eta^2 \overline{T}(x,\eta,t)\right)\\ &\quad + a_2(1-\alpha)^{CF} D_t^{\alpha}\left(\frac{\partial^2 \overline{T}(x,\eta,t)}{\partial x^2} - \eta^2 \overline{T}(x,\eta,t)\right). \quad (17)\end{aligned}$$

with the transformed initial and boundary conditions:

$$\begin{aligned} t=0:&\quad \overline{T}(x,\eta,0)=0,\\ x=0:&\quad \overline{T}(0,\eta,t)=\overline{f}(\eta,t). \end{aligned}$$

Then, the sine-Fourier transform is applied according to the spatial coordinate x under the relation Eq. (13) and the result is obtained as follows:

$$\begin{aligned} \frac{\partial \tilde{\overline{T}}(\xi,\eta,t)}{\partial t} &= a_1\left(-\left(\xi^2+\eta^2\right)\tilde{\overline{T}}(\xi,\eta,t) + \xi\overline{f}(\eta,t)\right)\\ &\quad + a_2(1-\alpha)^{CF} D_t^{\alpha}\left(-\left(\xi^2+\eta^2\right)\tilde{\overline{T}}(\xi,\eta,t) + \xi\overline{f}(\eta,t)\right). \quad (18)\end{aligned}$$

Finally, applying the Laplace transform to the time variable t gives the transformed solution:

$$\tilde{\overline{T}}^{*}(\xi,\eta,s) = \xi\frac{(a\beta s + a_1\alpha)\overline{f}^{*}(\eta,s) - a_2\beta\overline{f}(\eta,0)}{\beta s^2 + \left[a\beta\left(\xi^2+\eta^2\right)+\alpha\right]s + a_1\alpha\left(\xi^2+\eta^2\right)}, \quad (19)$$

where α is the order of the Caputo-Fabrizio derivative, a_1 and a_2 are some real constants such that

$$a_1 = \frac{k_1}{C_p\rho},\ a_2 = \frac{k_2}{C_p\rho}, \quad (20)$$

for effective thermal conductivity k_1 and elastic conductivity k_2. Also, C_p is the specific heat and ρ is the density of particles on the plate.

$$\beta = 1-\alpha,\ a = a_1 + a_2. \quad (21)$$

Inversion of the transformations gives the closed-form solution

$$T(x,y,t) = \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_{-\infty}^{\infty} \int_{0}^{\infty} \tilde{\tilde{T}}(\xi,\eta,t) e^{-iy\eta} \sin(x\xi)\, d\xi d\eta, \tag{22}$$

which can be arranged using Euler's formula as

$$T(x,y,t) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_{0}^{\infty} \int_{0}^{\infty} \tilde{\tilde{T}}(\xi,\eta,t) \cos(y\eta) \sin(x\xi)\, d\xi d\eta. \tag{23}$$

To simulate the two-dimensional Cattaneo-Hristov diffusion equation, we consider two particular cases. In the 1^{st} case, our the aim is to extend the original Cattaneo-Hristov heat diffusion problem considered for $x \in (0,\infty)$ to the half-real plane $(x,y) \in (0,\infty) \times (-\infty,\infty)$. In the 2^{nd} case, we intend to examine the effect of the harmonic temperature function at the boundary on the Cattaneo-Hristov model, which has also an important effect on both classical and fractional heat conduction problems.

Case 1: Fundamental solution to two-dimensional Cattaneo-Hristov heat diffusion

Here, we consider the Dirac delta pulse at the boundary given by Eq. (16) for Cattaneo-Hristov heat diffusion equation:

$$T(0,y,t) = f(y,t) = \delta(y). \tag{24}$$

Substituting the exponential Fourier and Laplace transforms of this condition in Eq. (19) gives

$$\tilde{\tilde{T}}^{*}(\xi,\eta,s) = \xi \frac{a_1(\beta s + \alpha)}{\beta s^3 + \left[a\beta\left(\xi^2+\eta^2\right)+\alpha\right]s^2 + a_1\alpha\left(\xi^2+\eta^2\right)s}. \tag{25}$$

