Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 48, Special Issue, 2022, Pages 84–99
https://doi.org/10.30546/2409-4994.48.2022.8499

MATHEMATICAL ANALYSIS AND SIMULATION OF A GIVING UP SMOKING MODEL WITHIN THE SCOPE OF NON-SINGULAR DERIVATIVE

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Abstract. Smoking has caused the illness and death of many people around the world for a long time. Therefore, many researchers have investigated many methods to quit smoking and reduce its use. In this paper, a smoking model with determination is first examined broadly with Caputo-Fabrizio derivative. We give the equilibrium points and their stability analysis for our fractional model. Taking advantage of fixed point theory, we give some solution properties for the fractional smoking model with determination. Our findings are highlighted by presenting some explanatory graphics.

1. Introduction

There are certain matters and values which render life meaningful in the eyes of the humankind, such as hope, love, home, family, etc. And on the other side of the coin, there are things stealing our lives away from our hands; these are matters and values such as poverty, pain, illness, alcohol, and cigarettes. What a pity it is to waste our lives with a substance called cigarettes and made of tobacco, inducing only temporary pleasures.

The hazards of smoking are taught in schools, shown on television, and even written on cigarette packs. Due to the harmful effects of carbon dioxide and hazardous heavy metals, the human body cannot endure and therefore deteriorates. Smoking causes cancer, heart attack, and skin diseases. Those who feel these hazardous effects at the utmost are, of course, smokers, because they sense the impact each inhalation has on their bodies. Their daily lives become harder, even climbing up the stairs turns into a challenge. Without cigarettes, smokers feel unhappy with the activities in which they used to take pleasure before. Smoking is no longer a means of pleasure, turning into a substance to which smokers are addicted.

Mathematical modeling becomes prominent when it comes to preventing the aforementioned health hazards and illnesses. There exists a good deal of studies analyzing numerous mathematical models [8, 9, 11, 14, 16, 20, 24, 30, 33, 34]. In 1993 Perelson et al. [19] studied a model for the interaction of HIV with CD4

2010 *Mathematics Subject Classification.* 34A08, 34A34, 47H10.
Key words and phrases. fractional differential equations, fixed point theory, Caputo-Fabrizio derivative.

+ T cells. Straughan [21] studied e-cigarette smoking with peer pressure model, Ahn et al. [1] gave basic SEIRA model related to computer viruses, Mulone et al. [15] examined a two-stage model for youths with critical drinking problems and their cure, Han [6] presented dynamical behavior of computer virus on Internet.

Following upon that, fractional analysis has lately come to be a notion of vital importance concerning the questions how the said models are built and what their influences are. In the light of the relevant works authored by numerous scholars, it can be seen that the fractional models illustrate even the actual construction more effectively compared to the conventional ones. In 2020, Ullah et al. [28] considered fractional tuberculosis infection disease, Uçar et al. [26] touched on fractional glucose-insulin regulatory system for the first time. Baleanu et al. [2] presented the existence of a unique solution of human liver model by using exponential kernel derivative and Picard-Lindelof approach. Other excellent papers can be found in [4, 5, 7, 10, 17, 18, 23, 25, 27, 32].

In the present work, observing the great significance of CF fractional derivative we intend to promote the application of the CF derivative to the smoking model and present the detailed stability analysis and solution properties. After that, we interpret the effect of this fractional derivative enriching several numerical examples and also consider the description memory and hereditary properties of fractional order models. We deal with a smoking model with determination presented by [31] of the following integer form:

$$\begin{aligned}
\frac{dP}{dt} &= \mu N - \beta \frac{PS}{N} + \alpha (1-\varepsilon) S - \mu P, \\
\frac{dS}{dt} &= \beta \frac{PS}{N} - \mu S - \alpha S, \\
\frac{dQ}{dt} &= \alpha \varepsilon S - \mu Q,
\end{aligned} \tag{1.1}$$

where we believe that during the modeling phase the total population (N) is constant for all periods t. The whole population is divided into three subclassess: Potential Smokers (P), Smokers (S) and Quitters (Q). Potential smokers are those people who are susceptible to smoking; those who regularly smoke are Smokers, and those who have stopped smoking are Quitters. Therefore, the cumulative population $N = P + S + Q$. Let μ be inflow rate of people in the group of potential smokers. It also demonstrates per capita natural death in each unit. Let β be the rate of smoking habit transmission, so that the smoking incidence rate is denoted by $\beta \frac{PS}{N}$. We further believe that smokers quit smoking at a ratio αS. A fraction $\alpha (1-\varepsilon) S$ of these quitters will return to potential smoker community due to a poor degree of determination and the remaining $\alpha \varepsilon S$ will lead to community of quitters. ε is the size of determination. It should be remembered that if the determination of the quitter is 100%, then those who quit will switch to Q.

Since N is constant, we can take $p = \frac{P}{N}$, $s = \frac{S}{N}$, $q = \frac{Q}{N}$ to obtain:

$$\begin{aligned} \frac{dp}{dt} &= \mu - \beta ps + \alpha(1-\varepsilon)s - \mu p \\ \frac{ds}{dt} &= \beta ps - \mu s - \alpha s, \\ \frac{dq}{dt} &= \alpha\varepsilon s - \mu q \end{aligned} \tag{1.2}$$

with the initial conditions $p(0) \geq 0,\ s(0) \geq 0,\ q(0) \geq 0$.

