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Revisiting Some Popular Contractive Conditions For The Fixed-Circle Problem*

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Abstract

The main aim of this paper is to present some fixed-disc theorems as new solutions to the fixed-circle problem. For this purpose, we define the notions of Moradi type a_0-contraction, Geraghty type a_0-contraction, Jleli-Samet type a_0-contraction, Skof type a_0-contraction and Li-Jiang type a_0-contraction modifying the some known contractive conditions which are used to obtain fixed-point theorems. Also, we give an equivalent theorem of some contractions. Finally, we present an application to the rectified linear unit (ReLU) activation function.

1 Introduction and Motivation

The fixed-point theory is one of the most powerful tools of mathematical studies. This theory is a beautiful mixture of topology, analysis, and geometry which has many applications in various fields such as applied mathematics, engineering, activation functions etc. It has gained importance and developed rapidly for the last one and half century. In mathematics, this theory started with the Banach fixed-point theorem [1]. This theorem is also known as the contraction mapping theorem and is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points of self-mappings on metric spaces and provides a method to find those fixed points.

Let X be a nonempty set. A function $T : X \to X$ is called a self-mapping on X. A point $x \in X$ is called a fixed point of a self-mapping $T : X \to X$ if

$$Tx = x.$$

Now, let us consider the following self-mappings $T_1 : \mathbb{R} \to \mathbb{R}$ and $T_2 : \mathbb{R} \to \mathbb{R}$ defined as

$$T_1 a = \frac{\sqrt{a^2+1}-1}{2} + a$$

and

$$T_2 a = \begin{cases} \beta a & , \quad a < 0 \\ a & , \quad a \geq 0 \end{cases},$$

for all $a \in \mathbb{R}$ with parameter β. Then the fixed point set of T_1 is $Fix(T_1) = \{0\}$ and the fixed point set of T_2 is $Fix(T_2) = \{a \in \mathbb{R} : a \geq 0\}$. We can easily say that the self-mapping T_1 has a unique fixed point and the self-mapping T_2 has infinitely number of fixed point. Also, these self-mappings T_1 and T_2 are activation functions. T_1 is a Bent identity activation function and T_2 is a Parametric rectified linear unit activation function (see [4] and the references therein).

On the other hand, if we consider the self-mapping $T_3 : \mathbb{R} \to \mathbb{R}$ defined as

$$T_3 x = \ln(1 + e^x),$$

for all $a \in \mathbb{R}$, then the fixed point set of T_3 is $Fix(T_3) = \emptyset$, that is, T_3 does not have a fixed point. Also, T_3 is a Softplus activation function (see [3]).

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Based on the above reasons, "*Fixed-circle problem*" follows as a geometric generalization to the fixed-point theory when the self-mapping $T : X \to X$ has more than one fixed point [10].

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. Then the circle is defined by

$$C_{a_0,\rho} = \{a \in X : d(a, a_0) = \rho\} .$$

If $Ta = a$ for every $a \in C_{a_0,\rho}$ then $C_{a_0,\rho}$ is called as the fixed circle of T (see [10]).

As a natural consequence of the fixed circle, the notion of a fixed disc was given as follows:

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. Then the disc is defined by

$$D_{a_0,\rho} = \{a \in X : d(a, a_0) \leq \rho\} .$$

If $Ta = a$ for every $a \in D_{a_0,\rho}$ then $D_{a_0,\rho}$ is called as the fixed disc of T (see [11]).

Recently, some solutions was presented by many authors using various techniques and contractive conditions (see, for example [9], [11], [12], [13], [14] [17] and the references therein).

In this paper, we obtain new fixed-disc theorems as new solutions to the fixed-circle problem on metric spaces. To do this, we inspire some popular contractive conditions (see [2], [5], [6], [8], [15] and [16]). We introduce the notions of Moradi type a_0-contraction, Geraghty type a_0-contraction, Jleli-Samet type a_0-contraction, Skof type a_0-contraction and Li-Jiang type a_0-contraction. Using these new notions, we prove five fixed-disc theorems. On the other hand, we give an example to show the validity of our obtained results. Finally, we obtain an equivalent theorem of some contractions and give an application to rectified linear unit (ReLU) activation functions.

