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Research Article

Exponential approximation in variable exponent Lebesgue spaces on the real line

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ABSTRACT. Present work contains a method to obtain Jackson and Stechkin type inequalities of approximation by integral functions of finite degree (IFFD) in some variable exponent Lebesgue space of real functions defined on $\boldsymbol{R} := (-\infty, +\infty)$. To do this, we employ a transference theorem which produce norm inequalities starting from norm inequalities in $\mathcal{C}(\boldsymbol{R})$, the class of bounded uniformly continuous functions defined on $\boldsymbol{R}$. Let $B \subseteq \boldsymbol{R}$ be a measurable set, $p(x) : B \to [1, \infty)$ be a measurable function. For the class of functions f belonging to variable exponent Lebesgue spaces $L_{p(x)}(B)$, we consider difference operator $(I - T_\delta)^r f(\cdot)$ under the condition that $p(x)$ satisfies the log-Hölder continuity condition and $1 \leq \operatorname{ess\,inf}_{x\in B} p(x)$, $\operatorname{ess\,sup}_{x\in B} p(x) < \infty$, where I is the identity operator, $r \in \mathrm{N} := \{1, 2, 3, \cdots\}$, $\delta \geq 0$ and

$$(*) \qquad T_\delta f(x) = \frac{1}{\delta} \int_0^\delta f(x+t)\, dt, \quad x \in \boldsymbol{R}, \quad T_0 \equiv I,$$

is the forward Steklov operator. It is proved that

$$(**) \qquad \|(I - T_\delta)^r f\|_{p(\cdot)}$$

is a suitable measure of smoothness for functions in $L_{p(x)}(B)$, where $\|\cdot\|_{p(\cdot)}$ is Luxemburg norm in $L_{p(x)}(B)$. We obtain main properties of difference operator $\|(I - T_\delta)^r f\|_{p(\cdot)}$ in $L_{p(x)}(B)$. We give proof of direct and inverse theorems of approximation by IFFD in $L_{p(x)}(\boldsymbol{R})$.

Keywords: Variable exponent Lebesgue space, one sided Steklov operator, integral functions of finite degree, best approximation, direct theorem, inverse theorem, modulus of smoothness, Marchaud inequality, K-functional.

2020 Mathematics Subject Classification: 41A10, 41A25, 41A27, 41A65.

1. INTRODUCTION

Some inequalities of Approximation Theory in a Homogenous Banach Spaces (HBS) can be obtained their uniform-norm counterparts. This information is known for a long time, (see e.g., [20] for definition of HBS). This elegant method was generalized to some variable exponent Lebesgue spaces functions defined on $\boldsymbol{R}$ (see Theorem 1 of [9]). Generally, these scale of function classes are non-translation invariant with respect to the ordinary translation $x \to f(x+a)$. Here, we give several uniform-norm inequalities on $C(\boldsymbol{R})$ and apply them to obtain several inequalities of approximation by IFFD in some variable exponent Lebesgue spaces $L_{p(x)}(\boldsymbol{R})$. Under some condition on $p(x)$ of $L_{p(x)}(\boldsymbol{R})$, we obtain main inequalities of exponential approximation by IFFD such as Jackson-Stechkin-Timan type estimates and equivalence of K-functional with suitable modulus of smoothness $(**)$ given in abstract for functions of $L_{p(x)}(\boldsymbol{R})$. Note that many results of approximation by IFFD can be obtained easily their uniform-norm counterparts in $C(\boldsymbol{R})$.

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Consider an entire function $f(z)$ and put $M(r) = \max_{|z|=r} |f(z)|$ for $z = x + iy$. We say that an entire function f is of exponential type σ if $\limsup_{r\to\infty} r^{-1} \ln M(r) \leq \sigma, \quad \sigma < \infty$.

The approximation by entire function of finite degree in the real line was originated in the beginning of twentieth century by Serge Bernstein [15] and became a separate branch of analysis due to the efforts of many mathematicians such as N. Wiener and R. Paley [45], N.I. Achiezer [4], S.M. Nikolskii [42], I.I. Ibragimov [29], A.F. Timan [52], M.F. Timan [53], R. Taberski [54, 55], F.G. Nasibov [41], V. Yu. Popov [46], A.A. Ligun [43], and others.

Studying function spaces with variable exponent is now an extensively developed field after their applications in elasticity theory [58], fluid mechanics [47, 48], differential operators [19, 48], nonlinear Dirichlet boundary value problems [40], nonstandard growth [58], and variational calculus. See the books [16, 18, 51] for more references. Nowadays, many mathematician solved many problems for the approximation of function in these type spaces defined on $[0, 2\pi] \subset \boldsymbol{R}$ (see e.g., [7, 8, 26, 30, 31, 34], [1, 2, 3, 11, 12], [5, 6, 9, 13, 14],[22, 24, 25, 28, 32, 33, 36],[37, 38, 44, 49, 50, 56]). In this paper, we propose generalized our last results in [10] which we obtained a direct and inverse theorems for approximation by entire functions of finite degree in variable exponent Lebesgue spaces on the whole real axis $\boldsymbol{R}$ with

$$\sup_{0<h\leq\delta} \|(I - T_h)f\|_{p(\cdot)} \tag{1.1}$$

as modulus of continuity $\Omega_1(f,\delta)_{p(\cdot)}$. Instead of (1.1), here we will use

$$\|(I - T_\delta)^r f\|_{p(\cdot)} \tag{1.2}$$

as modulus smoothness $\Omega_r(f,\delta)_{p(\cdot)}$ and we obtain stronger Jackson inequality than obtained in [10].

Let $B \subseteq \boldsymbol{R}$ be a measurable set and $p(x) : B \to [1,\infty)$ be a measurable function. We define $\tilde{P}(B)$ as the class of measurable functions $p(x)$ satisfying the conditions

$$1 \leq p_B^- := \operatorname{ess\,inf}_{x\in B} p(x), \quad p_B^+ := \operatorname{ess\,sup}_{x\in B} p(x) < \infty. \tag{1.3}$$

We also set $p^- := p_{\boldsymbol{R}}^-$ and $p^+ := p_{\boldsymbol{R}}^+$. We define the $L_{p(\cdot)}(B)$ as the set of all functions $f : B \to \boldsymbol{R}$ such that

$$I_{p(\cdot),B}\left(\frac{f}{\lambda}\right) := \int_B \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy < \infty \tag{1.4}$$

for some $\lambda > 0$. We set $I_{p(\cdot)}(f) := I_{p(\cdot),\boldsymbol{R}}(f)$. The set of functions $L_{p(\cdot)}(B)$, with norm

$$\|f\|_{p(\cdot),B} := \inf\left\{\eta > 0 : I_{p(\cdot),B}\left(\frac{f}{\eta}\right) < 1\right\}$$

is Banach space. We set $L_{p(\cdot)} := L_{p(\cdot)}(\boldsymbol{R})$.

For $i \in \mathrm{N}$, all constants $c_i(x, y, \cdots)$ will be some positive number such that they depend on the parameters $x, y, \cdots$ given in the brackets. Also, constants $c_i(x, y, \cdots)$ can be change only when the parameters $x, y, \cdots$ change. Absolute constants $\mathbf{c}_1, \mathbf{c}_2, \ldots$ will not change in each occurrence.

Definition 1.1. *For a measurable set $B \subseteq \boldsymbol{R}$, a measurable function $p(\cdot) : B \to \boldsymbol{R}$ is said to locally log-Hölder continuous on B if there is a positive constant $c_1(p)$ such that*

$$|p(x) - p(y)| \log(e + 1/|x - y|) \leq c_1(p) < \infty \tag{1.5}$$

for any $x, y \in B$. We say that p satisfies log-Hölder decay condition if there is a constant $c_2(p) > 0$ and $p_\infty > 1$ such that

$$|p(x) - p_\infty| \log(e + |x|) \leq c_2(p) < \infty \tag{1.6}$$

for any $x \in B$.

Define the class $P^{Log}(B) := \left\{ p \in \tilde{P}(B) : \frac{1}{p} \text{ is satisfy (1.5)-(1.6)} \right\}$. *We set* $c_3(p) := \max\{c_1(p), c_2(p)\}$.

Definition 1.2. *([27, p.96]) Let* $\mathrm{N} := \{1, 2, 3, \cdots\}$ *be natural numbers and* $\mathrm{N}_0 := \mathrm{N} \cup \{0\}$.

(a) *A family* Q *of measurable sets* $E \subset \boldsymbol{R}$ *is called locally* N*-finite* ($N \in \mathrm{N}$) *if*

$$\sum_{E \in Q} \chi_E(x) \leq N$$

almost everywhere in $\boldsymbol{R}$*, where* χ_U *is the characteristic function of the set* U.

(b) *A family* Q *of open bounded sets* $U \subset \boldsymbol{R}$ *is locally* 1*-finite if and only if the sets* $U \in Q$ *are pairwise disjoint.*

(c) *Let* $U \subset \boldsymbol{R}$ *be a measurable set and*

$$A_U f := \frac{1}{|U|} \int_U |f(t)|\, dt.$$

(d) *For a family* Q *of open sets* $U \subset \boldsymbol{R}$*, we define averaging operator by*

$$T_Q : L^1_{loc} \to L^0,$$

$$T_Q f(x) := \sum_{U \in Q} \chi_U(x) A_U f = \sum_{U \in Q} \frac{\chi_U(x)}{|U|} \int_U |f(y)|\, dy, \quad x \in \boldsymbol{R},$$

where L^0 *is the set of measurable functions on* $\boldsymbol{R}$.

For a measurable set $A \subset \boldsymbol{R}$, symbol $|A|$ will represent the Lebesgue measure of A. We consider Transference result.

Definition 1.3. *For* $0 < \delta < \infty$, $\tau \in \boldsymbol{R}$*, we define family of Steklov operators*

$$\mathsf{S}_\delta f(x) := \frac{1}{\delta} \int_{x-\delta/2}^{x+\delta/2} f(t)\, dt = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} f(x+t)\, dt, \quad x \in \boldsymbol{R}, \tag{1.7}$$

where f *is a locally integrable function defined on* $\boldsymbol{R}$.

The following result was obtained by Drihem for every cubes or balls in $\boldsymbol{R}^d$. We write below its restricted version with constants. The proof of this is the same with Theorem 2 of [23].

Proposition 1.1. *([23]) Suppose that* $p \in P^{Log}(\boldsymbol{R})$ *and* Q *is a bounded interval of* $\boldsymbol{R}$ *having Lebesgue measure* ≥ 1. *For every* $m > 0$*, there is* $c_4(m, c_3(p)) := \exp(-4mc_3(p)) \in (0,1)$ *such that*

$$\left(\frac{c_4(m, c_3(p))}{|Q|} \int_Q |f(y+\tau)|\, dy \right)^{p(x)} \leq \frac{3^{p^+}}{|Q|} \int_Q |f(y+\tau)|^{p(y+\tau)}\, dy + \frac{3^{p^+-1}}{(e+|x|)^m} + 3^{p^+-1} \int_Q \frac{dy}{(e+|y+\tau|)^m}$$

holds for all $x \in Q$, $\tau \in \boldsymbol{R}$ *and all* $f \in L_{p(\cdot)} + L_\infty(\boldsymbol{R})$ *with* $\|f\|_{p(\cdot)} + \|f\|_\infty \leq 1$.

Theorem 1.1. *Suppose that* $p \in P^{Log}(\boldsymbol{R})$. *Then, the family of operators* $\{\mathcal{U}_\tau f\}_{\tau \in \boldsymbol{R}}$ *defined by*

$$\mathcal{U}_\tau f(x) := \mathsf{S}_1 f(x+\tau) = \int_{-1/2}^{+1/2} f(x+\tau+t)\, dt, \quad x \in \boldsymbol{R}, \quad \tau \in \boldsymbol{R}$$

is uniformly bounded (in τ) in $L_{p(\cdot)}$, namely,

$$\|\mathcal{U}_\tau f\|_{p(\cdot)} \le c_5\left(p^+, c_3(p)\right)\|f\|_{p(\cdot)}$$

holds with $c_5\left(p^+, c_3(p)\right) := 2^{p^+ +1}3^{p^+}\left(1 + 2\cdot 3^{p^+}\left[\sum_{k=2}^{\infty} 2^{-k} + 2\right]\right)\exp\left(8c_3(p)\right)$.