The paper proceeds as follows. In Section 2, we give some basic definitions related to Caputo-Fabrizio (CF) derivative that we used. Section 3 introduce equilibrium points and their stability analysis for this fractional model. Section 4 presents existence and uniqueness conditions for the fractional smoking model with determination. In Section 5, the simulation results are given and briefly interpreted. Finally, this work is concluded in Section 6.

2. Some preliminaries

In this part, we give basic definitions about to the CF derivative.

Definition 2.1. [3] Let $a < b$, $g \in H^1(a,b)$ and $\eta \in [0,1]$, the Caputo-Fabrizio derivative is defined as

$$D_t^\eta g(t) = \frac{M(\eta)}{1-\eta} \int_a^t g'(x) \exp\left[-\eta \frac{t-x}{1-\eta}\right] dx, \tag{2.1}$$

where $M(\eta)$ is a normalization function with $M(0) = M(1) = 1$. If $g \notin H^1(a,b)$, this derivative can be written of the following form:

$$D_t^\eta g(t) = \frac{\eta M(\eta)}{1-\eta} \int_a^t (g(t) - g(x)) \exp\left[-\eta \frac{t-x}{1-\eta}\right] dx. \tag{2.2}$$

Remark 2.1. If $\varsigma = \frac{1-\eta}{\eta} \in [0,\infty)$, $\eta = \frac{1}{1+\varsigma} \in [0,1]$, then Eq. (5.2) is given by:

$$D_t^\eta g(t) = \frac{N(\varsigma)}{\varsigma} \int_a^t g'(x) \exp\left[-\frac{t-x}{\varsigma}\right] dx,$$

with $N(0) = N(\infty) = 1$. Moreover,

$$\lim_{\varsigma \to 0} \frac{1}{\varsigma} \exp\left[-\frac{t-x}{\varsigma}\right] = \delta(x-t).$$

Definition 2.2. Let $0 < \eta < 1$ and g be a function. The fractional integral of order η is given by [13]:

$$I_t^\eta g(t) = \frac{2(1-\eta)}{(2-\eta)M(\eta)} g(t) + \frac{2\eta}{(2-\eta)M(\eta)} \int_0^t g(s)\, ds, \quad t \geq 0. \tag{2.3}$$

Additionally, the below result satisfy

$$\frac{2(1-\eta)}{(2-\eta)M(\eta)} + \frac{2\eta}{(2-\eta)M(\eta)} = 1,$$

then $M(\tau) = \frac{2}{2-\eta}$ for $0 < \eta < 1$.

Using these results, another form of the new Caputo derivative of order $0 < \eta < 1$ given as [13]:

$$D_t^{\eta} g(t) = \frac{1}{1-\eta} \int_a^t g'(x) \exp\left[-\eta \frac{t-x}{1-\eta}\right] dx. \tag{2.4}$$

Let $0 < \eta < 1$. The time fractional ordinary differential equation

$${}_0^{CF}D_t^{\eta} f(t) = u(t),$$

has a unique solution using the inverse Laplace transform and from the convolution theorem below:

$$f(t) = \frac{2(1-\eta)}{(2-\eta)M(\eta)} u(t) + \frac{2\eta}{(2-\eta)M(\eta)} \int_a^t u(s)\, ds, \quad t \geq 0.$$

3. Equilibrium points and their stability analysis

Here, we extend the model (1.2) using CF derivative:

$$\begin{aligned} {}_0^{CF}D_t^{\eta} p(t) &= \mu^{\eta} - \beta^{\eta} ps + \alpha^{\eta}(1-\varepsilon^{\eta}) s - \mu^{\eta} p, \\ {}_0^{CF}D_t^{\eta} s(t) &= \beta^{\eta} ps - \mu^{\eta} s - \alpha^{\eta} s, \\ {}_0^{CF}D_t^{\eta} q(t) &= \alpha^{\eta}\varepsilon^{\eta} s - \mu^{\eta} q. \end{aligned} \tag{3.1}$$

with the initial condition $p(0) = m_1, s(0) = m_2, q(0) = m_3$ where ${}_0^{CF}D_t^{\eta}$ is CF derivative.

In order to get the equilibrium points of fractional order model (3.1), we obtain the points satisfying the following equations:

$${}_0^{CF}D_t^{\eta} p(t) = {}_0^{CF}D_t^{\eta} s(t) = {}_0^{CF}D_t^{\eta} q(t) = 0.$$

The smoking-free steady state $E^0 = (1, 0, 0)$ and the smoking-persistent steady state $E^* = (p^*, s^*, q^*)$ where $p^* = \frac{\mu^{\eta}+\alpha^{\eta}}{\beta^{\eta}}$, $s^* = \frac{\mu(p^*-1)}{-\beta^{\eta}p^*+\alpha^{\eta}(1-\varepsilon^{\eta})}$, $q^* = \frac{\alpha^{\eta}\varepsilon^{\eta}}{\mu^{\eta}} s^*$. In order to investigate basic reproduction number we benefit from the method given in [29]. Using the said method, the matrices $\widetilde{F}$ and $\widetilde{V}$ given by

$$\widetilde{F} = \begin{bmatrix} \beta^{\eta} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \widetilde{V} = \begin{bmatrix} \mu^{\eta}+\alpha^{\eta} & 0 \\ \alpha^{\eta}\varepsilon^{\eta} & \mu^{\eta} \end{bmatrix}.$$

