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Research Article

Some Fixed-Circle Results with Different Auxiliary Functions

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As the generalization of the fixed-point theory, the fixed-circle problems are interesting and notable geometric constructions. In this paper, we prove that some new necessary conditions are investigated for the existence of a fixed circle of a given self-mapping in $\mathbb{G}$-metric spaces. The well-known Braincari and Chatterjea contractive conditions are generalized for proving the uniqueness of obtained theorems. Finally, an application to parametric rectified linear unit activation functions are given to show the importance of studying the fixed-circle problem.

1. Introduction and Preliminaries

Recently, there has been a trend to work fixed-circle problems in both metric spaces and some generalized metric spaces [1–17]. For some self mappings, when the fixed point is not unique, it is an open question about the geometric shape and in some cases the set of fixed point form a circle. For example, in establishing some applicable areas such as neural networks, besides many others. This approach was initiated in [6, 7] to examine the geometry of the set of fixed-points when the number of the fixed-points of self-mappings is more than one on both metric and S-metric spaces. Fixed-circle theorems were proved and extended with various aspects and were applied to discontinuous activation functions (for example, see [18–20] and the references therein), to rectified linear units activation functions used in the neural networks [21].

In this paper, we establish various fixed-circle theorems in $\mathbb{G}$-metric spaces. Different examples and application to parametric rectified linear unit activation functions are considered to illustrate the usability of our obtained results.

Firstly, we recall the concept of a $\mathbb{G}$-metric space.

Definition 1.1 (see [22]). Consider the set $\mathfrak{F} \neq \emptyset$ and $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow \mathbb{R} \cup \{0\}$ such that, for all $\xi, \zeta, \varpi, \eta \in \mathfrak{F}$, the following conditions are satisfying:

($\mathbb{G}$1) $\mathbb{G}(\xi, \zeta, \varpi) = 0$ if and only if $\xi = \zeta = \varpi$;

($\mathbb{G}$2) $0 < \mathbb{G}(\xi, \xi, \zeta)$ for all $\xi, \zeta \in \mathfrak{F}$ with $\xi \neq \zeta$;

($\mathbb{G}$3) $\mathbb{G}(\xi, \xi, \zeta) \leq \mathbb{G}(\xi, \zeta, \varpi)$ for all $\xi, \zeta, \varpi \in \mathfrak{F}$ with $\eta \neq \varpi$;

($\mathbb{G}$4) $\mathbb{G}(\xi, \zeta, \varpi) = \mathbb{G}(\xi, \varpi, \zeta) = \mathbb{G}(\zeta, \varpi, \xi) = \cdots$, (symmetry in all three variables);

($\mathbb{G}$5) $\mathbb{G}(\xi, \zeta, \varpi) \leq \mathbb{G}(\xi, \eta, \eta) + \mathbb{G}(\eta, \zeta, \varpi)$ for all $\xi, \zeta, \varpi, \eta \in \mathfrak{F}$, (rectangle inequality).

Then, the function $\mathbb{G}$ is called a $\mathbb{G}$-metric on $\mathfrak{F}$.

Definition 1.2 (see [22]). A $\mathbb{G}$ -metric space $(\mathfrak{F}, \mathbb{G})$ is called be symmetric if

$$\mathbb{G}(\xi, \zeta, \zeta) = \mathbb{G}(\zeta, \xi, \xi), \tag{1}$$

for all $\xi, \zeta \in \mathfrak{F}$.

In [23], Kaplan and Tas introduced the notion of circle on a $\mathbb{G}$-metric space. More precisely, let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space and $\xi_0 \in \mathfrak{F}$, $r \in (0,\infty)$. The circle of center ξ_0 and radius $r > 0$ is defined as

$$C_{\mathbb{G}}(\xi_0, r) = \{\xi \in \mathfrak{F} : \mathbb{G}(\xi_0, \xi, \xi) = r\}. \quad (2)$$

Example 1.1. Let $\mathfrak{F} = \mathbb{R}$ and d be a metric space. Let the function $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \max \{d(\xi, \zeta), d(\zeta, \varpi), d(\varpi, \xi)\} \quad (3)$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$ [22]. Then, $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space. Let us consider the function $d : \mathfrak{F} \times \mathfrak{F} \longrightarrow \mathbb{R}$ as

$$d(\xi, \zeta) = \left| e^{\xi} - e^{\zeta} \right| \quad (4)$$

for all $\xi, \zeta \in \mathfrak{F}$. Then, we get

$$C_{\mathbb{G}}(\ln 2, \ln 4) = \ln 6 \quad (5)$$

the circle of center ln 2 and radius ln 4.