2 Some Fixed-Disc Results

In this section, we present new solutions to the fixed-circle problem using new types of contractions as follows:

Definition 1 *Let (X, d) be a metric space and $T : X \to X$ a self-mapping. If there exists $a_0 \in X$ such that*

$$d(Ta, a) > 0 \Longrightarrow \psi\left(d(Ta, a)\right) \leq F\left(\psi\left(d(a, a_0)\right)\right),$$

for all $a \in X$, where the functions $\psi, F : [0, \infty) \to [0, \infty)$ are such that

(i) ψ is nondecreasing with $\psi(0) = 0$ and $0 < \psi(t) < t$ for all $t > 0$,

(ii) F is a function with $F(0) = 0$ and $0 < F(t) < t$ for all $t > 0$,

then T is called Moradi type a_0-contraction.

Theorem 1 (Moradi type fixed-disc theorem) *Let (X, d) be a metric space, $T : X \to X$ Moradi type a_0-contraction with $a_0 \in X$ and ρ defined as*

$$\rho = \inf\left\{d(Ta, a) : Ta \neq a, a \in X\right\} . \tag{1}$$

Then we have

(i) $Ta_0 = a_0$,

(ii) T fixes the circle $C_{a_0,\rho}$,

(iii) T fixes the disc $D_{a_0,\rho}$.

Proof. (i) Assume that $a_0 \neq Ta_0$, that is, $d(Ta_0, a_0) > 0$. Using the Moradi type a_0-contraction hypothesis, we obtain

$$\psi\left(d(Ta_0, a_0)\right) \leq F\left(\psi\left(d(a_0, a_0)\right)\right) = 0,$$

which is a contradiction. So it should be $Ta_0 = a_0$.

(ii) At first, we suppose $\rho = 0$. Then we get $C_{a_0,\rho} = \{a_0\}$. By the condition (i), we have $Ta_0 = a_0$, that is, T fixes the circle $C_{a_0,\rho}$.

Now, let $\rho > 0$ and $a \in C_{a_0,\rho}$ be any point such that $Ta \neq a$. Using the Moradi type a_0-contraction condition, we obtain

$$\psi\left(d(Ta, a)\right) \leq F\left(\psi\left(d(a, a_0)\right)\right) < \psi\left(d(a, a_0)\right) = \psi\left(\rho\right) \leq \psi\left(d(Ta, a)\right),$$

which is a contradiction. Hence, it should be $Ta = a$, that is, T fixes the circle $C_{a_0,\rho}$.

(iii) By the similar arguments used in the proof of the condition (ii), it can be easily proved. ■

Definition 2 *Let (X, d) be a metric space and $T : X \to X$ a self-mapping. If there exists $a_0 \in X$ such that*

$$d(Ta, a) > 0 \Longrightarrow \psi\left(d(Ta, a)\right) \leq \alpha\left(d(a, a_0)\right)\psi\left(d(a, a_0)\right),$$

for all $a \in X$, where the functions $\psi : (0, \infty) \to (0, \infty)$ and $\alpha : (0, \infty) \to (0, 1)$ are such that ψ is nondecreasing then T is called Geraghty type a_0-contraction.

Theorem 2 (Geraghty type fixed-disc theorem) *Let (X, d) be a metric space, $T : X \to X$ Geraghty type a_0-contraction with $a_0 \in X$ and ρ defined as in (1). Then we have*

(i) $Ta_0 = a_0$,

(ii) T fixes the circle $C_{a_0,\rho}$,

(iii) T fixes the disc $D_{a_0,\rho}$.

Proof. (i) Let $d(Ta_0, a_0) > 0$, that is, $a_0 \neq Ta_0$. Using the Geraghty type a_0-contraction hypothesis, we get

$$\psi\left(d(Ta_0, a_0)\right) \leq \alpha\left(d(a_0, a_0)\right)\psi\left(d(a_0, a_0)\right) = \alpha\left(0\right)\psi\left(0\right),$$

which is a contradiction with the definitions of the functions α and ψ. Hence, it should be $Ta_0 = a_0$.