To get the eigenvalues of the matrix $\widetilde{F}\widetilde{V}^{-1}$ at the point at E^0, we need the solve this equation:

$$\left|\widetilde{F}\widetilde{V}^{-1} - jI\right| = 0,$$

where j are the eigenvalues of the matrix $\widetilde{F}\widetilde{V}^{-1}$ and I is the identity matrix. From Theorem 2 in [29], the reproduction number is

$$R_0 = \frac{\beta^\eta}{\mu^\eta + \alpha^\eta}. \tag{3.2}$$

The Jacobian matrix for the model (3.1) is given by

$$J = \begin{bmatrix} \beta^\eta s - \mu^\eta & -\beta^\eta p + \alpha^\eta (1 - \varepsilon^\eta) & 0 \\ \beta^\eta & \beta^\eta p - (\mu^\eta + \alpha^\eta) & 0 \\ 0 & \alpha^\eta \varepsilon^\eta & -\mu^\eta \end{bmatrix}.$$

The corresponding Jacobian matrix of the point E^0 is

$$J1 = \begin{bmatrix} \mu^\eta & -\beta^\eta + \alpha^\eta (1 - \varepsilon^\eta) & 0 \\ 0 & \beta^\eta - \mu^\eta - \alpha^\eta & 0 \\ 0 & \alpha^\eta \varepsilon^\eta & -\mu^\eta \end{bmatrix}.$$

The characteristic equation of $J1$ is

$$(\lambda + \mu^\eta)(\lambda + \mu^\eta)(\lambda - \beta^\eta + \mu^\eta + \alpha^\eta) = 0.$$

It is clear that all roots are negative if $\beta^\eta < \mu^\eta + \alpha^\eta$namely $R_0 < 1$. From Theorem 1 in [12], if $R_0 < 1$ the smoking-free steady state E^0 of model (3.1) is asymptotically stable.

The Jacobian matrix of the point E^* is

$$J2 = \begin{bmatrix} -\beta^\eta s^* - \mu^\eta & -\beta^\eta p^* + \alpha^\eta (1 - \varepsilon^\eta) & 0 \\ \beta^\eta s^* & \beta^\eta p^* - \mu^\eta - \alpha^\eta & 0 \\ 0 & \alpha^\eta \varepsilon^\eta & -\mu^\eta \end{bmatrix}.$$

The characteristic equation of $J2$ is

$$(\lambda + \mu^\eta)\left[(\lambda + \mu^\eta + \beta^\eta s^*)(\lambda - \beta^\eta p^* + \mu^\eta + \alpha^\eta) - \beta^\eta s^* (\alpha^\eta (1 - \varepsilon^\eta) - \beta^\eta p^*)\right] = 0. \tag{3.3}$$

One of the root of the Eq. (3.3) is $\lambda_1 = -\mu^\eta$ and the remaining of the solution is the root of the following quadratic equation

$$\lambda^2 + A\lambda + B = 0$$

where $A = \mu^\eta + \beta^\eta p^* > 0$, if $R_0 > 1$ then $B = \beta^\eta s^* (\mu^\eta + \alpha^\eta \varepsilon^\eta) > 0$. So all roots of the Eq. (3.3) are negative or with negative real parts. From Theorem 1 in [12], if $R_0 > 1$ the smoking-persistent steady state E^* of model (3.1) is asymptotically stable.

4. Existence and uniqueness of the smoking model with determination

Applying fractional integral to Eq. (3.1), we have

$$\begin{aligned} p(t) - p(0) &= {}_0^{CF} I_t^\eta \left\{ \mu^\eta - \beta^\eta p s + \alpha^\eta (1 - \varepsilon^\eta) s - \mu^\eta p \right\}, \\ s(t) - s(0) &= {}_0^{CF} I_t^\eta \left\{ \beta^\eta p s - \mu^\eta s - \alpha^\eta s \right\}, \\ q(t) - q(0) &= {}_0^{CF} I_t^\eta \left\{ \alpha^\eta \varepsilon^\eta s - \mu^\eta q \right\}. \end{aligned}$$

Using the notation introduced by Losada and Nieto [13], we have

$$
\begin{aligned}
p(t)-p(0) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}\left\{\mu^{\eta}-\beta^{\eta}ps+\alpha^{\eta}(1-\varepsilon^{\eta})s-\mu^{\eta}p\right\} \\
&+\frac{2\eta}{(2-\eta)M(\eta)}\int_{0}^{t}\left\{\mu^{\eta}-\beta^{\eta}ps+\alpha^{\eta}(1-\varepsilon^{\eta})s-\mu^{\eta}p\right\}d\lambda, \\
s(t)-s(0) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}\left\{\beta^{\eta}ps-\mu^{\eta}s-\alpha^{\eta}s\right\} \\
&+\frac{2\eta}{(2-\eta)M(\eta)}\int_{0}^{t}\left\{\beta^{\eta}ps-\mu^{\eta}s-\alpha^{\eta}s\right\}d\lambda \\
q(t)-q(0) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}\left\{\alpha^{\eta}\varepsilon^{\eta}s-\mu^{\eta}q\right\} \\
&+\frac{2\eta}{(2-\eta)M(\eta)}\int_{0}^{t}\left\{\alpha^{\eta}\varepsilon^{\eta}s-\mu^{\eta}q\right\}d\lambda, \qquad (4.1)
\end{aligned}
$$