They also introduced the notion of fixed circle on a $\mathbb{G}$-metric space [23]. Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\xi_0, r)$ be a circle. For a self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$, if $\mathfrak{T}\xi = \xi$ for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$ then, the circle $C_{\mathbb{G}}(\xi_0, r)$ is said to be a fixed circle of $\mathfrak{T}$.

2. Some New Existence Conditions for Fixed Circles with Auxiliary Functions

Now, we present some new existence theorems for fixed circles of self-mappings.

Theorem 2.1. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$. Consider $\mathbb{M}_r : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ as*

$$\mathbb{M}_r(\eta) = \begin{cases} \eta - r & if \eta > 0 \\ 0 & if \eta = 0 \end{cases}, \quad (6)$$

for all $\eta \in \mathbb{R}^+ \cup \{0\}$. If the self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ is a function such that, for all $\xi \in \mathfrak{F}$, the following conditions are fulfilled:

(1) $\mathbb{G}(\xi_0, \mathfrak{T}\xi, \mathfrak{T}\xi) = r$ for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$,

(2) $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \mathfrak{T}\zeta) > r$ for all $\xi, \zeta \in C_{\mathbb{G}}(\xi_0, r)$ with $\xi \neq \zeta$,

(3) $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \mathfrak{T}\zeta) \leq \mathbb{G}(\xi, \xi, \zeta) - \mathbb{M}_r(\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi))$ for all $\xi, \zeta \in C_{\mathbb{G}}(\xi_0, r)$.

Then, the circle $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$.

Proof. Fix $\xi \in C_{\mathbb{G}}(\xi_0, r)$. By hypothesis (1), we have $\mathfrak{T}\xi \in C_{\mathbb{G}}(\xi_0, r)$ for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$. We claim that $\xi = \mathfrak{T}\xi$, that is, ξ is a fixed point of $\mathfrak{T}$. Now, let us suppose that $\xi \neq \mathfrak{T}\xi$. Firstly, using the condition (2), we obtain

$$\mathbb{G}\left(\mathfrak{T}^2\xi, \mathfrak{T}^2\xi, \mathfrak{T}\xi\right) > r. \quad (7)$$

Using the condition (3), we have

$$\begin{aligned} \mathbb{G}\left(\mathfrak{T}^2\xi, \mathfrak{T}^2\xi, \mathfrak{T}\xi\right) &\leq \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi) - \mathbb{M}_r(\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi)) \\ &= \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi) - \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi) + r = r. \end{aligned} \quad (8)$$

Then, it follows from the inequalities (7) and (8), which is a contradiction. Hence, it should be $\xi = \mathfrak{T}\xi$. As a consequence, $\mathfrak{T}$ fixes the circle $C_{\mathbb{G}}(\xi_0, r)$. □

Remark 2.1.

(1) Note that, in Theorem 2.1, the center of $C_{\mathbb{G}}(\xi_0, r)$ need not to be fixed

(2) Theorem 2.1 generalizes Theorem 3 given in [9].

(3) Since the notion of a $\mathbb{G}$-metric and an S-metric are independent (see, [24] for more details), then Theorem 2.1 is independent from Theorem 4.1 given in [1].

Example 2.1. Let $\mathfrak{F} = [0,\infty)$ be the interval of nonnegative real numbers and let $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \begin{cases} 0 & if \xi = \zeta = \varpi \\ \max \{\xi, \zeta, \varpi\} & otherwise \end{cases} \quad (9)$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$. Then, $\mathbb{G}$ is a $\mathbb{G}$-metric on $\mathfrak{F}$.

The circle $C_{\mathbb{G}}(1, 3)$ is obtained as follows:

$$C_{\mathbb{G}}(1, 3) = \{\xi \in \mathfrak{F} : \mathbb{G}(1, \xi, \xi) = 3\} = \{3\}. \quad (10)$$

If $\mathfrak{T}_1 : \mathfrak{F} \longrightarrow \mathfrak{F}$ is defined by

$$\mathfrak{T}_1\xi = \begin{cases} \kappa & if \xi = 1 \\ 3 & if \xi \neq 1 \end{cases}, \quad (11)$$

for all $\xi \in \mathfrak{F}$ and $\kappa \neq 1$, then $\mathfrak{T}_1$ satisfies all the hypotheses of Theorem 2.1 and the circle $C_{\mathbb{G}}(1, 3)$ is fixed by $\mathfrak{T}_1$. That is, the self-mapping $\mathfrak{T}_1$ has the unique fixed point $\xi = 3$. Notice that the center 1 of the circle $C_{\mathbb{G}}(1, 3)$ is not fixed by the self-mapping $\mathfrak{T}_1$.