Proof of Theorem 1.1. Let us consider $f \in L_{p(\cdot)}$ with $\|f\|_{p(\cdot)} \le 1/2$. Suppose that

$$Q := \{U \subset \boldsymbol{R} : U \text{ open interval and } |U| = 1\}$$

be a locally 1-finite family of partition of $\boldsymbol{R}$. Choose $m = 2 > 1$ (constant $c_6\left(p^+\right)$ below becomes a finite number)

$$c_6\left(p^+\right) = 2^{p^+}3^{p^+}\left(1 + 2\cdot 3^{p^+}\left[\sum_{k=2}^{\infty} 2^{-k} + 2\right]\right) < \infty.$$

We can select $c_4\left(2, c_3(p)\right) = \exp\left(-8c_3(p)\right) \in (0,1)$ as in Proposition 1.1. Then, using Corollary 2.2.2 of [27, p.20] we obtain

$$\begin{aligned}
\rho_{p(\cdot)}\left(\frac{c_4\left(2,c_3(p)\right)}{c_6\left(p^+\right)}\mathcal{U}_\tau f\right) &= \frac{1}{c_6\left(p^+\right)}\int_{\boldsymbol{R}}\left|c_4\left(2,c_3(p)\right)\int_{-1/2}^{+1/2} f(x+\tau+t)\,dt\right|^{p(x)} dx \\
&\le \frac{1}{c_6\left(p^+\right)}\sum_{U\in Q}\int_U\left|c_4\left(2,c_3(p)\right)\int_{-1/2}^{+1/2} f(x+\tau+t)\,dt\right|^{p(x)} dx \\
&\le \frac{2^{p^+}}{c_6\left(p^+\right)}\sum_{U\in Q}\int_U\left|\frac{c_4\left(2,c_3(p)\right)}{|2U|}\int_{2U}\chi_{2U}(y)\, f(y+\tau)\,dy\right|^{p(x)} dx \\
&\le \frac{2^{p^+}}{c_6\left(p^+\right)}\sum_{U\in Q}\int_U\left[\frac{3^{p^+}\chi_{2U}(y)}{|2U|}\int_{2U}|f(y+\tau)|^{p(y+\tau)}\,dy + \right. \\
&\left. + \frac{3^{p^+-1}}{(e+|x|)^2} + \frac{\chi_{2U}(y)}{|2U|}\int_{2U}\frac{3^{p^+-1}dy}{(e+|y+\tau|)^2}\right] dx \\
&\le \frac{2^{p^+-1}3^{p^+}}{c_6\left(p^+\right)}\sum_{U\in Q}\int_U\left[\chi_{2U}(y)\int_{2U+\tau}|f(s)|^{p(s)}\,ds \right. \\
&\left. + \frac{3^{p^+-1}2}{(e+|x|)^2} + \int_{2U+\tau}\frac{3^{p^+-1}ds}{(e+|s|)^2}\right] dx \\
&\le \frac{2^{p^+-1}3^{p^+}}{c_6\left(p^+\right)}\left(\sum_{U\in Q}\chi_{2U}\right)\left(1 + 3^{p^+}\int_{\boldsymbol{R}}\frac{ds}{(e+|s|)^2}\right) \\
&= \frac{2^{p^+}3^{p^+}}{c_6\left(p^+\right)}\left(1 + 3^{p^+}\int_{\boldsymbol{R}}\frac{ds}{(e+|s|)^2}\right) \\
&\le \frac{2^{p^+}3^{p^+}}{c_6\left(p^+\right)}\left(1 + 2\cdot 3^{p^+}\left[\sum_{k=2}^{\infty}\frac{1}{2^k} + 2\right]\right) = 1
\end{aligned}$$

and hence

$$\|\mathcal{U}_\tau f\|_{p(\cdot)} \leq 2^{-1} c_5 \left(p^+, c_3(p)\right).$$

General case $f \in L_{p(\cdot)}$ can be obtained easily by re-scaling:

$$\|\mathcal{U}_\tau f\|_{p(\cdot)} \leq c_5 \left(p^+, c_3(p)\right) \|f\|_{p(\cdot)}.$$

□

Theorem 1.2. ([18, Theorem 4.4.8]) *Suppose that $p \in P^{Log}(\boldsymbol{R})$ and $f \in L_{p(\cdot)}$. If Q is locally 1-finite family of open bounded subintervals of $\boldsymbol{R}$ having Lebesgue measure 1, then the averaging operator T_Q is uniformly bounded in $L_{p(\cdot)}$, namely,*

$$\|T_Q f\|_{p(\cdot)} \leq c_7 (c_3(p)) \|f\|_{p(\cdot)}$$

holds with $c_7(c_3(p)) := 2\exp(8c_3(p))$.

Let $C(\boldsymbol{R})$ be the class of continuous functions defined on $\boldsymbol{R}$. For $r \in \mathrm{N}$, we define C^r consisting of every member $f \in C(\boldsymbol{R})$ such that the derivative $f^{(k)}$ exists and is continuous on $\boldsymbol{R}$ for $k = 1, ..., r$. We set $C^\infty := \{f \in C^r$ for any $r \in \mathrm{N}\}$. We denote by $C_c(\boldsymbol{R})$, the collection of real valued continuous functions on $\boldsymbol{R}$ and support of f is compact set in $\boldsymbol{R}$. We define $C_c^r := C^r \cap C_c(\boldsymbol{R})$ for $r \in \mathrm{N}$ and $C_c^\infty := C^\infty \cap C_c(\boldsymbol{R})$. Let $L_p(\boldsymbol{R})$, $1 \leq p \leq \infty$ be the classical Lebesgue space of functions on $\boldsymbol{R}$.

Theorem 1.3. [18, Corollary 4.6.6] *Let $p \in P^{Log}(\boldsymbol{R})$ and $f \in L_{p(\cdot)}$. Then*

$$\frac{\|f\|_{p(\cdot)}}{12c_7(c_3(p))} \leq \sup_{g \in L_{p'(\cdot)} \cap C_c^\infty : \|g\|_{p'(\cdot)} \leq 1} \int_{\boldsymbol{R}} |f(x) g(x)| \, dx \leq 2 \|f\|_{p(\cdot)}. \tag{1.8}$$

Definition 1.4. *Let $p \in P^{Log}(\boldsymbol{R})$. For an $f \in L_{p(\cdot)}$, we define*

$$F_f(u) := \int_{\boldsymbol{R}} (\mathsf{S}_1 f)(x+u) |G(x)| \, dx, \quad u \in \boldsymbol{R}, \tag{1.9}$$

where $G \in L_{p'(\cdot)} \cap C_c^\infty$ and $\|G\|_{p'(\cdot)} \leq 1$.

Let $W_{p(\cdot)}^r$, $r \in \mathrm{N}$, be the class of functions $f \in L_{p(\cdot)}$ such that derivatives $f^{(k)}$ exist for $k = 1, ..., r-1$, $f^{(r-1)}$ absolutely continuous and $f^{(r)} \in L_{p(\cdot)}$.

Some properties of the function $F_f(\cdot)$ is given in the following theorem.

Theorem 1.4. *Let $p \in P^{Log}(\boldsymbol{R})$, $0 < \delta < \infty$, and $f \in L_{p(\cdot)}$. Then,*

(a) the function $F_f(\cdot)$ defined in (1.9) is a bounded, uniformly continuous on $\boldsymbol{R}$,

(b) $(\mathsf{S}_\delta f)' = \mathsf{S}_\delta(f')$ on $\boldsymbol{R}$ for $f \in W_{p(\cdot)}^1$.

Main theorem of this section is as follows.

Theorem 1.5. *Let $p \in P^{Log}(\boldsymbol{R})$. If $f, g \in L_{p(\cdot)}$ and*

$$\|F_f\|_{C(\boldsymbol{R})} \leq \mathbf{c}_1 \|F_g\|_{C(\boldsymbol{R})}$$

holds with an absolute constant $\mathbf{c}_1 > 0$, then norm inequality

$$\|f\|_{p(\cdot)} \leq c_8 \left(\mathbf{c}_1, p^+, c_3(p)\right) \|g\|_{p(\cdot)} \tag{1.10}$$

also holds with $c_8(\mathbf{c}_1, p^+, c_3(p)) := 48c_7(c_3(p)) \mathbf{c}_1 c_5(p^+, c_3(p))$.

Remark 1.1. *Theorem 1.5 is a powerful tool to obtain norm inequalities in $L_{p(\cdot)}$ (and other non-translation invariant Banach spaces of functions) for $p \in P^{Log}(\boldsymbol{R})$. In this work, we will use it frequently. See for example the following result.*

As a corollaries of Theorem 1.5, we get the following two results:

Theorem 1.6. *Suppose that* $p \in P^{Log}(\boldsymbol{R})$, $0<\delta<\infty$ *and* $\tau \in \boldsymbol{R}$. *Then, the family of operators* $\{\mathcal{S}_{\delta,\tau} f\}$ *defined by*

$$\mathcal{S}_{\delta,\tau} f(x) := \mathsf{S}_{\delta} f(\cdot+\tau) = \frac{1}{\delta}\int_{x+\tau-\delta/2}^{x+\tau+\delta/2} f(s)\,ds, \quad x \in \boldsymbol{R}$$

is uniformly bounded (in δ *and* τ*) in* $L_{p(\cdot)}$, *namely,*

$$\|\mathcal{S}_{\delta,\tau} f\|_{p(\cdot)} \le 48 c_7(c_3(p))\, c_5\left(p^+, c_3(p)\right) \|f\|_{p(\cdot)}$$

holds.

Corollary 1.1. *Let* $p \in P^{Log}(\boldsymbol{R})$, $0<\delta<\infty$, *and* $f \in L_{p(\cdot)}$. *If* $\tau=\delta/2$, *then*

$$\begin{gathered}\mathcal{S}_{\delta,\delta/2} f(x) = \frac{1}{\delta}\int_0^{\delta} f(x+t)\,dt = T_{\delta} f(x), \\ \|T_{\delta} f\|_{p(\cdot)} \le 48 c_7(c_3(p))\, c_5\left(p^+, c_3(p)\right)\|f\|_{p(\cdot)}, \\ \|(I-T_{\delta})^r f\|_{p(\cdot)} \le \left(1+48 c_7(c_3(p))\, c_5\left(p^+, c_3(p)\right)\right)^r \|f\|_{p(\cdot)}. \end{gathered} \tag{1.11}$$

For the proof of these results, we will need the following Propositions.

Proposition 1.2. *(a)* $C_c(\boldsymbol{R})$ *and* C_c^{∞} *are dense subsets of* $L_p(\boldsymbol{R})$, $1 \le p < \infty$ *(Theorems 17.10 and 23.59 of* [57, p. 415 and p. 575]*).*

(b) $C_c(\boldsymbol{R})$ *contained* $L_{\infty}(\boldsymbol{R})$, *but not dense (Remark 17.11 of* [57, p.416]*) in* $L_{\infty}(\boldsymbol{R})$.

(c) If $r \in \mathrm{N}$ *and* $f \in C_c^r$, *then* $\mathsf{S}_{\delta}(f) \in C_c^r$.

Proof of Proposition 1.2. (a) and (b) are known. (c) is follows from definitions. □

Proposition 1.3. *(*[18, Theorem 2.26]*) Let* $B \subseteq \boldsymbol{R}$ *be a measurable set. If* $1 \le p(x) < p_B^+ < \infty$, $p'(x)=p(x)/(p(x)-1)$, $f \in L_{p(\cdot)}(B)$, *and* $g \in L_{p'(\cdot)}(B)$, *then Hölder's inequality*

$$\int_B f(x)g(x)dx \le 2\|f\|_{p(\cdot),B}\|g\|_{p'(\cdot),B} \tag{1.12}$$

holds.