To simplify, we write

$$
\begin{aligned}
K_1(t,p) &= \mu^{\eta}-\beta^{\eta}ps+\alpha^{\eta}(1-\varepsilon^{\eta})s-\mu^{\eta}p, \\
K_2(t,s) &= \beta^{\eta}ps-\mu^{\eta}s-\alpha^{\eta}s, \qquad (4.2)\\
K_3(t,q) &= \alpha^{\eta}\varepsilon^{\eta}s-\mu^{\eta}q.
\end{aligned}
$$

Theorem 4.1. *The kernel K_1 satisfies Lipschitz condition and contraction if the following inequality holds:*

$$0<\beta^{\eta}b+\mu^{\eta}\leq 1.$$

Proof. Let p and p_1 be two functions, then we have

$$
\begin{aligned}
\left\|K_1(t,p)-K_1(t,p_1)\right\| &= \left\|(-\beta^{\eta}s+\mu^{\eta})(p(t)-p_1(t))\right\| \\
&\leq \left[\beta^{\eta}\left\|s(t)\right\|+\mu^{\eta}\right]\left\|p(t)-p_1(t)\right\|.
\end{aligned}
$$

Let $\phi_1=\beta^{\eta}b+\mu^{\eta}$ and $\|p(t)\|\leq a, \|s(t)\|\leq b, \|q(t)\|\leq c$. So we find

$$\left\|K_1(t,p)-K_1(t,p_1)\right\|\leq\phi_1\left\|p(t)-p_1(t)\right\|. \qquad (4.3)$$

Thus, the Lipschitz condition is valid for p_1 and $0<\beta^{\eta}b+\mu^{\eta}\leq 1$ yields K_1 is contraction. □

Similarly, the other kernels K_2 and K_3 satisfy the Lipschitz condition and contraction.

$$
\begin{aligned}
\left\|K_2(t,s)-K_2(t,s_1)\right\| &\leq \phi_2\left\|s(t)-s_1(t)\right\|, \\
\left\|K_3(t,q)-K_3(t,q_1)\right\| &\leq \phi_3\left\|q(t)-q_1(t)\right\|. \qquad (4.4)
\end{aligned}
$$

Using the so-called kernels, Eq. (4.1) becomes

$$\begin{aligned}
p(t) &= p(0) + \frac{2(1-\eta)}{(2-\eta)M(\eta)}K_1(t,p) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t K_1(\lambda,p)\,d\lambda,\\
s(t) &= s(0) + \frac{2(1-\eta)}{(2-\eta)M(\eta)}K_2(t,s) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t K_2(\lambda,s)\,d\lambda,\\
q(t) &= q(0) + \frac{2(1-\eta)}{(2-\eta)M(\eta)}K_3(t,q) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t K_3(\lambda,q)\,d\lambda.
\end{aligned} \tag{4.5}$$

Now, we concentrate the following recursive formula:

$$\begin{aligned}
p_n(t) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}K_1(t,p_{n-1}) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t K_1(\lambda,p_{n-1})\,d\lambda,\\
s_n(t) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}K_2(t,s_{n-1}) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t K_2(\lambda,s_{n-1})\,d\lambda,\\
q_n(t) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}K_3(t,q_{n-1}) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t K_3(\lambda,q_{n-1})\,d\lambda.
\end{aligned} \tag{4.6}$$

The initial conditions are given below

$$\begin{aligned}
p_0(t) &= p(0),\\
s_0(t) &= s(0),\\
q_0(t) &= q(0).
\end{aligned} \tag{4.7}$$

We give the difference between successive terms as follows:

$$\begin{aligned}
\Psi_{1n}^*(t) = p_n(t) - p_{n-1}(t) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}\{K_1(t,p_{n-1}) - K_1(t,p_{n-2})\}\\
&+ \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \{K_1(\lambda,p_{n-1}) - K_1(\lambda,p_{n-2})\}\,d\lambda,\\
\Psi_{2n}^*(t) = s_n(t) - s_{n-1}(t) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}\{K_2(t,s_{n-1}) - K_2(t,s_{n-2})\}\\
&+ \frac{2\eta}{(2-\eta)M(\eta)}\left\{\int_0^t K_2(\lambda,s_{n-1}) - K_2(\lambda,s_{n-2})\right\}d\lambda,\\
\Psi_{3n}^*(t) = q_n(t) - q_{n-1}(t) &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}\{K_3(t,q_{n-1}) - K_3(t,q_{n-2})\}\\
&+ \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \{K_3(\lambda,q_{n-1}) - K_3(\lambda,q_{n-2})\}\,d\lambda.
\end{aligned} \tag{4.8}$$