Theorem 2.2. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$-metric space, $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$ and let define $\varphi : \mathfrak{F} \longrightarrow [0,\infty)$ by*

$$\varphi(\xi) = \mathbb{G}(\xi, \xi, \xi_0), \quad (12)$$

for $\xi \in \mathfrak{F}$. Suppose that the following conditions hold:

(1) $\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq \varphi(\xi) + \varphi(\mathfrak{T}\xi) - 2r$,

(2) $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \leq r$,

for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$ *such that* $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$. *Then,* $C_{\mathbb{G}}(\xi_0, r)$ *is a fixed circle of* $\mathfrak{T}$.

Proof. Let $\xi_0 \in C_{\mathbb{G}}(\xi_0, r)$ be any arbitrary point. Together with (1), we obtain

$$\begin{aligned}\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) &\le \varphi(\xi) + \varphi(\mathfrak{T}\xi) - 2r \\ &\le \mathbb{G}(\xi, \xi, \xi_0) + \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - 2r \\ &= \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0).\end{aligned} \tag{13}$$

From (2), the point $\mathfrak{T}\xi$ should lie on or interior of the circle $C_{\mathbb{G}}(\xi_0, r)$. If $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) < r$, which leads to a contradiction by the inequality (2.5). Therefore, it should be $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) = r$. If $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) < r$, then by the inequality (13) we have

$$\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \le \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - r = r - r = 0 \tag{14}$$

and we obtain $\mathfrak{T}\xi = \xi$. As a consequence, the circle $C_{\mathbb{G}}(\xi_0, r)$ is fixed circle of $\mathfrak{T}$. □

Remark 2.2. Notice that the condition (1) implies that $\mathfrak{T}\xi$ is not inside $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Similarly, (2) guarantees that $\mathfrak{T}\xi$ is not outside of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Thus, $\mathfrak{T}\xi \in C_{\mathbb{G}}(\xi_0, r)$ for any $\xi \in C_{\mathbb{G}}(\xi_0, r)$ and so we get $\mathfrak{T}(C_{\mathbb{G}}(\xi_0, r)) \subset C_{\mathbb{G}}(\xi_0, r)$.

(1) Theorem 2.2 generalizes Theorem 2.2 given in [7].

(2) Theorem 2.2 is independent from Theorem 3.11 given in [6].

Example 2.2. Let $\mathfrak{F} = \mathbb{R}$ and the mapping $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = |\xi - \zeta| + |\xi - \varpi| + |\zeta - \varpi|, \tag{15}$$

for each $\xi, \zeta, \varpi \in \mathfrak{F}$ [25]. Then, $(\mathfrak{F}, \mathbb{G})$ is a $\mathbb{G}$-metric space. Let us take the circle $C_{\mathbb{G}}(0, 6)$. If we define $\mathfrak{T}_2 : \mathfrak{F} \longrightarrow \mathfrak{F}$ by

$$\mathfrak{T}_2\xi = \frac{7\xi + 9\sqrt{3}}{\sqrt{3}\xi + 7}, \tag{16}$$

for all $\xi \in \mathfrak{F}$, then $\mathfrak{T}_2$ confirms that the conditions (1) and (2) in Theorem 2.2. Hence, the circle $C_{\mathbb{G}}(0, 6)$ is a fixed circle of $\mathfrak{T}_2$.

In the following example, we present an example of a self-mapping that satisfies the condition (1) and does not satisfy the condition (2).

Example 2.3. Let $\mathfrak{F} = \mathbb{R}$ and $(\mathfrak{F}, \mathbb{G})$ be the $\mathbb{G}$ -metric space defined in Example 2.2. Let us consider the circle $C_{\mathbb{G}}(-2, 4)$ and define the self-mapping $\mathfrak{T}_3 : \mathfrak{F} \longrightarrow \mathfrak{F}$ by

$$\mathfrak{T}_3\xi = \begin{cases} -5 & \xi = -4 \\ 5 & \xi = 0 \\ 10 & \text{otherwise} \end{cases}, \tag{17}$$

for all $\xi \in \mathfrak{F}$. Then, the self-mapping $\mathfrak{T}_3$ satisfies the condition (1) in Theorem 2.2 but does not satisfy the condition (2) in Theorem 2.2. Obviously, $\mathfrak{T}_3$ does not fix the circle $C_{\mathbb{G}}(-2, 4)$.