Proof of Theorem 1.4. (a) Since $C_c(\boldsymbol{R})$ is a dense subset ([39, Theorem 4.1 (I)]) of $L_{p(\cdot)}$, we consider functions $H \in C_c(\boldsymbol{R})$ and prove that $F_H(\cdot)=\int_{\boldsymbol{R}}(S_1 H)(x+u_1)\,|G(x)|\,dx$ is bounded and uniformly continuous on $\boldsymbol{R}$, where $G \in L_{p'(\cdot)} \cap C_c^{\infty}$ and $\|G\|_{p'(\cdot)} \le 1$. Boundedness of $F_H(\cdot)$ is easy consequence of the Hölder's inequality (1.12) and Theorem 1.1. On the other hand, note that H is uniformly continuous on $\boldsymbol{R}$, see e.g. Lemma 23.42 of [57, pp.557-558]. Take $\varepsilon>0$ and $u_1, u_2, x \in \boldsymbol{R}$. Then, there exists a $\delta := \delta(\varepsilon)>0$ such that

$$|H(x+u_1)-H(x+u_2)| \le \frac{\varepsilon}{2(1+|\mathrm{supp}(G)|)}$$

for $|u_1-u_2|<\delta$. Then, for $|u_1-u_2|<\delta$, $u_1,u_2 \in \boldsymbol{R}$ we have

$$\begin{aligned}|F_H(u_1)-F_H(u_2)| &= \left|\int_{\boldsymbol{R}} S_1(H(x+u_1)-H(x+u_2))\,|G(x)|\,dx\right| \\ &\le \frac{1}{2(1+|\mathrm{supp}(G)|)}\int_{\boldsymbol{R}} |S_1(\varepsilon)|\,|G(x)|\,dx = \frac{\varepsilon}{2(1+|\mathrm{supp}(G)|)}\int_{\boldsymbol{R}} |G(x)|\,dx \\ &\le \frac{\varepsilon}{(1+|\mathrm{supp}(G)|)}(1+|\mathrm{supp}(G)|)\,\|G\|_{p'(\cdot)} \le \varepsilon.\end{aligned}$$

Now, the conclusion of Theorem 1.4 follows for the class $C_c(\boldsymbol{R})$. For the general case $f \in L_{p(\cdot)}$, there exists an $H \in C_c(\boldsymbol{R})$ so that

$$\|f - H\|_{p(\cdot)} < \xi / \left(8c_5\left(p^+, c_3(p)\right)\right)$$

for any $\xi > 0$. Then, for this ξ,

$$\begin{aligned}|F_f(u_1) - F_f(u_2)| &\leq \left|\int_{\boldsymbol{R}} \mathsf{S}_1(f-H)(x+u_1)|G(x)|\,dx\right| \\ &+ \left|\int_{\boldsymbol{R}} \mathsf{S}_1\left(H(x+u_1) - H(x+u_2)\right)|G(x)|\,dx\right| \\ &+ \left|\int_{\boldsymbol{R}} \mathsf{S}_1(H-f)(x+u_2)|G(x)|\,dx\right| \\ &\leq 2\left\|\mathsf{S}_1(f-H)(\cdot+u_1)\right\|_{p(\cdot)} + \left|\int_{\boldsymbol{R}} \mathsf{S}_1\left(H(x+u_1) - H(x+u_2)\right)|G(x)|\,dx\right| \\ &+ 2\left\|\mathsf{S}_1(f-H)(\cdot+u_2)\right\|_{p(\cdot)} \\ &\leq 4c_5\left(p^+, c_3(p)\right)\|f-H\|_{p(\cdot)} + \xi/2 \leq \xi/2 + \xi/2 = \xi.\end{aligned}$$

As a result F_f is bounded, uniformly continuous function defined on $\boldsymbol{R}$.

(b) can be obtained easily from definition. □

Proof of Theorem 1.5. Let $f \in L_{p(\cdot)}$ be non-negative. If $\|f\|_{p(\cdot)} = 0$, then the result (1.10) is obvious. So we assume that $\infty > \|f\|_{p(\cdot)} > 0$. In this case

$$\begin{aligned}\|F_f\|_{C(\boldsymbol{R})} \leq \mathbf{c}_1\|F_g\|_{C(\boldsymbol{R})} &= \mathbf{c}_1\left\|\int_{\boldsymbol{R}} \mathsf{S}_1(g)(u+x)|G(x)|\,dx\right\|_{C(\boldsymbol{R})} \\ &= \mathbf{c}_1 \max_{u\in\boldsymbol{R}}\left|\int_{\boldsymbol{R}} \mathsf{S}_1(g)(u+x)|G(x)|\,dx\right| \\ &\leq 2\mathbf{c}_1 \max_{u\in\boldsymbol{R}}\left\|\mathsf{S}_1(g)(u+\cdot)\right\|_{p(\cdot)} \leq 2c_5\left(p^+, c_3(p)\right)\mathbf{c}_1\|g\|_{p(\cdot)},\end{aligned}$$

where we used hypothesis, Hölder's inequality and Theorem 1.1, respectively. On the other hand, for any $\varepsilon \in \left(0, \frac{\|f\|_{p(\cdot)}}{12c_7(c_3(p))}\right)$ and appropriately chosen $\tilde{G}_\varepsilon \in L_{p'(\cdot)}$ with $\left\|\tilde{G}_\varepsilon\right\|_{X'} \leq 1$ (see e.g. Theorem 1.3)

$$\int_{\boldsymbol{R}} |g(x)|\left|\tilde{G}_\varepsilon(x)\right|dx \geq \frac{1}{12c_7(c_3(p))}\|g\|_{p(\cdot)} - \varepsilon,$$

one can find

$$\begin{aligned}\|F_f\|_{C(\boldsymbol{R})} \geq |F_f(0)| &\geq \int_{\boldsymbol{R}} \mathsf{S}_1(f)(x)|G(x)|\,dx \\ &= \mathsf{S}_1\left(\int_{\boldsymbol{R}} f(x)|G(x)|\,dx\right) \geq \mathsf{S}_1\left(\frac{1}{12c_7(c_3(p))}\|f\|_{p(\cdot)} - \varepsilon\right) \\ &= \frac{1}{12c_7(c_3(p))}\|f\|_{p(\cdot)} - \varepsilon.\end{aligned}$$

In the last inequality, we take as $\varepsilon \to 0+$ and obtain

$$\|F_f\|_{C(\boldsymbol{R})} \geq \frac{1}{12c_7(c_3(p))}\|f\|_{p(\cdot)}.$$

Then for $f \in L_{p(\cdot)}$, we get

$$\|f\|_{p(\cdot)} \leq 24c_7 (c_3 (p)) \|F_f\|_{C(\boldsymbol{R})} \leq 24c_7 (c_3 (p)) \mathbf{c}_1 \|F_g\|_{C(\boldsymbol{R})}$$
$$\leq 48c_7 (c_3 (p)) \mathbf{c}_1 c_5 \left(p^+, c_3 (p)\right) \|g\|_{p(\cdot)} .$$

□

Definition 1.5. *For $p \in P^{Log}(\boldsymbol{R})$, $f \in L_{p(\cdot)}$, $0 < \delta < \infty$, $r \in \mathrm{N}_0$, we can define modulus of smoothness as*

$$\Omega_r(f,\delta)_{p(\cdot)} = \|(I - T_\delta)^r f\|_{p(\cdot)},$$
$$\Omega_0(f,\delta)_{p(\cdot)} := \|f\|_{p(\cdot)} =: \Omega_r(f,0)_{p(\cdot)}.$$

2. Uniform norm estimates

In this section, let $\Omega \subseteq \boldsymbol{R}$ be a measurable set and $C(\Omega)$ be the collection of functions continuous on Ω. If $\Omega \neq \boldsymbol{R}$ and $f \in C(\Omega)$, we will extend f to whole $\boldsymbol{R}$ by "$f(s) \equiv 0$ whenever $s \notin \Omega$." when necessary. For $f \in C(\Omega)$ and $\delta \geq 0$, we define the modulus of smoothness as

$$\Omega_r(f,\delta)_{C(\Omega)} := \|(I - T_\delta)^r f\|_{C(\Omega)}, \quad r \in \mathrm{N}, \tag{2.13}$$
$$\Omega_0(f,\cdot)_{C(\Omega)} := \|f\|_{C(\Omega)}$$

with $T_\delta f$ of ($*$).

Lemma 2.1. *Let $0 \leq \delta < \infty$, $r \in \mathrm{N}$ and $f \in C^r(\Omega)$. Then*

$$\frac{d^r}{dx^r} T_\delta f(x) = T_\delta \frac{d^r}{dx^r} f(x) \text{ on } \Omega. \tag{2.14}$$

The following theorem states the main properties of (2.13).

Theorem 2.7. *For $f \in C(\Omega)$, $0 \leq \delta < \infty$, and $r \in \mathrm{N}$, the following properties hold.*

(1) $\Omega_r(f,\delta)_{C(\Omega)}$ is non-negative, non-decreasing function of δ,
(2) $\Omega_r(f,\delta)_{C(\Omega)}$ is sub-additive with respect to f,
(3) $\|T_\delta f\|_{C(\Omega)} \leq \|f\|_{C(\Omega)}$,
(4) $\Omega_r(f,\delta)_{C(\Omega)} \leq 2\Omega_{r-1}(f,\delta)_{C(\Omega)} \leq \cdots \leq 2^{r-1}\Omega_1(f,\delta)_{C(\Omega)} \leq 2^r \|f\|_{C(\Omega)}$, (***)
(5) $\Omega_r(f,\delta)_{C(\Omega)} \leq 2^{-1}\delta\Omega_{r-1}(f',\delta)_{C(\Omega)} \leq \cdots \leq 2^{-r}\delta^r \left\|f^{(r)}\right\|_{C(\Omega)}$, if $f \in C^r(\Omega)$.

Let X be a Banach space with a norm $\|\cdot\|_X$ and $r \in \mathrm{N}$. We define Peetre's K-functional for the pair X and W_X^r as follows :

$$K_r(f,\delta,X)_X := \inf_{g \in W_X^r} \left\{\|f-g\|_X + \delta^r \left\|g^{(r)}\right\|_X\right\}, \quad \delta > 0.$$

We set $T_\delta^r f := (T_\delta f)^r$.

Lemma 2.2. *Let $0 \leq \delta < \infty$, $r-1 \in \mathrm{N}$, and $f \in C^r(\Omega)$ be given. Then*

$$\frac{d^r}{dx^r} T_\delta^r f(x) = \frac{d}{dx} T_\delta \frac{d^{r-1}}{dx^{r-1}} T_\delta^{r-1} f(x) \quad \text{on } \Omega. \tag{2.15}$$

Lemma 2.3. *(see e.g.[17, p.177]) Let $\Omega \subseteq \boldsymbol{R}$ be a measurable set, $\delta > 0$, $f \in C(\Omega)$ and $\tilde{T}_\delta f(\cdot) = f(\cdot + \delta)$. Then, for any $r \in \mathrm{N}$, there holds*

$$\frac{1}{r^r+2^r} \leq \frac{\sup_{|h|\leq\delta}\left\|\left(I-\tilde{T}_h\right)^r f\right\|_{C(\Omega)}}{K_r\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}} \leq 2^r.$$

Main result of this section is the following theorem.