It is evident that

$$p_n(t) = \sum_{i=0}^{n} \Psi_{1i}^{*}(t),$$
$$s_n(t) = \sum_{i=0}^{n} \Psi_{2i}^{*}(t), \tag{4.9}$$
$$q_n(t) = \sum_{i=0}^{n} \Psi_{3i}^{*}(t).$$

Taking the norm Eq. (4.8) and using triangular identity, we have

$$\begin{aligned} \left\|\Psi_{1n}^{*}(t)\right\| &= \left\|p_n(t) - p_{n-1}(t)\right\| \\ &= \left\| \frac{2(1-\eta)}{(2-\eta)M(\eta)} \left\{K_1(t, p_{n-1}) - K_1(t, p_{n-2})\right\} \right. \\ &\left. + \frac{2\eta}{(2-\eta)M(\eta)} \int_0^t \left\{K_1(\lambda, p_{n-1}) - K_1(\lambda, p_{n-2})\right\} d\lambda \right\|. \end{aligned} \tag{4.10}$$

Because the kernel verifies Lipschitz condition, we gain

$$\begin{aligned} \left\|\Psi_{1n}^{*}(t)\right\| &= \left\|p_n(t) - p_{n-1}(t)\right\| \\ &\leq \frac{2(1-\eta)}{(2-\eta)M(\eta)} \phi_1 \left\|p_{n-1} - p_{n-2}\right\| + \frac{2\eta}{(2-\eta)M(\eta)} \phi_1 \int_0^t \left\|p_{n-1} - p_{n-2}\right\| d\lambda. \end{aligned} \tag{4.11}$$

and

$$\left\|\Psi_{1n}^{*}(t)\right\| \leq \frac{2(1-\eta)}{(2-\eta)M(\eta)} \phi_1 \left\|\Psi_{1(n-1)}^{*}(t)\right\| + \frac{2\eta}{(2-\eta)M(\eta)} \phi_1 \int_0^t \left\|\Psi_{1(n-1)}^{*}(\lambda)\right\| d\lambda. \tag{4.12}$$

Analogously, we get the below identities:

$$\begin{aligned} \left\|\Psi_{2n}^{*}(t)\right\| &\leq \frac{2(1-\eta)}{(2-\eta)M(\eta)} \phi_2 \left\|\Psi_{2(n-1)}^{*}(t)\right\| + \frac{2\eta}{(2-\eta)M(\eta)} \phi_2 \int_0^t \left\|\Psi_{2(n-1)}^{*}(\lambda)\right\| d\lambda, \\ \left\|\Psi_{3n}^{*}(t)\right\| &\leq \frac{2(1-\eta)}{(2-\eta)M(\eta)} \phi_3 \left\|\Psi_{3(n-1)}^{*}(t)\right\| + \frac{2\eta}{(2-\eta)M(\eta)} \phi_3 \int_0^t \left\|\Psi_{3(n-1)}^{*}(\lambda)\right\| d\lambda. \end{aligned} \tag{4.13}$$

In the light of above results, we give the following theorem.

Theorem 4.2. *Fractional model given in (3.1) has a solution, if we can find* t_0 *such that*

$$\frac{2(1-\eta)}{(2-\eta)M(\eta)} \phi_i + \frac{2\eta}{(2-\eta)M(\eta)} \phi_i t_0 < 1,$$

for $i = 1, 2, 3$.

Proof. We know that $p(t)$, $s(t)$, $q(t)$ are bounded functions and the kernels provide Lipschitz condition. Using Eqs. (4.12) and (4.13), we have the succeeding relations as below:

$$\begin{aligned}
\left\|\Psi_{1n}^{*}(t)\right\| &\leq \left\|p_n(0)\right\| \left[\frac{2(1-\eta)}{(2-\eta)M(\eta)}\phi_1 + \frac{2\eta}{(2-\eta)M(\eta)}\phi_1 t\right]^n, \\
\left\|\Psi_{2n}^{*}(t)\right\| &\leq \left\|s_n(0)\right\| \left[\frac{2(1-\eta)}{(2-\eta)M(\eta)}\phi_2 + \frac{2\eta}{(2-\eta)M(\eta)}\phi_2 t\right]^n, \\
\left\|\Psi_{3n}^{*}(t)\right\| &\leq \left\|q_n(0)\right\| \left[\frac{2(1-\eta)}{(2-\eta)M(\eta)}\phi_3 + \frac{2\eta}{(2-\eta)M(\eta)}\phi_3 t\right]^n. \qquad (4.14)
\end{aligned}$$

So, the existence and continuity of the above solutions are showed. We aim to show that the above functions are solution of Eq. (3.1), assume that

$$\begin{aligned}
p(t) - p(0) &= p_n(t) - d_{1n}(t), \\
s(t) - s(0) &= s_n(t) - d_{2n}(t) \\
q(t) - q(0) &= q_n(t) - d_{3n}(t). \qquad (4.15)
\end{aligned}$$