In the next example, we present an example of a self-mapping that satisfies the condition (2) and does not satisfy the condition (1).

Example 2.4. Let $\mathfrak{F} = \mathbb{R}$ and the mapping $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \max\{|\xi - \zeta|, |\xi - \varpi|, |\zeta - \varpi|\}, \tag{18}$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$ [25]. Then, $(\mathfrak{F}, \mathbb{G})$ is a $\mathbb{G}$-metric space. Let us take the circle $C_{\mathbb{G}}(0, 1/2)$. If we define $\mathfrak{T}_4 : \mathfrak{F} \longrightarrow \mathfrak{F}$ by

$$\mathfrak{T}_4\xi = \begin{cases} -\frac{1}{2} & \mathit{if}\,\xi = -1 \\ \frac{1}{2} & \mathit{if}\,\xi = 1 \\ 3 & \mathit{otherwise} \end{cases}, \tag{19}$$

for all $\xi \in \mathfrak{F}$, then $\mathfrak{T}_4$ confirms that condition (2) in Theorem 2.2 but does not satisfy the condition (1) in Theorem 2.2. Clearly, $\mathfrak{T}_4$ does not fix the circle $C_{\mathbb{G}}(0, 1/2)$.

Now, we present the following theorem.

Theorem 2.3. *Let* $(\mathfrak{F}, \mathbb{G})$ *be a* $\mathbb{G}$ *-metric space and* $C_{\mathbb{G}}(\xi_0, r)$ *be any circle on* $\mathfrak{F}$. *Let the mapping* φ *be defined as Theorem 2.1. If the self-mapping* $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ *is a function such that for all* $\xi \in C_{\mathbb{G}}(\xi_0, r)$ *and* $k \in [0, 1)$, *the following conditions are satisfied:*

(1) $\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \le \varphi(\xi) - \varphi(\mathfrak{T}\xi)$,

(2) $k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) + \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \ge r$,

then the circle $C_{\mathbb{G}}(\xi_0, r)$ *is a fixed circle of* $\mathfrak{T}$.

Proof. Let $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Conversely, suppose that $\xi \ne \mathfrak{T}\xi$. Then, take into account the conditions (1) and (2), we conclude that

$$\begin{aligned}\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) &\le \varphi(\xi) - \varphi(\mathfrak{T}\xi) \\ &= \mathbb{G}(\xi, \xi, \xi_0) - \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \\ &= r - \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \le k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \\ &\quad + \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) \\ &= k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi),\end{aligned} \tag{20}$$

which is a contradiction $k \in 0, 1)$. As a result, we get $\xi = \mathfrak{T}\xi$ and $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$. □

Remark 2.3. Notice that the condition (1) guarantees that $\mathfrak{T}\xi$ is not in the exterior of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Similarly, the condition (2) guarantees that $\mathfrak{T}\xi$ can lies on or exterior or interior of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Hence $\mathfrak{T}\xi$ should lies on or interior of the circle $C_{\mathbb{G}}(\xi_0, r)$.

(1) Theorem 2.3 generalizes Theorem 2.3 given in [7].

(2) Theorem 2.3 is independent from Theorem 3.2 given in [8].

Now, we present some examples concerning with self-mappings which have a fixed circle.

Example 2.5. Let $\mathfrak{F} = \mathbb{R}$ and $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space defined in Example 2.4. Let us consider the circle $C_{\mathbb{G}}(1, 3) = 3$ and define the self-mapping $\mathfrak{T}_5 : \mathfrak{F} \longrightarrow \mathfrak{F}$ by

$$\mathfrak{T}_5\xi = \begin{cases} 2\xi - 3 & \xi = 3 \\ 5 & \text{otherwise} \end{cases}, \tag{21}$$

for all $\xi \in \mathfrak{F}$. Then, the self-mapping $\mathfrak{T}_5$ satisfies the condition (1) and (2) in Theorem 2.3. So, $C_{\mathbb{G}}(1, 3)$ is a fixed circle of $\mathfrak{T}_5$.