Next, inverting the Laplace transform reveals

$$\begin{aligned} \tilde{\tilde{T}}(\xi,\eta,t) \;=\; & \frac{\xi}{\xi^2+\eta^2}\left\{\frac{1}{2}\left(\frac{C(\xi,\eta)}{B(\xi,\eta)}-1\right)\exp\left(\frac{B(\xi,\eta)-A(\xi,\eta)}{2\beta}t\right)+1\right. \\ & \left. -\frac{1}{2}\left(\frac{C(\xi,\eta)}{B(\xi,\eta)}+1\right)\exp\left(\frac{-B(\xi,\eta)-A(\xi,\eta)}{2\beta}t\right)\right\}, \end{aligned} \tag{26}$$

where the notations defined in the following are used only for convenience

$$A(\xi,\eta) = a\beta\left(\xi^2+\eta^2\right)+\alpha, \tag{27}$$

$$B(\xi,\eta) = \sqrt{a\beta^2\left(\xi^2+\eta^2\right)^2 + 2\left(a_2-a_1\right)a\beta\left(\xi^2+\eta^2\right)+\alpha^2}, \tag{28}$$

$$C(\xi,\eta) = \left(a_2-a_1\right)\beta\left(\xi^2+\eta^2\right)-\alpha. \tag{29}$$

To make the solution suitable for numerical calculations, we need to reduce the double integral in Eq. (23) to a single integral by converting it to polar coordinates. For this purpose, we first suppose that

$$\xi = \rho\cos\theta, \eta = \rho\sin\theta, \tag{30}$$

and so we obtain

$$T(x,y,t) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^\infty \int_0^{\frac{\pi}{2}} \tilde{\tilde{\overline{T}}}_1(\rho,t)\cos(y\rho\sin\theta)\sin(x\rho\cos\theta)\,\rho^2\cos\theta d\theta d\rho, \tag{31}$$

where $\tilde{\tilde{\overline{T}}}_1 = \tilde{\tilde{\overline{T}}}/\xi$ and since $\rho^2 = \xi^2 + \eta^2$ from Eq. (30), $\tilde{\tilde{\overline{T}}}_1$ can be written as the function of (ρ, t) according to Eqs. (26)-(29). By using the change of the variable $v = \sin\theta$ and considering the following integral relation [51, 52]:

$$\int_0^1 \cos(y\rho v)\sin\left(x\rho\sqrt{1-v^2}\right) dv = \frac{\pi}{2}\frac{x}{\sqrt{x^2+y^2}} J_1\left(\sqrt{x^2+y^2}\right), \tag{32}$$

in which J_1 is the first kind Bessel function of order 1. Thereby, the closed-form solution is arrived at as

$$T(x,y,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{\tilde{\overline{T}}}_1(\rho,t)\frac{x}{\sqrt{x^2+y^2}} J_1\left(\sqrt{x^2+y^2}\right)\rho^2 d\rho. \tag{33}$$

and solved by numerical computation of the improper integral. Then, the results are depicted in Figure 1.

In Figure 1(a), we aim to illustrate the dependence of heat diffusion on the variation of order of fractional derivatives. The 2D graphics show the cross-section of the temperature surface for the arbitrary values of $x = y = 0.5$. Note that the α parameter plays two critique roles in the discussed model, one as a coefficient and the other to determine the influence of fading memory. As α approaches 1, the damping memory effect weakens due to the coefficient role of α, and the temperature function tends to behave as in the classical heat equation. In the case of $\alpha = 1$, the elastic conductivity constant k_1 in the coefficient a_1 also loses its effect.

In Figure 1(b), the behavior of the temperature surface is shown for the arbitrary values of $\alpha = 0.6$ and $t = 0.5$. In this graph, the instantaneous Dirac heat pulse at the boundary of the region is clearly visible. For evolution equations such as heat conduction, examining the effects of instantaneous changes at the beginning or at the boundary is important both in obtaining fundamental solutions and in the sense of physical behavior. Due to this importance, the Dirac delta pulse effect is examined in different classical or fractional heat conduction models, as in the current study.