Thus,we have

$$\begin{aligned}
\left\|d_{1n}(t)\right\| &= \left\| \frac{2(1-\eta)}{(2-\eta)M(\eta)} \left\{K_1(t,p) - K_1(t,p_{n-1})\right\} \right. \\
&\quad \left. + \frac{2\eta}{(2-\eta)M(\eta)} \int_0^t \left\{K_1(\lambda,p) - K_1(\lambda,p_{n-1})\right\} d\lambda \right\| \\
&\leq \frac{2(1-\eta)}{(2-\eta)M(\eta)} \left\|K_1(t,p) - K_1(t,p_{n-1})\right\| \\
&\quad + \frac{2\eta}{(2-\eta)M(\eta)} \int_0^t \left\|K_1(\lambda,p) - K_1(\lambda,p_{n-1})\, d\lambda\right\| \\
&\leq \frac{2(1-\eta)}{(2-\eta)M(\eta)} \phi_1 \left\|p - p_{n-1}\right\| + \frac{2\eta}{(2-\eta)M(\eta)} \phi_1 t \left\|p - p_{n-1}\right\|. \qquad (4.16)
\end{aligned}$$

By continuing this process recursively, it gives at t_0

$$\left\|d_{1n}(t)\right\| \leq \left(\frac{2(1-\eta)}{(2-\eta)M(\eta)} + \frac{2\eta}{(2-\eta)M(\eta)} t_0\right)^{n+1} \phi_1^{n+1} a. \qquad (4.17)$$

As n approaches to ∞, $\left\|d_{1n}(t)\right\|$ tends to 0. Similarly, we obtain

$$\left\|d_{2n}(t)\right\| \to 0 \text{ and } \left\|d_{3n}(t)\right\| \to 0. \qquad (4.18)$$

□

To prove the uniqueness of the solutions for the model (3.1), we give the below steps. Let $p_1(t)$, $s_1(t)$ and $q_1(t)$ be another solutions, we have

$$p(t) - p_1(t) = \frac{2(1-\eta)}{(2-\eta)M(\eta)}\{K_1(t,p) - K_1(t,p_1)\} + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \{K_1(\lambda,p) - K_1(\lambda,P_1)\}\,d\lambda. \tag{4.19}$$

Applying norm to Eq. (4.19) and utilizing the fact that the kernel satisfies Lipschitz condition, we find

$$\|p(t) - p_1(t)\| \leq \frac{2(1-\eta)}{(2-\eta)M(\eta)}\phi_1\|p(t) - p_1(t)\| + \frac{2\eta}{(2-\eta)M(\eta)}\phi_1 t\|p(t) - p_1(t)\|. \tag{4.20}$$

This gives

$$\|p(t) - p_1(t)\|\left(1 - \frac{2(1-\eta)}{(2-\eta)M(\eta)}\phi_1 - \frac{2\eta}{(2-\eta)M(\eta)}\phi_1 t\right) \leq 0. \tag{4.21}$$

If the following inequality holds

$$\left(1 - \frac{2(1-\eta)}{(2-\eta)M(\eta)}\phi_1 - \frac{2\eta}{(2-\eta)M(\eta)}\phi_1 t\right) \geq 0$$

then $\|p(t) - p_1(t)\| = 0$.Thus we have

$$p(t) = p_1(t).$$

Benefiting from same steps, we find

$$s(t) = s_1(t) \text{ and } q(t) = q_1(t).$$

5. Numerical Results

In order to get solution for nonlinear differential equation in the sense of CF operator, Toh et al. [22] has enhanced a new simple numerical method. We take account of the following fractional ordinary differential equation to give their scheme:

$${}_0^{CF}D_t^{\eta} g(x) = f(x, g(x)), \tag{5.1}$$

with the initial condition $i = 0, 1, ..., n-1$ where $n = \lceil \eta \rceil$ and $g^{(i)}(0) = g_{(0)}^{(i)}$.

Theorem 5.1. [22] *The initial value problem in Eq. (5.1) is*

$$g(x) = T_{n-1}(x) + \frac{1-(\eta)}{M(\eta)(n-2)!}\int_0^x (x-t)^{n-2} f(t, g(t))\,dt + \frac{\eta}{M(\eta)(n-1)!}\int_0^x (x-t)^{n-1} f(t, g(t))\,dt, \tag{5.2}$$

where $T_{n-1}(x)$ *is the Taylor expansion of* $g(x)$ *centered at* $x_0 = 0$

$$T_{n-1}(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!} g_{(0)}^{(i)}. \tag{5.3}$$

Proof. Applying Laplace transform to both sides of Eq. (5.1), we get

$$L\left[{}_0^{CF}D_t^{\eta} g(x)\right] = L\left[f(x)\right], \ s > 0,$$

$$\frac{M(\eta)}{s(1-\eta)+\eta}\left(s^n F(s) - \sum_{i=0}^{n-1} s^{n-k} g^{(k-1)}(0)\right) = G(s).$$

So,

$$\begin{aligned} F(s) &= \frac{1}{s^n}\sum_{i=0}^{n-1} s^{n-k} g^{(k-1)}(0) \frac{s(1-\eta)}{s^n M(\eta)} G(s) + \frac{\eta}{s^n M(\eta)} G(s) \\ &= \sum_{i=0}^{n-1} \frac{1}{s^k} g^{(k-1)}(0) + \frac{1-\eta}{M(\eta)}\left(\frac{1}{s^{n-1}} G(s)\right) + \frac{\eta}{M(\eta)}\left(\frac{1}{s^n} G(s)\right). \end{aligned}$$