Example 2.6. Let $\mathfrak{F} = \mathbb{R}$ and the function $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \left|e^{\xi} - e^{\zeta}\right| + \left|e^{\zeta} - e^{\varpi}\right| + \left|e^{\xi} - e^{\varpi}\right|, \tag{22}$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$. Then, it can be easily checked that $(\mathfrak{F}, \mathbb{G})$ is a $\mathbb{G}$-metric space. Let us consider the circle $C_{\mathbb{G}}(0, 2) = \{\ln 2\}$ and define the self-mapping $\mathfrak{T}_6 : \mathfrak{F} \longrightarrow \mathfrak{F}$ as

$$\mathfrak{T}_6\xi = \begin{cases} \xi & \xi \in C_{\mathbb{G}}(0, 2) \\ \ln 5 & \text{otherwise} \end{cases}, \tag{23}$$

for all $\xi \in \mathfrak{F}$. So, the self-mapping $\mathfrak{T}_6$ provides the condition (1) and (2) in Theorem 2.3. Hence, $C_{\mathbb{G}}(0, 2)$ is a fixed circle of $\mathfrak{T}_6$.

Next, we give an example of a self-mapping which provides the condition (1) and does not provide the condition (2).

Example 2.7. Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r)$ be a circle on $\mathfrak{F}$. If we take $\mathfrak{T}_7\xi = \xi_0$ as the self-mapping on $\mathfrak{F}$, then we deduce that the self-mapping $\mathfrak{T}_7$ satisfies the condition (1) in Theorem 2.3 but does not satisfy the condition (2) in Theorem 2.3. So, it can be easily shown that $\mathfrak{T}_7$ does not fix a circle $C_{\mathbb{G}}(\xi_0, r)$.

In the next example, we present an example of a self-mapping which satisfies the condition (2) and does not satisfy the condition (1).

Example 2.8. Let $\mathfrak{F} = \mathbb{R}$ and let the function $\mathbb{G} : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,\infty)$ be defined by

$$\mathbb{G}(\xi, \zeta, \varpi) = \max\{|\xi - \zeta|, |\zeta - \varpi|, |\xi - \varpi|\}, \tag{24}$$

for all $\xi, \zeta, \varpi \in \mathfrak{F}$ [25]. Let us consider the circle $C_{\mathbb{G}}(0, 5)$ and define the self-mapping $\mathfrak{T}_8 : \mathfrak{F} \longrightarrow \mathfrak{F}$ as $\mathfrak{T}_8\xi = 5$ for all $\xi \in \mathfrak{F}$. Then, the self-mapping $\mathfrak{T}_8$ provides the condition (2) in Theorem 2.3 but does not provide the condition (1) in Theorem 2.3. It can be easily shown that $\mathfrak{T}_8$ does not fix the circle $C_{\mathbb{G}}(0, 5)$.

Theorem 2.4. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$. Let the mapping φ be defined as Theorem 2.1. If the self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ is a function such that for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$ and $k \in [0, 1)$, the following conditions are satisfied:*

(1) $\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq \max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} - r$,

(2) $\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq r$,

then the circle $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$.

Proof. Let $\xi \in C_{\mathbb{G}}(\xi_0, r)$ such that $\xi \neq \mathfrak{T}\xi$. We show $\xi = \mathfrak{T}\xi$ under the following two cases:

Case 1: Let $\max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} = \varphi(\xi)$. Then, we get

$$\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \leq \max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} - r = \varphi(\xi) - r = r - r = 0, \tag{25}$$

a contradiction. Hence, we get $\xi = \mathfrak{T}\xi$.

Case 2: Let $\max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} = \varphi(\mathfrak{T}\xi)$. Then, we obtain

$$\begin{aligned} \mathbb{G}(\xi, \xi, \mathfrak{T}\xi) &\leq \max\{\varphi(\xi), \varphi(\mathfrak{T}\xi)\} - r = \varphi(\mathfrak{T}\xi) - r \\ &= \mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \xi_0) - r \leq r + k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \\ &\quad - r = k\mathbb{G}(\xi, \xi, \mathfrak{T}\xi), \end{aligned} \tag{26}$$

a contradiction with $k \in 0, 1)$. Therefore, we have $\xi = \mathfrak{T}\xi$. □

Consequently, the circle $C_{\mathbb{G}}(\xi_0, r)$ is a fixed circle of $\mathfrak{T}$.