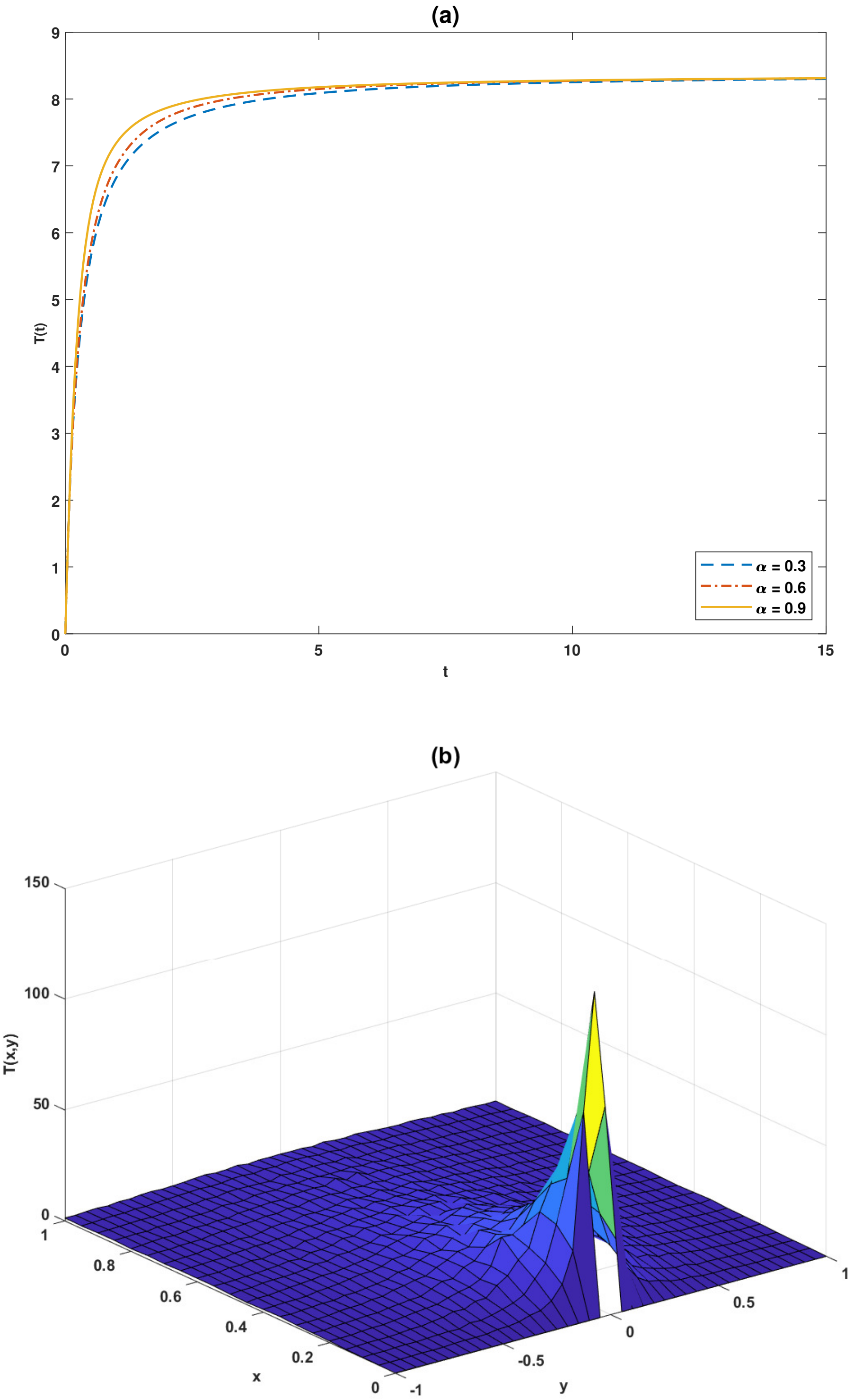

Figure 1. Temperature function for Dirac delta pulse at the boundary

Case 2: Non-moving harmonic temperature at the boundary

In this case, the behavior of the Cattaneo-Hristov heat diffusion is investigated under the effect of a time non-moving harmonic boundary temperature which is described by

$$T\left(0, y, t\right) = f\left(y, t\right) = \delta\left(y\right)\exp\left(i\omega t\right), \tag{34}$$

where ω denotes the angular frequency. Substituting integral transforms of the condition Eq. (34) into the transformed solution Eq. (19) and rearranging the results give

$$\begin{aligned}\tilde{\tilde{T}}^{*}(\xi,\eta,s) &= \xi\left\{\left[a_1\beta s^2+a_1\alpha s-a_2\beta\omega^2\right]+i\left[a\beta\omega s+a_1\alpha\omega\right]\right\}\\ &\quad /\left\{\beta s^4+\left[a\beta\left(\xi^2+\eta^2\right)+\alpha\right]s^3+\left[\beta\omega^2+a_1\alpha\left(\xi^2+\eta^2\right)\right]s^2\right.\\ &\quad +\left.\left[a\beta\left(\xi^2+\eta^2\right)+\alpha\right]\omega^2 s+a_1\alpha\left(\xi^2+\eta^2\right)\omega^2\right\}. \end{aligned} \tag{35}$$