Taking into consideration inverse Laplace transform properties

$$\begin{aligned} g(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!} g^{(i)}(0) &+ \frac{1-\eta}{M(\eta)}\left(\frac{1}{\Gamma(n-1)} x^{n-2} * f(x)\right) \\ &+ \frac{\eta}{M(\eta)}\left(\frac{1}{\Gamma(n)} x^{n-1} * f(x)\right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} g(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!} g^{(i)}(0) &+ \frac{1-\eta}{(n-2)!M(\eta)} \int_0^x (x-t)^{n-2} f(t)\, dt \\ &+ \frac{\eta}{(n-1)!M(\eta)} \int_0^x (x-t)^{n-1} f(t)\, dt. \end{aligned}$$

This gives the proof of Theorem 5.1. □

Furthermore, the initial value problem in Eq. (5.2) can be approximated and obtained from Adams-Bashforth-Moulton method.

$$\begin{aligned} g(x_{k+1}) &= T_{n-1}(x_{k+1}) + \frac{1-\eta}{M(\eta)(n-2)!} \int_0^{x_{k+1}} (x_{k+1}-t)^{n-2} u(t)\, dt \\ &+ \frac{\eta}{M(\eta)(n-1)!} \int_0^{x_{k+1}} (x_{k+1}-t)^{n-1} u(t)\, dt. \end{aligned} \tag{5.4}$$

The integration part can be approximated by

$$\begin{aligned}\int_0^{x_{k+1}} (x_{k+1}-t)^{n-2}\, u(t)\, dt &\approx \int_0^{x_{k+1}} (x_{k+1}-t)^{n-2}\, \widetilde{u}_{k+1}(t)\, dt,\\ \int_0^{x_{k+1}} (x_{k+1}-t)^{n-1}\, u(t)\, dt &\approx \int_0^{x_{k+1}} (x_{k+1}-t)^{n-1}\, \widetilde{u}_{k+1}(t)\, dt, \end{aligned} \tag{5.5}$$

where $\widetilde{u}_{k+1}(t)$ is the approximation of $u(t)$.

By determining $\widetilde{u}_{k+1}(t)|_{[x_i,x_{i+1})} = u(x_i)$ with $0 \le i \le k$ or in order words, we instead integrate the part from Eq. (5.5) by the rectangle rule, we find the explicit method, fractional Euler method. The approximation solution, $g(x_i) \approx g_i$ at discrete space x_i.

Hence, the predictor formula, g^p_{k+1} can be determined by the fractional Adams-Bashforth method as below:

$$\begin{aligned} g^p_{k+1} = T_{n-1}(x_{k+1}) &+ \frac{1-\eta}{M(\eta)(n-2)!}\sum_{i=0}^{k} c_{i,k+1} f(x_i, g_i)\\ &+ \frac{\eta}{M(\eta)(n-1)!}\sum_{i=0}^{k} d_{i,k+1} f(x_i, g_i). \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} c_{i,k+1} &= \frac{h^{n-1}}{n-1}\left[(k-i+1)^{n-1} - (k-i)^{n-1}\right],\\ d_{i,k+1} &= \frac{h^{n}}{n}\left[(k-i+1)^{n} - (k-i)^{n-1}\right]. \end{aligned} \tag{5.7}$$

If $\widetilde{u}_{k+1}(s)|_{[x_i,x_{i+1})} = \frac{x_{k+1}-s}{h} u(x_i) + \frac{s-x_i}{h} u(x_{i+1})$, we get the implicit method, fractional trapezoidal rule. The corrector formula g_{k+1} can be determined by the fractional Adams-Moulton method as below:

$$\begin{aligned} g_{k+1} = T_{n-1}(x_{k+1}) &+ \frac{1-\eta}{M(\eta)(n-2)!}\left[\sum_{i=0}^{k} a_{i,k+1} f(x_i, g_i) + a_{k+1,k+1} f\left(x_{k+1}, g^p_{k+1}\right)\right]\\ &+ \frac{\eta}{M(\eta)(n-1)!}\left[\sum_{i=0}^{k} b_{i,k+1} f(x_i, g_i) + b_{k+1,k+1} f\left(x_{k+1}, g^p_{k+1}\right)\right], \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} a_{i,k+1} &= \frac{h^{n-1}}{n(n-1)}\begin{cases} k^n - (k+1)^{n-1}(k-n+1), & i=0\\ (k-i+2)^n - 2(k-i+1)^n + (k-i)^n, & 1 \le t \le k\\ 1, & i=k+1 \end{cases},\\ b_{i,k+1} &= \frac{h^{n}}{n(n+1)}\begin{cases} k^{n+1} - (k+1)^{n}(k-n), & i=0\\ (k-i+2)^{n+1} - 2(k-i+1)^{n+1} + (k-i)^{n+1}, & 1 \le t \le k\\ 1, & i=k+1 \end{cases}. \end{aligned} \tag{5.9}$$

We find the predictor in Eq. (5.6) by calculating an initial approximation g^p_{k+1} from current value f_k. Meanwhile, the corrector in Eq. (5.8) applies the approximation, g^p_{k+1} to get the refined corrector value of g_{k+1}, which is used in next iteration successively.