Theorem 2.8. *Let* $\Omega \subseteq \boldsymbol{R}$ *be a measurable set,* $0<\delta<\infty$*,* $f \in C\left(\Omega\right)$*,* $r \in \mathrm{N}$ *and* $g \in C^2\left(\Omega\right)$*. Then, the following inequalities*

$$\left\|\frac{d}{dx}T_\delta f\left(x\right)\right\|_{C(\Omega)} \leq \frac{2}{\delta}\left\|f\right\|_{C(\Omega)},$$

$$\left\|\frac{d^2}{dx^2}T_\delta f\left(x\right)\right\|_{C(\Omega)} \leq \frac{2}{\delta}\left\|\frac{d}{dx}T_\delta f\right\|_{C(\Omega)},$$

$$\left\|g\left(x\right)-T_\delta g\left(x\right)+\frac{\delta}{2}\frac{d}{dx}g\left(x\right)\right\|_{C(\Omega)} \leq \frac{\delta^2}{6}\left\|\frac{d^2}{dx^2}g\right\|_{C(\Omega)},$$

$$(c_8\left(r\right))^{-1}K_r\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)} \leq \left\|\left(I-T_\delta\right)^r f\right\|_{C(\Omega)} \leq 2^r K_r\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)} \quad (2.16)$$

are hold with $c_8\left(1\right)=36$*,* $c_8\left(r\right)=2^r\left(r^r+\left(34\right)^r\right)$ *for* $r>1$*.*

As a corollary of Theorem 2.8, we can state the following result.

Proposition 2.4. *If* $0<h\leq\delta<\infty$ *and* $f \in C\left(\Omega\right)$*, then*

$$\left\|\left(I-T_h\right)f\right\|_{C(\Omega)} \leq 72\left\|\left(I-T_\delta\right)f\right\|_{C(\Omega)}. \quad (2.17)$$

As a corollary of (2.16) and Lemma 2.3, we can write

Corollary 2.2. *Let* $\Omega \subseteq \boldsymbol{R}$ *be a measurable set,* $\delta>0$*,* $f \in C\left(\Omega\right)$ *and* $r \in \mathrm{N}$*. Then,*

(i) there holds

$$1+2^{-r}r^r \leq \frac{\sup_{|h|\leq\delta}\left\|\left(I-\tilde{T}_h\right)^r f\right\|_{C(\Omega)}}{\left\|\left(I-T_\delta\right)^r f\right\|_{C(\Omega)}} \leq 2^r c_8\left(r\right),$$

(ii) for $0<\delta_1\leq\delta_2$*, there holds*

$$\left(1+2^{-r}r^r\right)\Omega_r\left(f,\delta_1\right)_{C(\Omega)} \leq c_8\left(r\right)2^r\Omega_r\left(f,\delta_2\right)_{C(\Omega)}.$$

Remark 2.2. *From Theorem 23.62 of* [57, p.579]*, we have*

$$\lim_{\delta\searrow0}\Omega_1(f,\delta)_{C(\boldsymbol{R})} = \lim_{\delta\searrow0}\left\|\left(I-T_\delta\right)f\right\|_{C(\boldsymbol{R})} = 0. \quad (2.18)$$

Corollary 2.3. *If* $f \in C\left(\boldsymbol{R}\right)$*,* $0<\delta<\infty$*, and* $r \in \mathrm{N}$*, then, by (2.18) and (***),*

$$\lim_{\delta\searrow0}\Omega_r(f,\delta)_{C(\boldsymbol{R})} = \lim_{\delta\searrow0}\left\|\left(I-T_\delta\right)^r f\right\|_{C(\boldsymbol{R})} = 0$$

holds.

Let $\mathcal{G}_\sigma\left(X\right)$ be the subspace of entire function of exponential type σ that belonging to a Banach space X. The quantity

$$A_\sigma(f)_X := \inf_g\{\|f-g\|_X : g \in \mathcal{G}_\sigma\left(X\right)\} \quad (2.19)$$

is called the deviation of the function $f \in X$ from $\mathcal{G}_\sigma\left(X\right)$.

Let $\mathcal{G}_{\sigma,p(\cdot)} := \mathcal{G}_\sigma\left(L_{p(\cdot)}\right)$ be the subspace of integral function f of exponential type σ that belonging to $L_{p(\cdot)}$. The quantity

$$A_\sigma(f)_{p(\cdot)} := \inf_g\{\|f-g\|_{p(\cdot)} : g \in \mathcal{G}_{\sigma,p(\cdot)}\}$$

is the deviation of the function $f \in L_{p(\cdot)}$ from $\mathcal{G}_\sigma$.

Remark 2.3. *Let $\sigma > 0$, $1 \le p \le \infty$, $f \in L_p(\boldsymbol{R})$,*

$$\vartheta(x) := \frac{2}{\pi}\frac{\sin(x/2)\sin(3x/2)}{x^2}$$

and

$$J(f,\sigma) = \sigma\int_{\boldsymbol{R}} f(x-u)\,\vartheta(\sigma u)\,du$$

be the de la Valèe Poussin operator ([13, definition given in (5.3)]). *It is known (see (5.4)-(5.5) of* [13]*) that, if $f \in L_p(\boldsymbol{R})$, $1 \le p \le \infty$, then*

(i) $J(f,\sigma) \in \mathcal{G}_{2\sigma}(L_p(\boldsymbol{R}))$,
(ii) $J(g_\sigma,\sigma) = g_\sigma$ *for any* $g_\sigma \in \mathcal{G}_\sigma(L_p(\boldsymbol{R}))$,
(iii) $\|J(f,\sigma)\|_{L_p(\boldsymbol{R})} \le \frac{3}{2}\|f\|_{L_p(\boldsymbol{R})}$,
(iv) $(J(f,\sigma))^{(r)} = J\left(f^{(r)},\sigma\right)$ *for any* $r \in \mathrm{N}$ *and* $f \in (L_p(\boldsymbol{R}))^r$,
(v) $\|J\left(f,\frac{\sigma}{2}\right) - f\|_{L_p(\boldsymbol{R})} \to 0$ *(as* $\sigma \to \infty$*) and hence*

$$\|\left(J\left(f,\frac{\sigma}{2}\right)\right)^{(k)} - f^{(k)}\|_{L_p(\boldsymbol{R})} \to 0 \text{ as } \sigma \to \infty$$

for $f \in W^r_{L_p(\boldsymbol{R})}$ *and* $1 \le k \le r$.

Corollary 2.4. *Let $0 < \sigma < \infty$.*

(i) If $1 \le p < \infty$, $f \in L_p(\boldsymbol{R})$. Then, using (v) of the last remark, we conclude

$$\lim_{\sigma\to\infty} A_\sigma(f)_{L_p(\boldsymbol{R})} = 0.$$

(ii) Let $g : \boldsymbol{R} \to \mathbb{C}$ be bounded on the real axis $\boldsymbol{R}$. Then (see [14]*)*

$$\lim_{\sigma\to\infty} A_\sigma(g)_{C(\boldsymbol{R})} = 0$$

if and only if g is uniformly continuous on $\boldsymbol{R}$.

Theorem 2.9. *Let $r \in \mathrm{N}$, $\sigma > 0$, $\delta \in (0,1)$ and $f \in \mathcal{C}(\boldsymbol{R})$. Then, the following Jackson type inequality*

$$A_\sigma(f)_{\mathcal{C}(\boldsymbol{R})} \le 5\pi 4^{r-1} c_8(r)\,\Omega_r(f,1/\sigma)_{\mathcal{C}(\boldsymbol{R})}, \tag{2.20}$$

and its weak inverse

$$\Omega_r(f,\delta)_{\mathcal{C}(\boldsymbol{R})} \le \left(1+2^{2r-1}\right)2^{r-1}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})} + \int_{1/2}^{1/\delta} u^{r-1}A_u(f)_{\mathcal{C}(\boldsymbol{R})}\,du\right) \tag{2.21}$$

are hold.

We set $\lfloor\sigma\rfloor := \max\{n \in \mathbb{Z} : n \le \sigma\}$.

Theorem 2.10. *Let $r \in \mathrm{N}$, $f \in X^r_{\mathcal{C}(\boldsymbol{R})}$ and $\sigma > 0$. Then*

(a) (i) there exists (see [13, Proposition 25]*) a $g_\sigma \in \mathcal{G}_\sigma(\mathcal{C}(\boldsymbol{R}))$ such that*

$$A_\sigma(f)_{\mathcal{C}(\boldsymbol{R})} \le \|f-g_\sigma\|_{\mathcal{C}(\boldsymbol{R})} \le \frac{5\pi}{4}\frac{4^r}{\sigma^r}\|f^{(r)}\|_{\mathcal{C}(\boldsymbol{R})},$$

(ii) and its weak inverse

$$\|f^{(k)}\|_{\mathcal{C}(\boldsymbol{R})} \leq \left(1+2^{2k-1}\right) 2^{k+2}\pi^{k} c_8\left(k\right) \sum_{\nu=0}^{\infty} \frac{(\nu+1)^{r}}{\nu+1} A_{\nu}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}$$

holds whenever $k=1,2,\cdots,r$ *and* $\sum_{\nu=0}^{\infty}(\nu+1)^{r-1}A_{\nu}\left(f\right)_{C(\boldsymbol{R})}<\infty$.

(b) (i) the following inequality (see [29, p.397]*)*

$$A_{\sigma}\left(f\right)_{\mathcal{C}(\boldsymbol{R})} \leq \frac{(5\pi)^{r}}{\sigma^{r}} A_{\sigma}\left(f^{(r)}\right)_{\mathcal{C}(\boldsymbol{R})},$$

(ii) and its weak inverse

$$A_{\sigma}\left(f^{(r)}\right)_{\mathcal{C}(\boldsymbol{R})} \leq \left\|f^{(r)}-\left(J\left(f^{(r)},\frac{\sigma}{2}\right)\right)\right\|_{\mathcal{C}(\boldsymbol{R})}$$
$$\leq \left(1+2^{2r-1}\right) 2^{r+2}\pi^{r} c_8\left(r\right) \left(A_{\sigma}\left(f\right)_{\mathcal{C}(\boldsymbol{R})} \sum_{k=0}^{\lfloor\sigma\rfloor} \frac{k^{r}}{k} + \sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty} \frac{(\nu+1)^{r}}{\nu+1} A_{\nu}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}\right)$$

hold when $\sum_{\nu=0}^{\infty}(\nu+1)^{r-1}A_{\nu}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}<\infty$.

Theorem 2.11. *Let* $r,k\in\mathrm{N}$, $0<t\leq 1/2$, $0\leq\delta<\infty$ *and* $f\in\mathcal{C}(\boldsymbol{R})$. *Then*

(i) there holds

$$\Omega_{r+k}\left(f,\delta\right)_{\mathcal{C}(\boldsymbol{R})} \leq 2^{k}\Omega_{r}\left(f,\delta\right)_{\mathcal{C}(\boldsymbol{R})},$$

(ii) and its weak inverse (Marchaud inequality)

$$\Omega_{r}\left(f,t\right)_{\mathcal{C}(\boldsymbol{R})} \leq C_9\left(r,k\right) t^{r} \int_{t}^{1} \frac{\Omega_{r+k}\left(f,u\right)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}} du$$

with $C_9\left(r,k\right)=10\pi\left(1+2^{2r-1}\right)2^{2r+3k}c_8\left(r+k\right)$.

Theorem 2.12. *Let* $\sigma>0$ *and* $f\in\mathcal{C}(\boldsymbol{R})$. *If* $\sum_{\nu=0}^{\infty}(\nu+1)^{k-1}A_{\nu}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}<\infty$, *holds for some* $k\in\mathrm{N}$, *then*

(i) the following Jackson type inequality for derivatives

$$A_{\sigma}\left(f\right)_{\mathcal{C}(\boldsymbol{R})} \leq (5\pi)^{k+1} c_8\left(r\right)\sigma^{-k}\Omega_{r}\left(f^{(k)},\sigma^{-1}\right)_{\mathcal{C}(\boldsymbol{R})},$$

(ii) and its weak inverse (see Theorem 6.3.4 of [29, p.343]*)*

$$\Omega_{r}\left(f^{(k)},\frac{1}{\sigma}\right)_{\mathcal{C}(\boldsymbol{R})} \leq 2^{2k+r+1}\left(\frac{1}{\sigma^{r}}\sum_{\nu=0}^{\lfloor\sigma\rfloor}\frac{(\nu+1)^{r+k}}{\nu+1}A_{\nu}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty}\frac{\nu^{k}}{\nu}A_{\nu}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}\right)$$

are hold.