(ii) Assume that $\rho = 0$. Then we get $C_{a_0,\rho} = \{a_0\}$. By the condition (i), we have $Ta_0 = a_0$, that is, T fixes the circle $C_{a_0,\rho}$.

Now, let $\rho > 0$ and $a \in C_{a_0,\rho}$ be any point such that $Ta \neq a$. Using the Geraghty type a_0-contraction condition, we find

$$\begin{aligned}\psi\left(d(Ta, a)\right) &\leq \alpha\left(d(a, a_0)\right)\psi\left(d(a, a_0)\right) = \psi\left(d(Ta, a)\right) \leq \alpha\left(\rho\right)\psi\left(\rho\right) \\ &< \psi\left(\rho\right) \leq \psi\left(d(Ta, a)\right),\end{aligned}$$

which is a contradiction. So, it should be $Ta = a$, that is, T fixes the circle $C_{a_0,\rho}$.

(iii) It can be easily seen by the similar techniques used in the proof of the condition (ii). ■

Definition 3 *Let (X, d) be a metric space and $T : X \to X$ a self-mapping. If there exists $a_0 \in X$ such that*

$$d(Ta, a) > 0 \Longrightarrow \psi\left(d(Ta, a)\right) \leq \left[\psi\left(d(a, a_0)\right)\right]^{\alpha},$$

for all $a \in X$, where $\alpha \in (0, 1)$ and the function $\psi : (0, \infty) \to (1, \infty)$ is such that ψ is nondecreasing then T is called Jleli-Samet type a_0-contraction.

Theorem 3 (Jleli-Samet type fixed-disc theorem) *Let (X, d) be a metric space, $T : X \to X$ Jleli-Samet type a_0-contraction with $a_0 \in X$ and ρ defined as in (1). Then we have* *(i)* $Ta_0 = a_0$,

(ii) T *fixes the circle* $C_{a_0,\rho}$,

(iii) T *fixes the disc* $D_{a_0,\rho}$.

Proof. (i) Let $d(Ta_0, a_0) > 0$, that is, $a_0 \neq Ta_0$. Using the Jleli-Samet type a_0-contraction hypothesis, we obtain

$$\psi\left(d(Ta_0, a_0)\right) \leq \left[\psi\left(d(a_0, a_0)\right)\right]^{\alpha} = \left[\psi\left(0\right)\right]^{\alpha},$$

which is a contradiction with the definition of the function ψ. Thereby, it should be $Ta_0 = a_0$.

(ii) At first, let $\rho = 0$. Then we have $C_{a_0,\rho} = \{a_0\}$. By the condition (i), we get $Ta_0 = a_0$, that is, T fixes the circle $C_{a_0,\rho}$.

Now, suppose $\rho > 0$ and $a \in C_{a_0,\rho}$ is any point such that $Ta \neq a$. Using the Jleli-Samet type a_0-contraction condition, we find

$$\psi\left(d(Ta, a)\right) \leq \left[\psi\left(d(a, a_0)\right)\right]^{\alpha} = \left[\psi\left(\rho\right)\right]^{\alpha} \leq \left[\psi\left(d(Ta, a)\right)\right]^{\alpha},$$

which is a contradiction with $\alpha \in (0, 1)$. So, it should be $Ta = a$, that is, T fixes the circle $C_{a_0,\rho}$.

(iii) It can be easily proved. ∎

Definition 4 *Let* (X, d) *be a metric space and* $T : X \to X$ *a self-mapping. If there exists* $a_0 \in X$ *such that*

$$d(Ta, a) > 0 \Longrightarrow \psi\left(d(Ta, a)\right) \leq a\psi\left(d(a, a_0)\right) + b\psi\left(d(Ta, a)\right) + c\psi\left(d(Ta_0, a_0)\right),$$

for all $a \in X$, *where* $a, b, c \in [0, 1)$ *with* $0 \leq a + b + c < 1$ *and the function* $\psi : [0, \infty) \to [0, \infty)$ *is such that*

(i) ψ *is nondecreasing,*

(ii) $\psi(t) = 0 \Longleftrightarrow t = 0$,

then T *is called Skof type* a_0*-contraction.*

Theorem 4 (Skof type fixed-disc theorem) *Let* (X, d) *be a metric space,* $T : X \to X$ *Skof type* a_0*-contraction with* $a_0 \in X$ *and* ρ *defined as in (1). Then we have*

(i) $Ta_0 = a_0$,

(ii) T *fixes the circle* $C_{a_0,\rho}$,

(iii) T *fixes the disc* $D_{a_0,\rho}$.