Using the above numerical method [22], we illustrate some numerical simulations. With this design, we choose initial values $p(0) = 0.6$, $s(0) = 0.3$, $q(0) = 0.1$, and parameters $\mu = 0.02$, $\beta = 0.4$, $\alpha = 0.05$, $\delta = 0.1$, $\varepsilon = 0.2$ given in [31]. Fig. 1 shows that the number of individuals categorized as smokers quickly escalates in the beginning, then arrives at its maximal point possible in line with various fractional orders. Still, the smoker group diminishes in number when the fractional order also shows a decline. Therefore, in order to restrain the number of smokers and enlarge the group of potential smokers, one can make use of the above-mentioned model featuring small fractional orders, supporting an eventual diminution in the commonness of smoking in the long run. Fig. 2 shows that the attitude of the model components are displayed according to several values of the fractional order η. Furthermore, from Fig. 2 as the fractional order η decreases, the number of potential smokers increases, whereas the number of smokers and quitters reduce .

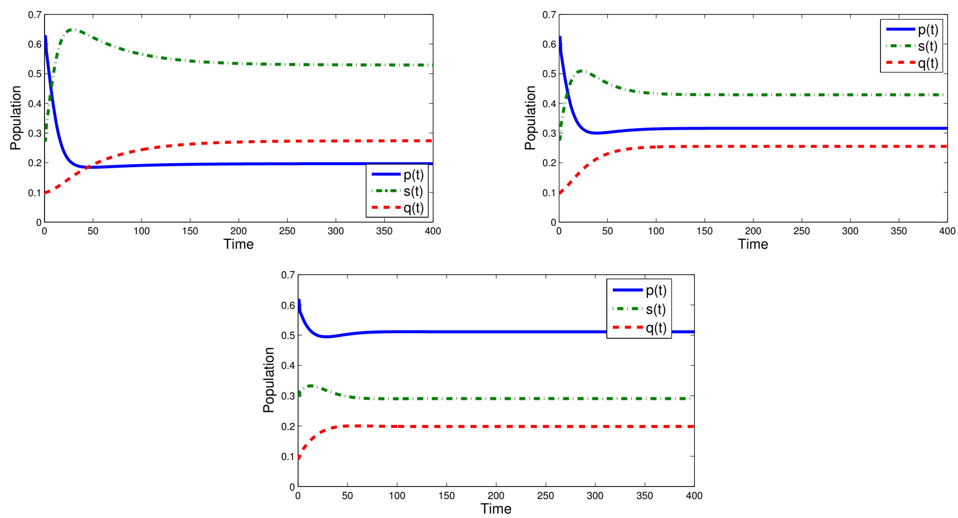

FIGURE 1. Numerical simulations for the Eq. (3.1) at $\eta = 0.95, \eta = 0.75, \eta = 0.55$.

6. Conclusion

The secret to quitting smoking is to be determined about it. There are millions of people in the world who have smoked before but are not smoking now, anyone who is willing can quit smoking. Paying attention to these facts, in this paper, we first touch on the smoking model with determination in [31] enlarging the CF derivative with exponential kernel. The equilibrium points of the fractional model have been calculated and the stability of these points have been investigated. Our theoretical studies give that the solution of the so-called model in Eq. (3.1) is

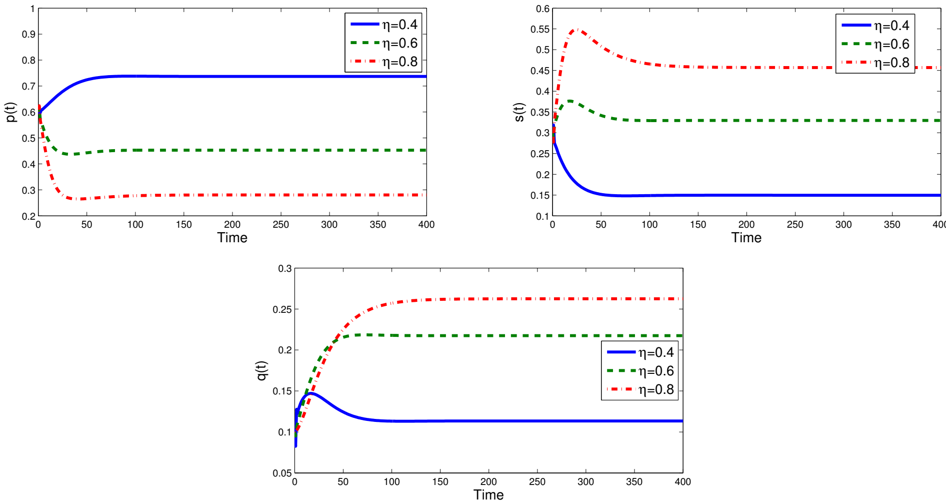

FIGURE 2. The behavior of the fractional smoking model with determination components for distinct values of η.

exist and unique. Then, several numerical graphics are depicted with distinct values of η and shortly interpreted. The use of fractional derivatives gives useful information about the complexity of the dynamics of the smoking model with determination.

Acknowledgements. This research is supported by Balikesir University Research Grant No. Bap 2019/099.

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Received: March 10, 2022; Revised: July 7, 2022; Accepted: July 30, 2022