2.1. Proofs of the results of section 2.

Proof of Lemma 2.1. For $\delta=0$ (2.14) is obvious. For $0<\delta<\infty$, and $r=1$, one can find

$$\frac{d}{dx}T_{\delta}f(x)=\frac{d}{dx}\left(\frac{1}{\delta}\int_{0}^{\delta}f\left(x+t\right)dt\right)=\frac{1}{\delta}\int_{0}^{\delta}\frac{d}{dx}f\left(x+\tau\right)d\tau$$
$$=\frac{1}{\delta}\int_{0}^{\delta}\left(\frac{d}{dx}f\right)\left(x+\tau\right)d\tau=T_{\delta}\frac{d}{dx}f(x). \tag{2.22}$$

For $r>1$, (2.14) follows from (2.22). □

Proof of Theorem 2.7. (1)-(3) is known. (4) is seen from binomial expansion. To prove (5), it is sufficient to note inequality (see [10])

$$\left\|(I-T_\delta)\, f\right\|_{C(\Omega)} \le 2^{-1}\delta \left\|f'\right\|_{C(\Omega)}, \quad \delta>0$$

for $f\in C^1(\Omega)$. Then

$$\left\|(I-T_\delta)^r f\right\|_{C(\Omega)} \le 2^{-1}\delta \left\|(I-T_\delta)^{r-1} f'\right\|_{C(\Omega)} \le \cdots \le 2^{-r}\delta^r \left\|f^{(r)}\right\|_{C(\Omega)}$$

for $f\in C^r(\Omega)$, because

$$\left[(I-T_\delta)^r f\right]' = (I-T_\delta)^r f'.$$

□

Proof of Lemma 2.2. For $r=2$, by Lemma 2.1,

$$\begin{aligned}\frac{d^2}{dx^2}T_\delta^2 f &= \frac{d}{dx}\frac{d}{dx}T_\delta T_\delta f = \frac{d}{dx}\frac{d}{dx}T_\delta\Psi, \qquad [\Psi := T_\delta f]\\ &= \frac{d}{dx}T_\delta\frac{d}{dx}\Psi = \frac{d}{dx}T_\delta\frac{d}{dx}T_\delta f\end{aligned}$$

and the result (2.15) follows. For $r=3$, by Lemma 2.1,

$$\begin{aligned}\frac{d^3}{dx^3}T_\delta^3 f &= \frac{d}{dx}\frac{d^2}{dx^2}T_\delta^2 T_\delta f = \frac{d}{dx}\frac{d^2}{dx^2}T_\delta^2\Psi = \frac{d}{dx}\frac{d}{dx}T_\delta\frac{d}{dx}T_\delta\Psi\\ &= \frac{d}{dx}\frac{d}{dx}T_\delta\frac{d}{dx}{T_\delta}^2 f = \frac{d}{dx}T_\delta\frac{d}{dx}\frac{d}{dx}{T_\delta}^2 f = \frac{d}{dx}T_\delta\frac{d^2}{dx^2}T_\delta^2 f\end{aligned}$$

and (2.15) holds. Let (2.15) holds for $k\in\mathbb{N}$:

$$\frac{d^k}{dx^k}T_\delta^k f = \frac{d}{dx}T_\delta\frac{d^{k-1}}{dx^{k-1}}T_\delta^{k-1} f. \tag{2.23}$$

Then, for $k+1$, (2.23) and Lemma 2.1 implies that

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}T_\delta^{k+1} f &= \frac{d}{dx}\frac{d^k}{dx^k}T_\delta^k T_\delta f = \frac{d}{dx}\frac{d^k}{dx^k}T_\delta^k\Psi = \frac{d}{dx}\frac{d}{dx}T_\delta\frac{d^{k-1}}{dx^{k-1}}T_\delta^{k-1}\Psi\\ &= \frac{d}{dx}\frac{d}{dx}T_\delta\frac{d^{k-1}}{dx^{k-1}}T_\delta^k f = \frac{d}{dx}T_\delta\frac{d}{dx}\frac{d^{k-1}}{dx^{k-1}}T_\delta^k f = \frac{d}{dx}T_\delta\frac{d^k}{dx^k}T_\delta^k f.\end{aligned}$$

□

Proof of Theorem 2.8. For $f\in C(\Omega)$, we have

$$\begin{aligned}\left\|\frac{d}{dx}T_\delta f(x)\right\|_{C(\Omega)} &= \left\|\frac{d}{dx}\frac{1}{\delta}\int_0^\delta f(x+t)\,dt\right\|_{C(\Omega)}\\ &= \left\|\frac{1}{\delta}\frac{d}{dx}\int_x^{x+\delta} f(\tau)\,d\tau\right\|_{C(\Omega)} = \left\|\frac{1}{\delta}\left(f(x+\delta)-f(x)\right)\right\|_{C(\Omega)} \le \frac{2}{\delta}\left\|f\right\|_{C(\Omega)}.\end{aligned} \tag{2.24}$$

Inequality (2.24) also implies

$$\left\|\left(\frac{d}{dx}\right)^2 T_\delta f(x)\right\|_{C(\Omega)} \le \frac{2}{\delta}\left\|\frac{d}{dx}T_\delta f\right\|_{C(\Omega)}$$

for $f\in C(\Omega)$. If $f\in C^2(\Omega)$, one can get

$$\left\|f(x)-T_\delta f(x)+\frac{\delta}{2}\frac{d}{dx}f(x)\right\|_{C(\Omega)} \le \frac{\delta^2}{6}\left\|\frac{d^2}{dx^2}f\right\|_{C(\Omega)}. \tag{2.25}$$

To obtain (2.25), we will use the Taylor formula

$$f\left(x+t\right)=f(x)+t\frac{d}{dx}f(x)+\frac{t^{2}}{2}\frac{d^{2}}{dx^{2}}f(\xi)$$

for some $\xi\leq[x,x+t]$. Then, integrating the last equation with respect to t

$$\frac{1}{\delta}\int_{0}^{\delta}f\left(x+t\right)dt=f(x)+\frac{1}{\delta}\int_{0}^{\delta}tdt\frac{d}{dx}f(x)+\frac{1}{2}\frac{1}{\delta}\int_{0}^{\delta}t^{2}dt\frac{d^{2}}{dx^{2}}f(\xi),$$
$$T_{\delta}f\left(x\right)=f(x)+\frac{\delta}{2}\frac{d}{dx}f\left(x\right)+\frac{\delta^{2}}{6}\frac{d^{2}}{dx^{2}}f(\xi)$$

and (2.25) holds.

Now, (2.24) and (2.25) imply that

$$(1/36)\,K_{1}\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}\leq\left\Vert \left(I-T_{\delta}\right)f\right\Vert_{C(\Omega)}\leq2K_{1}\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}. \tag{2.26}$$

Firstly, let us prove the right hand side of (2.26). For any $g\in C^{1}\left(\Omega\right)$

$$\begin{aligned}\left\Vert f-T_{\delta}f\right\Vert_{C(\Omega)}&\leq\left\Vert f-g\right\Vert_{C(\Omega)}+\left\Vert g-T_{\delta}g\right\Vert_{C(\Omega)}+\left\Vert T_{\delta}\left(g-f\right)\right\Vert_{C(\Omega)}\\&\leq2\left\Vert f-g\right\Vert_{C(\Omega)}+\frac{\delta}{2}\left\Vert g'\right\Vert_{C(\Omega)}\leq2K_{1}\left(f,\delta,C\left(\Omega\right)\right)_{C(\Omega)}.\end{aligned}$$

For the left hand side of inequality (2.26), we need inequalities

$$\left\Vert f-T_{\delta}^{2}f\right\Vert_{C(\Omega)}\leq2\left\Vert f-T_{\delta}f\right\Vert_{C(\Omega)}, \tag{2.27}$$

$$\delta\left\Vert \frac{d}{dx}T_{\delta}^{2}f\right\Vert_{C(\Omega)}\leq34\left\Vert f-T_{\delta}f\right\Vert_{C(\Omega)}. \tag{2.28}$$

First we prove (2.27). Then

$$\left\Vert f-T_{\delta}^{2}f\right\Vert_{C(\Omega)}\leq\left\Vert f-T_{\delta}f\right\Vert_{C(\Omega)}+\left\Vert T_{\delta}f-T_{\delta}T_{\delta}f\right\Vert_{C(\Omega)}\leq2\left\Vert f-T_{\delta}f\right\Vert_{C(\Omega)}.$$

Now, we consider inequality (2.28). In (2.25), we replace f by $T_{\delta}^{2}f$ and obtain

$$\left\Vert T_{\delta}^{2}f\left(x\right)-T_{\delta}T_{\delta}^{2}f\left(x\right)+\frac{\delta}{2}\frac{d}{dx}T_{\delta}^{2}f\left(x\right)\right\Vert_{C(\Omega)}\leq\frac{\delta^{2}}{6}\left\Vert \frac{d^{2}}{dx^{2}}T_{\delta}^{2}f\right\Vert_{C(\Omega)}.$$

On the other hand, by (2.24),

$$\begin{aligned}\left\Vert \frac{d^{2}}{dx^{2}}T_{\delta}^{2}f\right\Vert_{C(\Omega)}&\leq\frac{2}{\delta}\left\Vert \frac{d}{dx}T_{\delta}f\right\Vert_{C(\Omega)}\\&\leq\frac{2}{\delta}\left\{\left\Vert \frac{d}{dx}T_{\delta}^{2}f\right\Vert_{C(\Omega)}+\left\Vert \frac{d}{dx}T_{\delta}\left(T_{\delta}f-f\right)\right\Vert_{C(\Omega)}\right\}\\&\leq\frac{2}{\delta}\left\Vert \frac{d}{dx}T_{\delta}^{2}f\right\Vert_{C(\Omega)}+\frac{4}{\delta^{2}}\left\Vert T_{\delta}f-f\right\Vert_{C(\Omega)}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\delta}{2}\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} &\leq \left\|T_\delta^2 f - T_\delta T_\delta^2 f - \frac{\delta}{2}\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} + \left\|T_\delta^2 f - T_\delta T_\delta^2 f\right\|_{C(\Omega)} \\ &\leq \frac{\delta^2}{6}\left\|\frac{d^2}{dx^2}T_\delta^2 f\right\|_{C(\Omega)} + \left\|T_\delta^2 f - T_\delta T_\delta^2 f\right\|_{C(\Omega)} \\ &\leq \frac{\delta^2}{6}\frac{2}{\delta}\left\{\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} + \frac{2}{\delta}\left\|T_\delta f - f\right\|_{C(\Omega)}\right\} + \left\|T_\delta^2 f - f\right\|_{C(\Omega)} \\ &+ \left\|T_\delta\left(T_\delta^2 f - f\right)\right\|_{C(\Omega)} + \left\|T_\delta f - f\right\|_{C(\Omega)}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\delta}{6}\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} &\leq \frac{17}{3}\left\|T_\delta f - f\right\|_{C(\Omega)}, \\ \delta\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} &\leq 34\left\|T_\delta f - f\right\|_{C(\Omega)}.\end{aligned}$$

To finish proof of the left hand side of inequality (2.16) with $r = 1$, we proceed as

$$K_1\left(f, \delta, C(\Omega)\right)_{C(\Omega)} \leq \left\|f - T_\delta^2 f\right\|_{C(\Omega)} + \delta\left\|\frac{d}{dx}T_\delta^2 f\right\|_{C(\Omega)} \leq 36\left\|T_\delta f - f\right\|_{C(\Omega)}.$$

The proof of (2.16) with $r = 1$ now completed.