For demonstration purposes, we focus on the real part of the transformed temperature function for the subsequent calculations. Inverting the Laplace transform of the real part of Eq. (35), one can obtain

$$\begin{aligned}\tilde{\tilde{T}}(\xi,\eta,t) &= \frac{\xi}{D(\xi,\eta)}\left\{\frac{E(\xi,\eta)}{2B(\xi,\eta)}\left[\exp\left(\frac{B(\xi,\eta)-A(\xi,\eta)}{2\beta}t\right)-\exp\left(\frac{-B(\xi,\eta)-A(\xi,\eta)}{2\beta}t\right)\right]\right.\\ &\quad + F(\xi,\eta)\sin\omega t + G(\xi,\eta)\cos\omega t\\ &\quad -\left.\frac{G(\xi,\eta)}{2}\left[\exp\left(\frac{B(\xi,\eta)-A(\xi,\eta)}{2\beta}t\right)-\exp\left(\frac{-B(\xi,\eta)-A(\xi,\eta)}{2\beta}t\right)\right]\right\}, \end{aligned} \tag{36}$$

where the notations $A(\xi,\eta)$ and $B(\xi,\eta)$ are given by Eqs. (27) – (28) and the other abbreviations are as follows:

$$\begin{aligned} D(\xi,\eta) &= \left(a^2\beta^2\omega^2+a_1^2\alpha^2\right)\left(\xi^2+\eta^2\right)^2+2a_2\alpha\beta\omega^2\left(\xi^2+\eta^2\right)+\beta^2\omega^4+\alpha^2\omega^2,\\ E(\xi,\eta) &= \left[a^2(a_1-a_2)\beta^3\omega^2-a_1^2a_2\alpha^2\beta\right]\left(\xi^2+\eta^2\right)^2\\ &\quad+\left[a(a_1-2a_2)\alpha\beta^2\omega^2-2a_1(a_1-a_2)\alpha\beta^2\omega^2-a_1^2\alpha^3\right]\left(\xi^2+\eta^2\right)-2a_2\beta^3\omega^4-a_2\alpha^2\beta\omega^2,\\ F(\xi,\eta) &= a\beta^2\omega^3+a_1\alpha^2\omega,\\ G(\xi,\eta) &= \left(a^2\beta^2\omega^2+a_1^2\alpha^2\right)\left(\xi^2+\eta^2\right)^2+a_2\alpha\beta\omega^2. \end{aligned}$$

Substituting the function $\tilde{\tilde{T}}(\xi,\eta,t)$ into Eq. (23) and using the same calculations in Eqs. (30)-(32) by indicating $\tilde{\tilde{T}}_2(\xi,\eta,t)=\tilde{\tilde{T}}(\xi,\eta,t)/\xi$ led to the closed-form solution as

$$T(x,y,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty \tilde{\tilde{T}}_2(\rho,t)\frac{x}{\sqrt{x^2+y^2}}J_1\left(\sqrt{x^2+y^2}\right)\rho^2 d\rho, \tag{37}$$

which is also depicted by calculating the improper integral numerically.