Proof. (i) Let $a_0 \neq Ta_0$, that is, $d(Ta_0, a_0) > 0$. Using the Skof type a_0-contraction hypothesis, we get

$$\begin{aligned} \psi\left(d(Ta_0, a_0)\right) &\leq a\psi\left(d(a_0, a_0)\right) + b\psi\left(d(Ta_0, a_0)\right) + c\psi\left(d(Ta_0, a_0)\right) \\ &= a\psi\left(0\right) + (b + c)\psi\left(d(Ta_0, a_0)\right) \\ &= (b + c)\psi\left(d(Ta_0, a_0)\right) < \psi\left(d(Ta_0, a_0)\right), \end{aligned}$$

which is a contradiction with $b + c < 1$. Then it should be $Ta_0 = a_0$.

(ii) Let $\rho = 0$. Then we get $C_{a_0,\rho} = \{a_0\}$. By the condition (i), we have $Ta_0 = a_0$, that is, T fixes the circle $C_{a_0,\rho}$.

Now, let $\rho > 0$ and $a \in C_{a_0,\rho}$ be any point such that $Ta \neq a$. Using the Skof type a_0-contraction condition and the condition (i), we find

$$\begin{aligned} \psi\left(d(Ta, a)\right) &\leq a\psi\left(d(a, a_0)\right) + b\psi\left(d(Ta, a)\right) + c\psi\left(d(Ta_0, a_0)\right) \\ &= a\psi\left(\rho\right) + b\psi\left(d(Ta, a)\right) \\ &\leq a\psi\left(d(Ta, a)\right) + b\psi\left(d(Ta, a)\right) \\ &= (a + b)\psi\left(d(Ta, a)\right) < \psi\left(d(Ta, a)\right), \end{aligned}$$

which is a contradiction with $a + b < 1$. Therefore, it should be $Ta = a$, that is, T fixes the circle $C_{a_0,\rho}$.

(iii) It is obvious. ∎

Definition 5 *Let (X,d) be a metric space and $T : X \to X$ a self-mapping. If there exists $a_0 \in X$ such that*

$$d(Ta,a) > 0 \Longrightarrow \psi\left(d(Ta,a)\right) \leq \left[\psi\left(m(a,a_0)\right)\right]^{\alpha},$$

for all $a \in X$, where $\alpha \in (0,1)$, the function $\psi : (0,\infty) \to (1,\infty)$ is such that ψ is nondecreasing and

$$m(a,b) = \max\left\{d(a,b), d(a,Ta), d(b,Tb), \frac{d(a,Tb)+d(b,Ta)}{2}\right\},$$

then T is called Li-Jiang type a_0-contraction.

Theorem 5 (Li-Jiang type fixed-disc theorem) *Let (X,d) be a metric space, $T : X \to X$ Li-Jiang type a_0-contraction with $a_0 \in X$ and ρ defined as in (1). If $d(Ta,a_0) \leq \rho$ for all $a \in C_{a_0,\rho}$, then we have*

(i) $Ta_0 = a_0$,

(ii) T fixes the circle $C_{a_0,\rho}$,

(iii) T fixes the disc $D_{a_0,\rho}$.

Proof. (i) Let $a_0 \neq Ta_0$, that is, $d(Ta_0,a_0) > 0$. Using the Li-Jiang type a_0-contraction hypothesis and the symmetry property, we obtain

$$\psi\left(d(Ta_0,a_0)\right) \leq \left[\psi\left(m(a_0,a_0)\right)\right]^{\alpha} = \left[\psi\left(d(Ta_0,a_0)\right)\right]^{\alpha},$$

which is a contradiction with $\alpha \in (0,1)$. So, it should be $Ta_0 = a_0$.