Let $r > 1$ be a natural number and we define

$$g(\cdot) = \sum_{l=1}^{r}(-1)^{l-1}\binom{r}{l}T_\delta^{2rl} f(\cdot).$$

Then,

$$\|f - g\|_{C(\Omega)} = \left\|\left(I - T_\delta^{2r}\right)^r f\right\|_{C(\Omega)} \leq (2r)^r\left\|(I - T_\delta)^r f\right\|_{C(\Omega)}.$$

On the other hand,

$$\begin{aligned}\delta^r\left\|\frac{d^r}{dx^r}T_\delta^{2r} f\right\|_{C(\Omega)} &= \delta^{r-1}\delta\left\|\frac{d}{dx}T_\delta^2\left(\frac{d^{r-1}}{dx^{r-1}}\right)T_\delta^{2r-2} f\right\|_{C(\Omega)} \\ &\leq 34\delta^{r-1}\left\|(I - T_\delta)\frac{d^{r-1}}{dx^{r-1}}T_\delta^{2r-2} f\right\|_{C(\Omega)} \\ &\leq (34)^2\delta^{r-2}\left\|(I - T_\delta)^2\frac{d^{r-2}}{dx^{r-2}}T_\delta^{2r-4} f\right\|_{C(\Omega)} \\ &\leq \cdots \leq (34)^r\left\|(I - T_\delta)^r f\right\|_{C(\Omega)}.\end{aligned}$$

Then

$$\begin{aligned}\delta^r\left\|\frac{d^r}{dx^r}T_\delta^{2rl} f\right\|_{C(\Omega)} &\leq (34)^r\left\|(I - T_\delta)^r T_\delta^{2r(l-1)} f\right\|_{C(\Omega)} \\ &= (34)^r\left\|T_\delta^{2r(l-1)}(I - T_\delta)^r f\right\|_{C(\Omega)} \leq (34)^r\left\|(I - T_\delta)^r f\right\|_{C(\Omega)}.\end{aligned}$$

Using the last inequality, we find

$$\begin{aligned}\delta^r \left\| \frac{d^r}{dx^r} g \right\|_{C(\Omega)} &= \delta^r \left\| \frac{d^r}{dx^r} \sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} T_\delta^{2rl} f \right\|_{C(\Omega)} \\ &= \delta^r \left\| \sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} \frac{d^r}{dx^r} T_\delta^{2rl} f \right\|_{C(\Omega)} \\ &\le \sum_{l=1}^{r} \left| \binom{r}{l} \right| \delta^r \left\| \frac{d^r}{dx^r} T_\delta^{2rl} f \right\|_{C(\Omega)} \\ &\le 2^r (34)^r \left\| (I - T_\delta)^r f \right\|_{C(\Omega)}\end{aligned}$$

and

$$\begin{aligned}K_r (f, \delta, C(\Omega))_{C(\Omega)} &\le \|f - g\|_{C(\Omega)} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{C(\Omega)} \\ &\le 2^r (r^r + (34)^r) \left\| (I - T_\delta)^r f \right\|_{C(\Omega)} .\end{aligned}$$

For the opposite direction of the last inequality, when $g \in W^r_{p(\cdot)}$,

$$\begin{aligned}\Omega_r (f, \delta)_{C(\Omega)} &\le 2^r \|f - g\|_{C(\Omega)} + \Omega_r (g, \delta)_{C(\Omega)} \\ &\le 2^r \|f - g\|_{C(\Omega)} + 2^{-r} \delta^r \left\| g^{(r)} \right\|_{C(\Omega)} , \qquad (2.29)\end{aligned}$$

and taking infimum on $g \in W^r_{p(\cdot)}$ in (2.29), we get

$$\Omega_r (f, \delta)_{C(\Omega)} \le 2^r K_r (f, \delta, C(\Omega))_{C(\Omega)} .$$

□

Proof of Proposition 2.4. Let $f \in C(\Omega)$. Then

$$\begin{aligned}\|(I - T_h) f\|_{C(\Omega)} &\le 2K_1 (f, h, C(\Omega))_{C(\Omega)} \\ &\le 2K_1 (f, \delta, C(\Omega))_{C(\Omega)} \le 72 \|(I - T_\delta) f\|_{C(\Omega)} .\end{aligned}$$

□

Proof of Theorem 2.9. (i) We consider Jackson type inequality (2.20). For any $g \in X^r_{\mathcal{C}(\boldsymbol{R})}$, we have

$$\begin{aligned}A_\sigma (f)_{\mathcal{C}(\boldsymbol{R})} &\le A_\sigma (f - g)_{\mathcal{C}(\boldsymbol{R})} + A_\sigma (g)_{\mathcal{C}(\boldsymbol{R})} \\ &\le \|f - g\|_{\mathcal{C}(\boldsymbol{R})} + \frac{5\pi}{4} \frac{4^r}{\sigma^r} \left\| \frac{d^r}{dx^r} g \right\|_{\mathcal{C}(\boldsymbol{R})} .\end{aligned}$$

Taking infimum on $g \in X^r_{\mathcal{C}(\boldsymbol{R})}$ in the last inequality, we have

$$A_\sigma (f)_{\mathcal{C}(\boldsymbol{R})} \le \frac{5\pi 4^r}{4} K_r \left(f, \frac{1}{\sigma}, \mathcal{C}(\boldsymbol{R}) \right)_{\mathcal{C}(\boldsymbol{R})} \le \frac{5\pi}{4} c_8 (r) 4^r \left\| \left(I - T_{\frac{1}{\sigma}} \right)^r f \right\|_{\mathcal{C}(\boldsymbol{R})} .$$

(ii) We give the proof of inverse estimate (2.21). Let $\sigma > 0$ and $g_\sigma \in \mathcal{G}_\sigma (\mathcal{C}(\boldsymbol{R}))$ be the best approximating IFFD of $f \in \mathcal{C}(\boldsymbol{R})$. Suppose that $r \in \mathrm{N}$, $0 < \delta < 1$. Then, there exists a $m \in \mathrm{N}$ such that $\lfloor 1/\delta \rfloor = 2^{m-1}$. Hence, $2^{m-1} \le 1/\delta < 2^m$. Now, we have

$$\begin{aligned}\Omega_r (f, \delta)_{\mathcal{C}(\boldsymbol{R})} &\le \Omega_r (f - g_{2^m}, \delta)_{\mathcal{C}(\boldsymbol{R})} + \Omega_r (g_{2^m}, \delta)_{\mathcal{C}(\boldsymbol{R})} \\ &\le 2^r A_{2^m} (f)_{\mathcal{C}(\boldsymbol{R})} + 2^{-r} \delta^r \left\| \frac{d^r}{dx^r} g_{2^m} \right\|_{\mathcal{C}(\boldsymbol{R})} .\end{aligned}$$

On the other hand

$$\begin{aligned}\left\|\frac{d^r}{dx^r}g_{2^m}\right\|_{\mathcal{C}(\boldsymbol{R})} &= \left\|\sum_{\gamma=1}^{m}\left(\frac{d^r}{dx^r}g_{2^\gamma}-\frac{d^r}{dx^r}g_{2^{\gamma-1}}\right)+\left(\frac{d^r}{dx^r}g_1-\frac{d^r}{dx^r}g_0\right)\right\|_{\mathcal{C}(\boldsymbol{R})}\\
&\le \sum_{\gamma=1}^{m}2^{\gamma r}\left\|g_{2^\gamma}-g_{2^{\gamma-1}}\right\|_{\mathcal{C}(\boldsymbol{R})}+\left\|g_1-g_0\right\|_{\mathcal{C}(\boldsymbol{R})}\\
&\le A_0(f)_{\mathcal{C}(\boldsymbol{R})}+A_1(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}2^{\gamma r}\left(A_{2^\gamma}(f)_{\mathcal{C}(\boldsymbol{R})}+A_{2^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\right)\\
&\le A_0(f)_{\mathcal{C}(\boldsymbol{R})}+2^rA_1(f)_{\mathcal{C}(\boldsymbol{R})}+2\sum_{\gamma=1}^{m}2^{\gamma r}A_{2^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\\
&\le 2\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}2^{\gamma r}A_{2^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\right).\end{aligned}$$

Then,

$$\frac{\delta^r}{2^r}\left\|\frac{d^r}{dx^r}g_{2^m}\right\|_{\mathcal{C}(\boldsymbol{R})}\le\frac{2}{2^r}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}2^{\gamma r}A_{q^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\right).$$

Hence,

$$\begin{aligned}\Omega_r(f,\delta)_{C(\boldsymbol{R})} &\le \frac{2^{(m+1)r}}{2^{mr}}A_{2^m}(f)_{\mathcal{C}(\boldsymbol{R})}+\frac{2}{2^r}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}2^{\gamma r}A_{q^{\gamma-1}}(f)_{\mathcal{C}(\boldsymbol{R})}\right)\\
&\le\left(1+2^{2r-1}\right)2^{1-r}2^{2r}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\gamma=1}^{m}\int_{2^{\gamma-2}}^{2^{\gamma-1}}u^{r-1}A_u(f)_{\mathcal{C}(\boldsymbol{R})}\,du\right)\\
&\le\left(1+2^{2r-1}\right)2^{r-1}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\int_{1/2}^{2^{m-1}}u^{r-1}A_u(f)_{\mathcal{C}(\boldsymbol{R})}\,du\right)\\
&\le\left(1+2^{2r-1}\right)2^{r-1}\delta^r\left(A_0(f)_{\mathcal{C}(\boldsymbol{R})}+\int_{1/2}^{1/\delta}u^{r-1}A_u(f)_{\mathcal{C}(\boldsymbol{R})}\,du\right).\end{aligned}$$

□

Proof of Theorem 2.10. Results a) (i) and b) (i) are known. Let us consider a) (ii). Suppose that $\sum_{\nu=0}^{\infty}\frac{(\nu+1)^r}{\nu+1}A_\nu(f)_{\mathcal{C}(\boldsymbol{R})}<\infty$ and $k\in\{1,2,\cdots,r\}$. Then, using Nikolskii inequality, one gets

$$\begin{aligned}\|f^{(k)}\|_{\mathcal{C}(\boldsymbol{R})} &= \lim_{\sigma\to\infty}\left\|J\left(f^{(k)},\frac{\sigma}{2}\right)\right\|_{\mathcal{C}(\boldsymbol{R})}=\lim_{\sigma\to\infty}\left\|\left(J\left(f,\frac{\sigma}{2}\right)\right)^{(k)}\right\|_{\mathcal{C}(\boldsymbol{R})}\\
&\le\frac{\pi^k}{2^k}\frac{\sup_{|h|\le\delta}\left\|\left(I-\tilde{T}_h\right)^k\left(J\left(f,\frac{\sigma}{2}\right)\right)\right\|_{\mathcal{C}(\boldsymbol{R})}}{\delta^k}\le\frac{\pi^k}{2^k}\frac{2^kc_8(k)\,\Omega_k\left(J\left(f,\frac{\sigma}{2}\right),\delta\right)_{\mathcal{C}(\boldsymbol{R})}}{\delta^k}\end{aligned}$$

$$\leq \left(1+2^{2k-1}\right)2^{k+2}\pi^{k}c_{8}\left(k\right)\sum_{\nu=0}^{\lfloor 1/\delta\rfloor}\frac{\left(\nu+1\right)^{k}}{\nu+1}A_{\nu}\left(J\left(f,\frac{\sigma}{2}\right)\right)_{\mathcal{C}(\boldsymbol{R})}$$
$$\leq \left(1+2^{2k-1}\right)2^{k+2}\pi^{k}c_{8}\left(k\right)\sum_{\nu=0}^{\infty}\frac{\left(\nu+1\right)^{r}}{\nu+1}A_{\nu}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}.$$