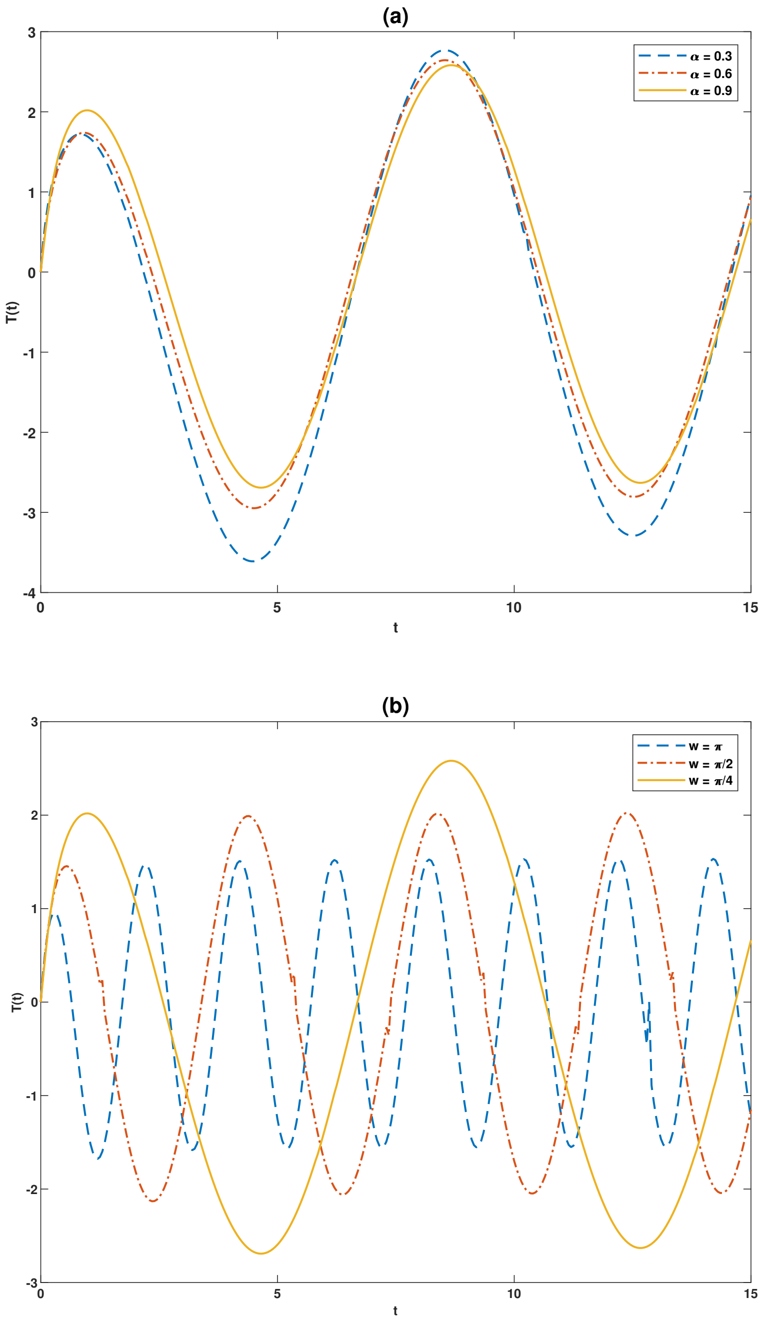

Figure 2. Time dependent temperature functions for non-moving harmonic boundary at $x = y = 0.5$ via variations of α and w, respectively.

Similar to Figure 1(a), Figure 2(a) shows also the dependence of temperature on the variation of α in the case of a non-moving harmonic temperature source at the boundary. Figure 2(b) shows the temperature response due to the change of angular frequency acting in the harmonic boundary temperature. As the angular frequency decreases, the wavelength of the temperature increases. It can be seen in both figures that temperature exhibits wave behavior similar to the boundary condition. This result clearly indicates Cattaneo's theory that wave phenomena may also occur in heat diffusion.