Remark 2.4. Notice that condition (1) guarantees that $\mathfrak{T}\xi$ is not in the interior of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Similarly, the condition (2) guarantees that $\mathfrak{T}\xi$ is not the exterior of the circle $C_{\mathbb{G}}(\xi_0, r)$ for $\xi \in C_{\mathbb{G}}(\xi_0, r)$. Hence $\mathfrak{T}\xi \in C_{\mathbb{G}}(\xi_0, r)$ for each $\xi \in C_{\mathbb{G}}(\xi_0, r)$ and so we get $\mathfrak{T}(C_{\mathbb{G}}(\xi_0, r)) \subset C_{\mathbb{G}}(\xi_0, r)$.

(1) Theorem 2.4 is independent from Theorem 4.2 given in [1].

(2) If we consider the self-mapping $\mathfrak{T}_5 : \mathfrak{F} \longrightarrow \mathfrak{F}$ defined in Example 2.5, then $\mathfrak{T}_5$ satisfies the conditions (1) and (2) in Theorem 2.4 and so $C_{\mathbb{G}}(1, 3)$ is a fixed circle of $\mathfrak{T}_5$.

Notice that the identity mapping $I_{\mathfrak{F}}$ defined as $I_{\mathfrak{F}}(\xi) = \xi$ for all $\xi \in \mathfrak{F}$ satisfies conditions (1) and (2) (resp., (1) and (2)) in Theorem 2.2 (resp., Theorem 2.3). Therefore, we need a condition which excludes the identity map in Theorem 2.2 (resp., Theorem 2.3). For this aim, we give in [23] the following theorem.

Theorem 2.5 (see [23]). *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space, $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ be a self-mapping having a fixed circle $C_{\mathbb{G}}(\xi_0, r)$ and the mapping φ be defined as 2.2. The self-mapping $\mathfrak{T}$ satisfies the condition*

$$(I_{\mathbb{G}})\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) \le h[\phi(\xi) - \phi(\mathfrak{T}\xi)], \tag{27}$$

for all $\xi \in \mathfrak{F}$ and some $h \in [0, 1/4)$ if and only if $\mathfrak{T} = I_{\mathfrak{F}}$.

Now we give the another theorem which excludes the identity map using the auxiliary function ξ_r defined in (6).

Theorem 2.6. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space, $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ be a self-mapping having a fixed circle $C_{\mathbb{G}}(\xi_0, r)$ and the mapping $\mathbb{M}_r$ defined in (6). The self-mapping $\mathfrak{T}$ satisfies the condition*

$$(I_{\mathbb{G}}^*)\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) < \mathbb{M}_r(\mathbb{G}(\xi, \xi, \mathfrak{T}\xi)) + r, \tag{28}$$

for all $\xi \in \mathfrak{F}$ if and only if $\mathfrak{T} = I_{\mathfrak{F}}$.

Proof. Let $\xi \in \mathfrak{F}$ be any point such that $\xi \ne \mathfrak{T}\xi$. Using the inequality $(I_{\mathbb{G}}^*)$, we get

$$\begin{aligned}\mathbb{G}(\xi, \xi, \mathfrak{T}\xi) &< \mathbb{M}_r(\mathbb{G}(\xi, \xi, \mathfrak{T}\xi)) + r \\ &= \mathbb{G}(\xi, \xi, \mathfrak{T}\xi) - r + r = \mathbb{G}(\xi, \xi, \mathfrak{T}\xi),\end{aligned} \tag{29}$$

a contradiction. Hence we get $\xi = \mathfrak{T}\xi$ and so $\mathfrak{T} = I_{\mathfrak{F}}$. □

The converse statement is clear.

3. Some New Uniqueness Conditions for Fixed Circles with Integral Type Contractions

In [26], Braincari gave an integral contractive condition which was a generalization of Banach contraction in a metric space. By the Braincari type contractive condition, we obtain a uniqueness theorem as follows.

Theorem 3.1. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r)$ be any circle on $\mathfrak{F}$. Let $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ be a self-mapping satisfying the inequalities of Theorem 2.1 (resp., Theorem 2.2, Theorem 2.3 and Theorem 2.4). If the contractive condition*

$$\int_0^{\mathbb{G}(\mathfrak{T}\xi, \mathfrak{T}\xi, \mathfrak{T}\zeta)} \omega(t)dt \le c\int_0^{\mathbb{G}(\xi, \xi, \zeta)} \omega(t)dt \tag{30}$$

is satisfied for all $\xi \in C_{\mathbb{G}}(\xi_0, r)$, $\zeta \in \mathfrak{F} - C_{\mathbb{G}}(\xi_0, r)$ where $c \in [0, 1)$ and $\omega : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesque measurable map which is summable (that is, with a finite integral) on each compact subset of $[0, \infty)$ such that $\int_0^{\varepsilon}\omega(t)dt > 0$ for each $\varepsilon > 0$, then $C_{\mathbb{G}}(\xi_0, r_0)$ is the unique fixed circle of $\mathfrak{T}$.