Note that (ii) b) is follow from (i) b). □

Proof of Theorem 2.11. (i) follows from properties of modulus of smoothness. We consider Marchaud type inequality (ii). Let $0<t<1/2$. Assume that $2^{m-1}\leq\frac{1}{t}<2^{m}$ for some $m\in\mathrm{N}$. Then,

$$\Omega_{r}(f,t)_{\mathcal{C}(\boldsymbol{R})}\leq\left(1+2^{2r-1}\right)2^{1-r}t^{r}\left(\sum_{\nu=1}^{m}2^{\nu r}A_{2^{\nu-1}}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}+A_{0}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}\right)$$
$$\leq\frac{5\pi}{2}\left(1+2^{2r-1}\right)2^{r+2k}c_{8}\left(r+k\right)t^{r}\left(A_{0}\left(f\right)_{\mathcal{C}(\boldsymbol{R})}+\sum_{\nu=1}^{m}2^{\nu r}\Omega_{k+r}(f,\frac{1}{2^{\nu}})_{\mathcal{C}(\boldsymbol{R})}\right)$$
$$\leq\frac{5\pi}{2}\left(1+2^{2r-1}\right)2^{2r+3k}c_{8}\left(r+k\right)t^{r}\left(\Omega_{k+r}(f,\frac{1}{2})_{\mathcal{C}(\boldsymbol{R})}+\sum_{\nu=1}^{m}\int_{2^{-v}}^{2^{-v+1}}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\right)$$
$$\leq\frac{5\pi}{2}\left(1+2^{2r-1}\right)2^{2r+3k}c_{8}\left(r+k\right)t^{r}\left(\Omega_{k+r}(f,\frac{1}{2})_{\mathcal{C}(\boldsymbol{R})}+\int_{2^{-1}}^{2^{-m+1}}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\right)$$
$$\leq 5\pi\left(1+2^{2r-1}\right)2^{2r+3k}c_{8}\left(r+k\right)t^{r}\left(\int_{1/2}^{1}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du+\int_{t}^{1}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\right)$$
$$\leq 10\pi\left(1+2^{2r-1}\right)2^{2r+3k}c_{8}\left(r+k\right)t^{k}\int_{t}^{1}\frac{\Omega_{k+r}(f,u)_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du.$$

□

Using this section's estimates and Transference result Theorem 1.5, in the next section we will give several results on difference operator $\left\|\left(I-T_{\delta}\right)^{r}f\right\|_{p(\cdot)}$ and approximation by IFFD in $L_{p(\cdot)}$.

3. Applications on Difference operator and Approximation

Notation . *Since the* $48c_{7}\left(c_{3}\left(p\right)\right)c_{5}\left(p^{+},c_{3}\left(p\right)\right)$ *of (1.11) will be used very frequently in the next parts, we will set* $c_{10}:=c_{10}\left(p^{+},c_{3}\left(p\right)\right):=48c_{7}\left(c_{3}\left(p\right)\right)c_{5}\left(p^{+},c_{3}\left(p\right)\right).$

Lemma 3.4. *Let* $p\in P^{Log}\left(\boldsymbol{R}\right)$, $r\in\mathrm{N}$, *and* $0<\delta<\infty$. *Then*

$$\left\|\left(I-T_{\delta}\right)^{r}f\right\|_{p(\cdot)}\leq c_{10}^{r}2^{-r}\delta^{r}\left\|f^{(r)}\right\|_{p(\cdot)},\quad f\in W_{L_{p(\cdot)}}^{r}$$

hold.

We will use notation $K_{r}\left(f,\delta,p\left(\cdot\right)\right):=K_{r}\left(f,\delta,L_{p(\cdot)}\right)_{L_{p(\cdot)}}$ for $r\in\mathrm{N}$, $p\in P^{Log}\left(B\right)$, $\delta>0$ and $f\in L_{p(\cdot)}\left(B\right)$.

As a corollary of Transference result, we can obtain the following Lemma.

Lemma 3.5. *Let $0 < h \leq \delta < \infty$, $p \in P^{Log}(\boldsymbol{R})$ and $f \in L_{p(\cdot)}$. Then*

$$\|(I - T_h) f\|_{p(\cdot)} \leq c_8 \left(72, p^+, c_3(p)\right) \|(I - T_\delta) f\|_{p(\cdot)} \tag{3.30}$$

holds.

In the following theorem, we show that K-functional $K_r(f, \delta, p(\cdot))$ and $\Omega_r(f,\delta)_{p(\cdot)}$ are equivalent.

Theorem 3.13. *Let $p(\cdot) \in P^{Log}(\boldsymbol{R})$. If $L_{p(\cdot)}$, then the K-functional $K_r(f, \delta, p(\cdot))$ and the modulus $\Omega_r(f,\delta)_{p(\cdot)}$ are equivalent, namely,*

$$\begin{aligned}\frac{1}{48c_7(c_3(p))\, 2^r c_5(p^+, c_3(p))} &\leq \frac{K_r(f, \delta, p(\cdot))}{\Omega_r(f,\delta)_{p(\cdot)}} \\ &\leq 48c_7(c_3(p)) \left\{(2r)^r + 2^r (34)^r\right\} c_5\left(p^+, c_3(p)\right).\end{aligned}$$

Theorem 3.14. *For $p(\cdot) \in P^{Log}(\boldsymbol{R})$, $f, g \in L_{p(\cdot)}$ and $\delta > 0$, the modulus of smoothness $\Omega_r(f,\delta)_{p(\cdot)}$ has the following properties:*

(1) $\Omega_r(f,\delta)_{p(\cdot)}$ is non-negative; non-decreasing function of δ.
(2) For $f, g \in L_{p(\cdot)}$ and $\delta > 0$,

$$\Omega_r(f+g,\delta)_{p(\cdot)} \leq \Omega_r(f,\delta)_{p(\cdot)} + \Omega_r(g,\delta)_{p(\cdot)}. \tag{3.31}$$

(3) For $f \in L_{p(\cdot)}$,

$$\lim_{\delta \to 0} \Omega_r(f,\delta)_{p(\cdot)} = 0. \tag{3.32}$$

As a corollary of Theorem 3.13,

Corollary 3.5. *Let $p(\cdot) \in P^{Log}(\boldsymbol{R})$. If $\delta, \lambda \in (0,1)$, $f \in L_{p(\cdot)}$, then*

$$\frac{\Omega_r(f, \lambda\delta)_{p(\cdot)}}{(1 + \lfloor \lambda \rfloor)^r \Omega_r(f,\delta)_{p(\cdot)}} \leq (48)^2 c_7^2(c_3(p))\, 2^r c_5^2\left(p^+, c_3(p)\right) \left((2r)^r + 2^r (34)^r\right)$$

holds.

Theorem 3.15. *Let $p(\cdot) \in P^{Log}(\boldsymbol{R})$, $r \in \mathrm{N}$, $\sigma > 0$ and $f \in L_{p(\cdot)}$. Then,*

$$A_\sigma(f)_{p(\cdot)} \leq c_{11} \left\| \left(I - T_{1/\sigma}\right)^r f \right\|_{p(\cdot)} \tag{3.33}$$

with $c_{11} := c_{11}(r, p^+, c_3(p)) := 30\pi 8^r c_5(p^+, c_3(p))\, c_7(c_3(p))\, c_8(r)$.

Now, we present the inverse theorem.

Theorem 3.16. *Let $p(\cdot) \in P^{Log}(\boldsymbol{R})$, $r \in \mathrm{N}$, $\delta \in (0,1)$ and $f \in L_{p(\cdot)}$. Then,*

$$\Omega_r(f,\delta)_{p(\cdot)} \leq c_{12}\delta^r \left(A_0(f)_{p(\cdot)} + \int_{1/2}^{1/\delta} u^{r-1} A_{u/2}(f)_{p(\cdot)}\, du \right)$$

holds with $c_{12} := c_{12}(r, p^+, c_3(p)) := c_{13} 12 c_7(c_3(p)) \left(1 + 2^{2r-1}\right) 2^r$, where $c_{13} := c_{13}(p^+, c_3(p)) := 2c_5(p^+, c_3(p)) \left(1 + 72c_7(c_3(p))\, c_5(p^+, c_3(p))\right).$

In this section, we obtain Marchaud inequality.

Theorem 3.17. *Let $r,k\in \mathrm{N}$, $p\in P^{Log}(\boldsymbol{R})$, $f\in L_{p(\cdot)}$ and $t\in(0,1/2)$. Then,*

$$\Omega_r(f,t)_{p(\cdot)} \leq c_{14} t^r \int_t^1 \frac{\Omega_{r+k}(f,u)_{p(\cdot)}}{u^{r+1}} du$$

holds with $c_{14}:=c_{14}(r,k,p^+,c_3(p)):=48c_7(c_3(p))\,C_9(r,k)\,c_5(p^+,c_3(p))$.

Theorem 3.18. *Let $p\in P^{Log}(\boldsymbol{R})$, $r\in \mathrm{N}$ and $f\in L_{p(\cdot)}$. If*

$$\sum_{\nu=0}^{\infty} \nu^{k-1} A_{\nu/2}(f)_{p(\cdot)} < \infty$$

holds for some $k\in\mathrm{N}$, then $f^{(k)}\in L_{p(\cdot)}$ and

$$\Omega_r\left(f^{(k)},\frac{1}{\sigma}\right)_{p(\cdot)} \leq c_{14}\left(\frac{1}{\sigma^r}\sum_{\nu=0}^{\lfloor\sigma\rfloor}(\nu+1)^{r+k-1}A_{\nu/2}(f)_{p(\cdot)} + \sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty}\nu^{k-1}A_{\nu/2}(f)_{p(\cdot)}\right) \tag{3.34}$$

with $c_{14}:=c_{14}(r,k,p^+,c_3(p)):=48c_7(c_3(p))\,c_5(p^+,c_3(p))\,2^{2k+r+2}$.

3.1. **Proofs of the results of section 3.**

Proof of Lemma 3.4. We note that (see [10]) the following inequality

$$\|(I-T_\delta)f\|_{p(\cdot)} \leq 2^{-1}c_{10}\delta\|f'\|_{p(\cdot)}, \quad \delta>0 \tag{3.35}$$

holds for $f\in L_{p(\cdot)}$. Then

$$\Omega_r(f,\delta)_{p(\cdot)} = \|(I-T_\delta)^r f\|_{p(\cdot)} \leq \ldots \leq 2^{-r}c_{10}^r\delta^r\left\|f^{(r)}\right\|_{p(\cdot)}, \delta>0$$

for $f\in W^r_{L_{p(\cdot)}}$. □

Proof of Theorem 3.13. For any $g\in W^r_{L_{p(\cdot)}}(\Omega)$, we have $F_g\in C^r(\Omega)$. Since F_f is linear in f,

$$(I-T_\delta)^r F_f = F_{(I-T_\delta)^r f} \text{ and } (F_g)^{(r)} = F_{g^{(r)}},$$

using Theorem 1.5 we obtain

$$\begin{aligned}
\|(I-T_\delta)^r f\|_{p(\cdot)} &\leq 24c_7(c_3(p))\left\|F_{(I-T_\delta)^r f}\right\|_{C(\Omega)} = 24c_7(c_3(p))\|(I-T_\delta)^r F_f\|_{C(\Omega)} \\
&\leq 24c_7(c_3(p))\,2^r K_r(F_f,\delta,C(\Omega))_{C(\Omega)} \\
&\leq 24c_7(c_3(p))\,2^r\left\{\|F_f-F_g\|_{C(\Omega)} + \delta^r\left\|(F_g)^{(r)}\right\|_{C(\Omega)}\right\} \\
&= 24c_7(c_3(p))\,2^r\left\{\left\|F_{(f-g)}\right\|_{C(\Omega)} + \delta^r\left\|F_{g^{(r)}}\right\|_{C(\Omega)}\right\} \\
&\leq 48c_7(c_3(p))\,2^r c_5(p^+,c_3(p))\left\{\|f-g\|_{p(\cdot)} + \delta^r\left\|g^{(r)}\right\|_{p(\cdot)}\right\}.
\end{aligned}$$

Taking infimum and considering definition of K-functional one gets,

$$\|(I-T_\delta)^r f\|_{p(\cdot)} \leq 48c_7(c_3(p))\,2^r c_5(p^+,c_3(p))\,K_r(f,\delta,p(\cdot)).$$