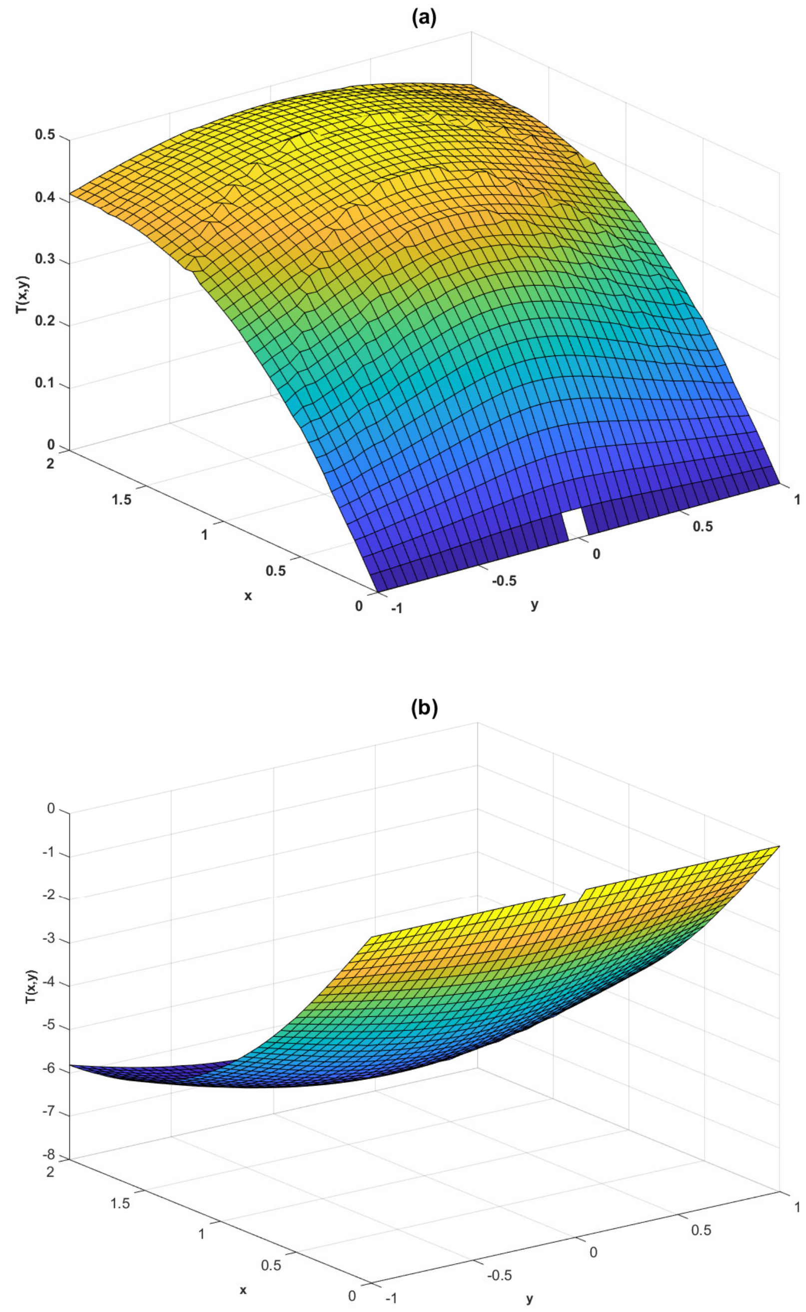

Figure 3. Temperature surfaces with non-moving harmonic boundary for $\alpha = 0.6$ at $t = 0.5$ and $t = 1.5$, respectively.

The time cross-section of the elastic heat diffusion at arbitrarily chosen times $t = 0.5$ and $t = 1.5$ is demonstrated in Figure 3. The wavelike temperature behaviour can be clearly seen for $\alpha = 0.6$ in both figures. This case can be similarly observed from the other α values depicted in Figure 2(a).

4 Conclusion

From the engineering point of view, it is important to know the mechanical and thermal behaviors of the materials under a heat force. These properties can be analyzed experimentally or with mathematical tools. In terms of mathematical analysis, it is crucial to exact modeling of the heat diffusion of the material. Although the Cattaneo-Hristov equation that models heat diffusion with fading memory was constructed on the real line that physically corresponds to a wire, it is also significant to know the heat diffusion of a plate or a film with a fading memory effect. This situation can be generalized according to the geometry of materials, such as cylinders, spheres, cubes etc., which vary via the application area of the engineering problems. Therefore, this paper concerns the Cattaneo-Hristov diffusion equation in the half-plane. Two types of boundary conditions have been considered for the Dirichlet problem which are Dirac delta and non-moving harmonic temperatures, respectively. The closed-form solutions are arrived at by applying Fourier and Laplace integral transforms. The temperature functions have been illustrated under the variations of the model parameters using MATLAB software. These analyses performed for the two boundary temperatures can also be considered for different boundary conditions and different coordinate systems in future works.

Declarations

List of abbreviations

Not applicable.

Ethical approval

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable.

Conflicts of interest

The author confirms that there is no competing interest in this study.

Data availability statement

Data availability is not applicable to this article as no new data were created or analysed in this study.

Funding

Not applicable.

Author's contributions

The author has made substantial contributions to the conception, design of the work, the acquisition, analysis, interpretation of data, and the creation of new software used in the work. Author has read and agreed to the published version of the manuscript.

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