Proof. Suppose that the self-mapping $\mathfrak{T}$ has two different fixed circles $C_{\mathbb{G}}(\xi_0, r_0)$ and $C_{\mathbb{G}}(\xi_1, r_1)$. Let $u \in C_{\mathbb{G}}(\xi_0, r_0)$ and $v \in C_{\mathbb{G}}(\xi_1, r_1)$ be arbitrary points such that $u \ne v$. We show that $\mathbb{G}(u, u, v) = 0$ and hence $u = v$. By the contractive condition of $\mathfrak{T}$, that is, using the inequality (30), we have

$$\int_0^{\mathbb{G}(u,u,v)} \omega(t)dt = \int_0^{\mathbb{G}(\mathfrak{T}u,\mathfrak{T}u,\mathfrak{T}v)} \omega(t)dt \le c\int_0^{\mathbb{G}(u,u,v)} \omega(t)dt \tag{31}$$

which is a contradiction $c \in [0, 1)$. Consequently, $C_{\mathbb{G}}(\xi_0, r_0)$ is the unique fixed circle of $\mathfrak{T}$. □

Taking into consideration that Chatterjea type contraction condition [27], we prove the following theorem.

Theorem 3.2. *Let $(\mathfrak{F}, \mathbb{G})$ be a $\mathbb{G}$ -metric space and $C_{\mathbb{G}}(\xi_0, r_0)$ be any circle on $\mathfrak{F}$. Let $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$ be a self-mapping satisfying the inequalities of Theorem 2.1 (resp., Theorem 2.2, Theorem 2.3 and Theorem 2.4). If the contractive condition*

$$\int_0^{\mathbb{G}(\mathfrak{T}\xi,\mathfrak{T}\xi,\mathfrak{T}\zeta)} \omega(t)dt \le \eta\left(\int_0^{\mathbb{G}(\xi,\xi,\mathfrak{T}\zeta)} \omega(t)dt + \int_0^{\mathbb{G}(\zeta,\zeta,\mathfrak{T}\xi)} \omega(t)dt\right) \tag{32}$$

is satisfied for all $\xi \in C_{\mathbb{G}}(\xi_0, r), \zeta \in \mathfrak{F} - C_{\mathbb{G}}(\xi_0, r)$ and $\eta \in [0, 1/2)$ where $\omega : [0,\infty) \longrightarrow 0,\infty)$ is a Lebesque measurable map which is summable (that is, with a finite integral) on each compact subset of $[0, \infty)$ such that $\int_0^{\varepsilon}\omega(t)dt > 0$ for each $\varepsilon > 0$, then the fixed circle of $\mathfrak{T}$ is unique.

Proof. Assume that there exist two different fixed-circles $C_{\mathbb{G}}(\xi_0, r_0)$ and $C_{\mathbb{G}}(\xi_1, r_1)$ of the self-mapping $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{F}$. Let $u \in C_{\mathbb{G}}(\xi_0, r_0)$ and $v \in C_{\mathbb{G}}(\xi_1, r_1)$ be arbitrary points such that $u \ne v$. Using the inequality (32) and the symmetric property of $\mathbb{G}$-metric, we obtain

$$\begin{aligned}\int_0^{\mathbb{G}(u,u,v)} \omega(t)dt &= \int_0^{\mathbb{G}(\mathfrak{T}u,\mathfrak{T}u,\mathfrak{T}v)} \omega(t)dt \\ &\le \eta\left(\int_0^{\mathbb{G}(u,u,\mathfrak{T}v)} \omega(t)dt + \int_0^{\mathbb{G}(v,v,\mathfrak{T}u)} \omega(t)dt\right) \\ &= \eta\left(\int_0^{\mathbb{G}(u,u,v)} \omega(t)dt + \int_0^{\mathbb{G}(v,v,u)} \omega(t)dt\right) \\ &= 2\eta\int_0^{\mathbb{G}(u,u,v)} \omega(t)dt,\end{aligned} \tag{33}$$