Now, we consider the opposite direction of the last inequality. For

$$g(\cdot) = \sum_{l=1}^{r}(-1)^{l-1}\binom{r}{l}T_\delta^{2rl}f(\cdot),$$

we have

$$\begin{aligned}
K_r(f,\delta,p(\cdot)) &\le \|f-g\|_{p(\cdot)} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{p(\cdot)} \\
&\le 24c_7(c_3(p)) \left\{ \left\| F_{(f-g)} \right\|_{C(\Omega)} + \delta^r \left\| F_{g^{(r)}} \right\|_{C(\Omega)} \right\} \\
&= 24c_7(c_3(p)) \left\{ \|F_f - F_g\|_{C(\Omega)} + \delta^r \left\| (F_g)^{(r)} \right\|_{C(\Omega)} \right\} \\
&\le 24c_7(c_3(p)) \left\{ \left\| \left(I - T_\delta^{2r}\right)^r F_f \right\|_{C(\Omega)} + \delta^r \left\| \left(\sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} T_\delta^{2rl} F_f \right)^{(r)} \right\|_{C(\Omega)} \right\} \\
&= 24c_7(c_3(p)) \left\{ \left\| \left(I - T_\delta^{2r}\right)^r F_f \right\|_{C(\Omega)} + \sum_{l=1}^{r} \left| \binom{r}{l} \right| \delta^r \left\| \left(T_\delta^{2rl} F_f \right)^{(r)} \right\|_{C(\Omega)} \right\} \\
&\le 24c_7(c_3(p)) \left\{ (2r)^r \left\| (I - T_\delta)^r F_f \right\|_{C(\Omega)} + 2^r (34)^r \left\| (I - T_\delta)^r F_f \right\|_{C(\Omega)} \right\} \\
&= 24c_7(c_3(p)) \left\{ (2r)^r + 2^r (34)^r \right\} \left\| F_{(I-T_\delta)^r f} \right\|_{C(\Omega)} \\
&\le 48c_7(c_3(p)) \left\{ (2r)^r + 2^r (34)^r \right\} c_5\left(p^+, c_3(p)\right) \left\| (I - T_\delta)^r f \right\|_{p(\cdot)}.
\end{aligned}$$

□

Proof of Theorem 3.14. Properties (1) and (2), by definition of $\Omega_r(f,\delta)_{p(\cdot)}$ and the triangle inequality of $L_{p(\cdot)}$ are clearly valid. By using [21, Theorem 10.1] and [35, Lemma 2], the relation (3.32) is satisfied. □

Proof of Corollary 3.5. We have

$$\begin{aligned}
\frac{\Omega_r(f,\lambda\delta)_{p(\cdot)}}{(1+\lfloor\lambda\rfloor)^r \Omega_r(f,\delta)_{p(\cdot)}} &\le \frac{48c_7(c_3(p))\, 2^r c_5(p^+, c_3(p))}{(1+\lfloor\lambda\rfloor)^r} \frac{K_r(f,\lambda\delta,p(\cdot))}{\Omega_r(f,\delta)_{p(\cdot)}} \\
&\le \frac{(48)^2 c_7^2(c_3(p))\, 2^r c_5^2(p^+, c_3(p))}{(1+\lfloor\lambda\rfloor)^r} \frac{(1+\lfloor\lambda\rfloor)^r}{1} \left\{ (2r)^r + 2^r(34)^r \right\} \\
&= (48)^2 c_7^2(c_3(p))\, 2^r c_5^2\left(p^+, c_3(p)\right) \left\{ (2r)^r + 2^r(34)^r \right\}.
\end{aligned}$$

□

Proof of Theorem 3.15. First we obtain

(3.36) $$A_{2\sigma}(f)_{p(\cdot)} \le 30\pi 8^r c_5\left(p^+, c_3(p)\right) c_7(c_3(p))\, c_8(r) \left\| \left(I - T_{1/(2\sigma)}\right)^r f \right\|_{p(\cdot)}$$

and (3.33) follows from (3.36). Let g_σ be an exponential type entire function of degree $\le \sigma$, belonging to $\mathcal{C}(\boldsymbol{R})$, as best approximation of $F_f \in \mathcal{C}(\boldsymbol{R})$. Since $F_{V_\sigma f} = V_\sigma F_f$ and $V_\sigma g_\sigma = g_\sigma$, there holds

$$\begin{aligned}A_{2\sigma}(f)_{p(\cdot)} \leq \|f-V_\sigma f\|_{p(\cdot)} &\leq 24c_7(c_3(p))\|F_{f-V_\sigma f}\|_{\mathcal{C}(\boldsymbol{R})}\\ &= 24c_7(c_3(p))\|F_f - V_\sigma F_f\|_{\mathcal{C}(\boldsymbol{R})}\\ &= 24c_7(c_3(p))\|F_f - g_\sigma + g_\sigma - V_\sigma F_f\|_{\mathcal{C}(\boldsymbol{R})}\\ &= 24c_7(c_3(p))\|F_f - g_\sigma + V_\sigma g_\sigma - V_\sigma F_f\|_{\mathcal{C}(\boldsymbol{R})}\\ &\leq 24c_7(c_3(p))\left(A_\sigma(F_f)_{\mathcal{C}(\boldsymbol{R})} + \frac{3}{2}A_\sigma(F_f)_{\mathcal{C}(\boldsymbol{R})}\right)\\ &= 12c_7(c_3(p))A_\sigma(F_f)_{\mathcal{C}(\boldsymbol{R})}.\end{aligned}$$

For any $g \in W^r_{\mathcal{C}(\boldsymbol{R})}$

$$\begin{aligned}A_\sigma(u)_{\mathcal{C}(\boldsymbol{R})} &\leq A_\sigma(u-g)_{\mathcal{C}(\boldsymbol{R})} + A_\sigma(g)_{\mathcal{C}(\boldsymbol{R})}\\ &\leq \|u-g\|_{\mathcal{C}(\boldsymbol{R})} + \frac{5\pi}{4}\frac{4^r}{\sigma^r}\left\|\frac{d^r}{dx^r}g\right\|_{\mathcal{C}(\boldsymbol{R})}\\ &\leq \frac{5\pi 4^r}{4}K_r\left(u,\frac{1}{\sigma},\mathcal{C}(\boldsymbol{R})\right)_{\mathcal{C}(\boldsymbol{R})} \leq \frac{5\pi 8^r}{4}K_r\left(u,\frac{1}{2\sigma},\mathcal{C}(\boldsymbol{R})\right)_{\mathcal{C}(\boldsymbol{R})}\\ &\leq \frac{5\pi 8^r}{4}c_8(r)\left\|\left(I-T_{\frac{1}{2\sigma}}\right)^r u\right\|_{\mathcal{C}(\boldsymbol{R})}.\end{aligned}$$

Therefore,

$$\begin{aligned}A_{2\sigma}(f)_{p(\cdot)} &\leq 12c_7(c_3(p))A_\sigma(F_f)_{\mathcal{C}(\boldsymbol{R})}\\ &\leq 15\pi 8^r c_7(c_3(p))c_8(r)\left\|\left(I-T_{\frac{1}{2\sigma}}\right)^r F_f\right\|_{\mathcal{C}(\boldsymbol{R})}\\ &= 15\pi 8^r c_7(c_3(p))c_8(r)\left\|F_{\left(I-T_{1/(2\sigma)}\right)^r f}\right\|_{\mathcal{C}(\boldsymbol{R})}\\ &\leq 30\pi 8^r c_5\left(p^+, c_3(p)\right)c_7(c_3(p))c_8(r)\left\|\left(I-T_{1/(2\sigma)}\right)^r f\right\|_{p(\cdot)}.\end{aligned}$$

□

Proof of Theorem 3.16. Let g_σ be an exponential type entire function of degree $\leq \sigma$, belonging to $L^{p(\cdot)}$, as best approximation of $f \in L^{p(\cdot)}$. Then

$$\begin{aligned}\Omega_r(f,\delta)_{p(\cdot)} &= \|(I-T_\delta)^r f\|_{p(\cdot)}\\ &\leq 24c_7(c_3(p))\left\|F_{(I-T_\delta)^r f}\right\|_{\mathcal{C}(\boldsymbol{R})}\\ &= 24c_7(c_3(p))\|(I-T_\delta)^r F_f\|_{\mathcal{C}(\boldsymbol{R})}\\ &\leq 12c_7(c_3(p))\left(1+2^{2r-1}\right)2^r\delta^r\left(A_0(F_f)_{\mathcal{C}(\boldsymbol{R})} + \int_{1/2}^{1/\delta} u^{r-1}A_u(F_f)_{\mathcal{C}(\boldsymbol{R})}\,du\right)\\ &\leq c_{13}12c_7(c_3(p))\left(1+2^{2r-1}\right)2^r\delta^r\left(A_0(f)_{p(\cdot)} + \int_{1/2}^{1/\delta} u^{r-1}A_{u/2}(f)_{p(\cdot)}\,du\right),\end{aligned}$$

because

$$A_{2\sigma}(F_f)_{\mathcal{C}(\boldsymbol{R})} \leq \|F_f - V_\sigma F_f\|_{\mathcal{C}(\boldsymbol{R})} = \|F_{f-V_\sigma f}\|_{\mathcal{C}(\boldsymbol{R})} \leq 2c_5\left(p^+, c_3(p)\right)\|f-V_\sigma f\|_{p(\cdot)}$$

$$
\begin{aligned}
&= 2c_5\left(p^+,c_3(p)\right)\|f-g_\sigma+g_\sigma-V_\sigma f\|_{p(\cdot)}\\
&\le 2c_5\left(p^+,c_3(p)\right)\left(\|f-g_\sigma\|_{p(\cdot)}+\|V_\sigma g_\sigma-V_\sigma f\|_{p(\cdot)}\right)\\
&\le 2c_5\left(p^+,c_3(p)\right)\left(\|f-g_\sigma\|_{p(\cdot)}+72c_7(c_3(p))\,c_5\left(p^+,c_3(p)\right)\|g_\sigma-f\|_{p(\cdot)}\right)\\
&= 2c_5\left(p^+,c_3(p)\right)\left(1+72c_7(c_3(p))\,c_5\left(p^+,c_3(p)\right)\right)A_\sigma(f)_{p(\cdot)}.
\end{aligned}
$$

□

Proof of Theorem 3.17. Let g_σ be an exponential type entire function of degree $\le\sigma$, belonging to $L^{p(\cdot)}$, as best approximation of $f\in L_{p(\cdot)}$. Then

$$
\begin{aligned}
\Omega_r(f,t)_{p(\cdot)} &= \|(I-T_t)^r f\|_{p(\cdot)} \le 24c_7(c_3(p))\left\|F_{(I-T_t)^r f}\right\|_{\mathcal{C}(\boldsymbol{R})}\\
&= 24c_7(c_3(p))\|(I-T_t)^r F_f\|_{\mathcal{C}(\boldsymbol{R})}\\
&\le 24c_7(c_3(p))\,C_9(r,k)\,t^r\int_t^1\frac{\left\|(I-T_u)^{r+k}F_f\right\|_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\\
&= 24c_7(c_3(p))\,C_9(r,k)\,t^r\int_t^1\frac{\left\|F_{(I-T_u)^{r+k}f}\right\|_{\mathcal{C}(\boldsymbol{R})}}{u^{r+1}}du\\
&\le 48c_7(c_3(p))\,C_9(r,k)\,c_5\left(p^+,c_3(p)\right)t^r\int_t^1\frac{\left\|(I-T_u)^{r+k}f\right\|_{p(\cdot)}}{u^{r+1}}du\\
&= 48c_7(c_3(p))\,C_9(r,k)\,c_5\left(p^+,c_3(p)\right)t^r\int_t^1\frac{\Omega_{r+k}(f,u)_{p(\cdot)}}{u^{r+1}}du.
\end{aligned}
$$

□

Proof of Theorem 3.18. Proof of (3.34) is similar to that of proof of Theorem 3.17. □

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