(ii) At first, we assume $\rho = 0$. Then we obtain $C_{a_0,\rho} = \{a_0\}$. By the condition (i), we have $Ta_0 = a_0$, that is, T fixes the circle $C_{a_0,\rho}$.

Now, we suppose $\rho > 0$ and $a \in C_{a_0,\rho}$ be any point such that $Ta \neq a$. Using the Li-Jiang type a_0-contraction hypothesis, the symmetry property and the condition (i), we find

$$\psi\left(d(Ta,a)\right) \leq \left[\psi\left(m(a,a_0)\right)\right]^{\alpha} \leq \left[\psi\left(\max\left\{\rho, d(a,Ta)\right\}\right)\right]^{\alpha} = \left[\psi\left(d(Ta,a)\right)\right]^{\alpha},$$

which is a contradiction with $\alpha \in (0,1)$. Hence, it should be $Ta = a$, that is, T fixes the circle $C_{a_0,\rho}$.

(iii) It is clear. ■

Now we give the following example.

Example 1 *Let $X = \mathbb{R}$ be the usual metric space with the usual metric d defined as*

$$d(a,b) = |a-b|,$$

for all $a,b \in \mathbb{R}$. Let us define the function $T : \mathbb{R} \to \mathbb{R}$ as

$$Ta = \begin{cases} a & , \quad a \in [-2,2] \\ a+1 & , \quad otherwise \end{cases},$$

for all $a \in \mathbb{R}$. Then the function T is Moradi type a_0-contraction with $a_0 = 0$, the function $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = \begin{cases} 0 & , \quad t = 0 \\ \frac{t}{6} & , \quad t > 0 \end{cases}$$

and the function $F : [0,\infty) \to [0,\infty)$ defined by

$$F(t) = \begin{cases} 0 & , \quad t = 0 \\ \frac{t}{2} & , \quad t > 0 \end{cases}.$$

Also, the function T is Geraghty type a_0-contraction with $a_0 = 0$, the function $\psi : (0,\infty) \to (0,\infty)$ defined by

$$\psi(t) = 2t$$

and the function $\alpha : (0,\infty) \to (0,1)$ defined by

$$\alpha(t) = \frac{3}{4}.$$

The function T is Jleli-Samet type a_0-contraction with $a_0 = 0$, the function $\psi : (0,\infty) \to (1,\infty)$ defined by

$$\psi(t) = t + 1$$

and $\alpha = 0.7$. On the other hand, the function T is Skof type a_0-contraction with $a_0 = 0$, the function $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = 3t$$

and $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = 0$. Finally, the function T is Li-Jiang type a_0-contraction with $a_0 = 0$, the function $\psi : (0,\infty) \to (1,\infty)$ defined by

$$\psi(t) = t + 1$$

and $\alpha = 0.8$. Consequently, we have $\rho = 1$ and so T fixes the circle $C_{0,1} = \{-1, 1\}$ and the disc $D_{0,1} = [-1, 1]$.

3 The Equivalence of Some Contractions

In the following theorem, we see the equivalence of some contractions used in the fixed-circle and fixed-disc results.

Theorem 6 *Let $X \neq \emptyset$, the functions $\gamma, \delta : X \times X \to \mathbb{R}_+$ be such that*

(i) $a = b$ implies $\gamma(a,b) = 0$,

(ii) $\delta(a,b) = 0$ implies $a = b$.

and T a self-mapping on X. Then the followings are equivalent:

(a) There exist $a_0 \in X$, a function $\psi : (0,\infty) \to (0,\infty)$ and $\lambda \in [0,1)$ such that

$$\gamma(a,Ta) > 0 \Longrightarrow \psi\left(\gamma(Ta,a)\right) \leq \lambda \psi\left(\delta(a,a_0)\right),$$

for all $a \in X$.

(b) There exist $a_0 \in X$, a function $\psi : (0,\infty) \to (1,\infty)$ and $\alpha \in [0,1)$ such that

$$\gamma(a,Ta) > 0 \Longrightarrow \psi\left(\gamma(Ta,a)\right) \leq \left[\psi\left(\delta(a,a_0)\right)\right]^{\alpha},$$

for all $a \in X$.

(c) There exist $a_0 \in X$, a function $\psi : (0,\infty) \to \mathbb{R}$ and $t > 0$ such that

$$\gamma(a,Ta) > 0 \Longrightarrow t + \psi\left(\gamma(Ta,a)\right) \leq \psi\left(\delta(a,a_0)\right),$$

for all $a \in X$.

Proof. $(a) \Longrightarrow (b)$: Let the condition (a) holds. From the inequality given in the condition (a), we have

$$\exp\left[\psi\left(\gamma(Ta,a)\right)\right] \leq \exp\left[\lambda\psi\left(\delta(a,a_0)\right)\right] = \exp\left[\psi\left(\delta(a,a_0)\right)\right]^{\lambda}. \tag{2}$$

If we define $\alpha \in [0,1)$ by $\alpha = \lambda$ and the function $\psi' : (0,\infty) \to (1,\infty)$ by $\psi'(t) = \exp\left[\psi(t)\right]$, then using the inequality (2), we get

$$\psi'\left(\gamma(Ta,a)\right) \leq \left[\psi'\left(\delta(a,a_0)\right)\right]^{\alpha},$$

which proves the condition (b).

$(b) \Longrightarrow (c)$: Let the condition (b) holds. By the condition (b), we obtain

$$\ln\left[\ln\left[\psi\left(\gamma(Ta,a)\right)\right]\right] \leq \ln\left[\ln\left[\left[\psi\left(\delta(a,a_0)\right)\right]^{\alpha}\right]\right] = \ln\left[\ln\left[\psi\left(\delta(a,a_0)\right)\right]\right] + \ln(\alpha). \tag{3}$$

If we define $t > 0$ by $t = -\ln(\alpha)$ and $\psi'' : (0,\infty) \to \mathbb{R}$ by $\psi''(t) = \ln\left(\ln(\psi(t))\right)$, then using the inequality (3), we get

$$t + \psi''\left(\gamma(Ta,a)\right) \leq \psi''\left(\delta(a,a_0)\right),$$

which proves the condition (c).

$(c) \Longrightarrow (a)$: Let the condition (c) holds. By the condition (c), find

$$\exp\left[\psi\left(\gamma(Ta,a)\right)\right] \leq \exp\left[\psi\left(\delta(a,a_0)\right) - t\right] = \exp\left[\psi\left(\delta(a,a_0)\right)\right]\exp(-t). \tag{4}$$

If we define $\lambda \in [0,1)$ by $\lambda = \exp(-t)$ and $\psi''' : (0,\infty) \to (0,\infty)$ by $\psi'''(t) = \exp(\psi(t))$, then using the inequality (4), we obtain

$$\psi'''\left(\gamma(Ta,a)\right) \leq \lambda\psi'''\left(\delta(a,a_0)\right),$$

which proves the condition (a). ■

From Theorem 6, we obtain the following remarks.

Remark 1 *(i) The condition (a) can be considered as Banach type contraction to obtain a fixed-circle result.*

(ii) The condition (b) can be considered as Jleli-Samet type contraction to obtain a fixed-circle result.

(iii) The condition (c) can be considered as Wardowski type contraction to obtain a fixed-circle result.

4 An Application to ReLU Activation Functions

In this section, we give an application to "Rectified Linear Unit Activation Function (ReLU)" (see [4], [7], [18] and the references therein). For this purpose, at first, we recall the notions of ReLU as follows:

$$ReLU(x) = \max\{0,x\} = \begin{cases} 0 & , \quad x \leq 0 \\ x & , \quad x > 0 \end{cases}.$$

Let $X = \{-1\} \cup [0,\infty)$ be the usual metric space with the usual metric. Then we obtain that

- The activation function $ReLU$ is Moradi type a_0-contraction with $a_0 = 1$, the function $\psi : [0,\infty) \to [0,\infty)$ defined by

  $$\psi(t) = \begin{cases} 0 & , \quad t = 0 \\ \frac{t}{8} & , \quad t > 0 \end{cases}$$

  and the function $F : [0,\infty) \to [0,\infty)$ defined by

  $$F(t) = \begin{cases} 0 & , \quad t = 0 \\ \frac{4t}{5} & , \quad t > 0 \end{cases}.$$

- The activation function $ReLU$ is Geraghty type a_0-contraction with $a_0 = 1$, the function $\psi : (0, \infty) \to (0, \infty)$ defined by

$$\psi(t) = \begin{cases} 1 & , \quad t \leq 1 \\ t+1 & , \quad t > 1 \end{cases}$$

and the function $\alpha : (0, \infty) \to (0, 1)$ defined by

$$\alpha(t) = \frac{2}{3}.$$

- The activation function $ReLU$ is Jleli-Samet type a_0-contraction and Li-Jiang type a_0-contraction with $a_0 = 1$, the function $\psi : (0, \infty) \to (1, \infty)$ defined by

$$\psi(t) = \begin{cases} t+1 & , \quad t \leq 1 \\ t^3 & , \quad t > 1 \end{cases}$$

and $\alpha = 0.5$.

- The activation function $ReLU$ is Skof type a_0-contraction with $a_0 = 1$, the function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = 4t$$

and $a = \frac{1}{2}$, $b = c = \frac{1}{8}$.

Consequently, we have $\rho = 1$ and so the activation function $ReLU$ fixes the circle $C_{1,1} = \{0, 2\}$ and the disc $D_{1,1} = [0, 2]$.

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References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fund. Math., 2(1922), 133–181.

[2] M. A. Geraghty, On contractive mappings, Proc. Am. Math. Soc., 40(1973), 604–608.

[3] X. Glorot, A. Bordes and Y. Bengio, Deep Sparse Rectifier Neural Networks, International Conference on Artificial Intelligence and Statistics, 2011.

[4] K. He, X. Zhang, S. Ren and J. Sun, Delving deep into rectifiers: surpassing human-level performance on imagenet classification, Proceedings of the IEEE International Conference on Computer Vision, (2015), 1026–1034.

[5] S. Jiang, L. Li and B. Damjanović, A note on 'Some fixed point theorems for generalized contractive mappings in complete metric spaces', Fixed Point Theory Appl., 2016(2016), 62.

[6] M. Jleli and B. Samet, A new generalization of the Banach contraction principle, J. Inequal. Appl., 2014(2014), 1–8.

[7] A. L. Maas, A. Y. Hannun and A. Y. Ng, Rectifier nonlinearities improve neural network acoustic models, In Proc. ICML, 30(2013), 3.

[8] S. Moradi, Fixed point of single-valued cyclic weak φ_F-contraction mappings, Filomat, 28(2014), 1747–1752.

[9] N. Y. Özgür and N. Taş, Some fixed-circle theorems and discontinuity at fixed circle, AIP Conference Proceedings, 1926, 020048 (2018).

[10] N. Y. Özgür and N. Taş, Some fixed-circle theorems on metric spaces, Bull. Malays. Math. Sci. Soc., 42(2019), 1433–1449.

[11] N. Özgür, Fixed-disc results via simulation functions, Turkish J. Math., 43(2019), 2794–2805.

[12] R. P. Pant, N. Y. Özgür and N. Taş, Discontinuity at fixed points with applications, Bull. Belg. Math. Soc.-Simon Stevin, 26(2019), 571–589.

[13] R. P. Pant, N. Y. Özgür and N. Taş, On discontinuity problem at fixed point, Bull. Malays. Math. Sci. Soc., 43(2020), 499–517.

[14] R. P. Pant, N. Özgür, N. Taş, A. Pant and M. C. Joshi, New results on discontinuity at fixed point, J. Fixed Point Theory Appl., 22(2020), 39.

[15] P. D. Proinov, Fixed point theorems for generalized contractive mappings in metric spaces, J. Fixed Point Theory Appl., 22(2020), 21.

[16] F. Skof, Theoremi di punto fisso per applicazioni negli spazi metrici, Atti. Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur., 111(1977), 323–329.

[17] N. Taş, Bilateral-type solutions to the fixed-circle problem with rectified linear units application, Turkish J. Math., 44(2020), 1330–1344.

[18] L. Toth, Phone recognition with deep sparse rectifier neural networks, IEEE International Conference on Acoustics, Speech and Signal Processing, (2013), 6985–6989.