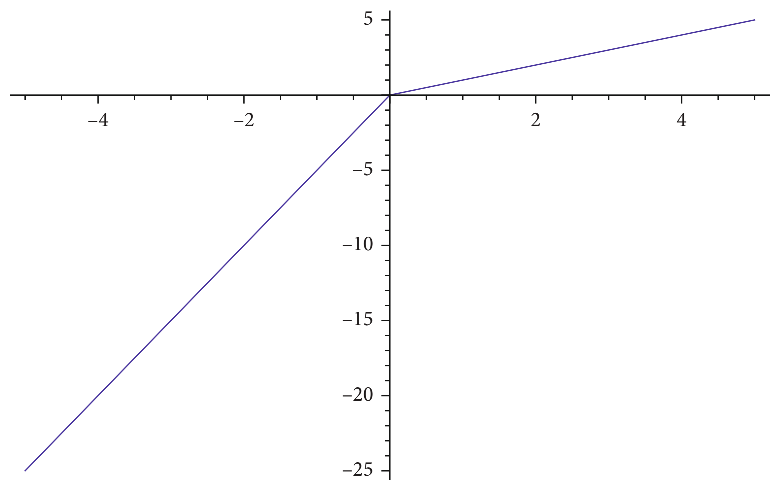

Figure 1: The activation function *PReLU*.

which is a contradiction. Consequently, it should be $u = v$ and thus $C_{\mathbb{G}}(\xi_0, r_0)$ is the unique fixed circle of $\mathfrak{T}$. □

Remark 3.1. The choice of used contractive condition in uniqueness theorem is not unique. Any contractive condition used to derive the fixed-point theorem can also be selected.

4. An Application to Parametric *ReLU*

In this section, we present a new application to "Parametric Rectified Linear Unit (*PReLU*)" using the obtained fixed-circle results. This activation function *PReLU* was defined to generalize the traditional rectified unit and it adaptively learns the parameters of the rectifiers (see [28] for more details). This activation function is defined by

$$P\,\mathrm{Re}\,LU(\xi) = \begin{cases} c\xi & if\,\xi < 0 \\ \xi & if\,\xi \geq 0 \end{cases}, \tag{34}$$

with parameter c. Let us take $\mathfrak{F} = [0,\infty)$ with the $\mathbb{G}$-metric defined as in Example 2.1 and $c = 5$. Then we have

$$P\,\mathrm{Re}\,LU(\xi) = \begin{cases} 5\xi & if\,\xi < 0 \\ \xi & if\,\xi \geq 0 \end{cases}, \tag{35}$$

for all $\xi \in 0,\infty)$ (see, Figure 1).

If we choose a circle $C_{\mathbb{G}}(0, 1) = \{1\}$, then *PReLU* satisfies the conditions of Theorem 2.1 (resp., Theorem 2.2, Theorem 2.3 and Theorem 2.4). Thereby, $C_{\mathbb{G}}(0, 1)$ is a fixed circle of *PReLU*. On the other hand, this activation function fixes all circles $C_{\mathbb{G}}(0, r)$ with $r > 0$, that is, the number of fixed circles of *PReLU* is infinite. In this case, it is important because it increases the learning capacity of the activation function.

Data Availability

The data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

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References

[1] U. Çelik and N. Y. Özgür, "On the fixed-circle problem," *Facta Universitatis, Series: Mathematics and Informatics*, vol. 35, no. 5, pp. 1273–1290, 2021.

[2] N. Mlaiki, N. Y. Özgür, and N. Taş, "New fixed-circle results related to Fc-contractive and Fc-expanding mappings on metric spaces," 2021, https://arxiv.org/abs/2101.10770.

[3] N. Mlaiki, U. Çelik, N. Taş, N. Y. Özgür, and A. Mukheimer, "Wardowski Type Contractions and the Fixed-Circle Problem on -Metric Spaces," *Journal of Mathematics*, vol. 2018, Article ID 9127486, 9 pages, 2018.

[4] N. Mlaiki, N. Y. Özgür, and N. Tas, "New Fixed-Point Theorems on an $S-metric$ Space via Simulation Functions," *Mathematics*, vol. 7, no. 7, p. 583, 2019.

[5] N. Mlaiki, N. Taş, and N. Y. Özgür, "On the fixed-circle problem and khan type contractions," *Axioms*, vol. 7, no. 4, p. 80, 2018.

[6] N. Y. Özgür, N. Taş, and U. Çelik, "New fixed-circle results on S-metric spaces," *Bulletin of Mathematical Analysis and Applications*, vol. 9, no. 2, pp. 10–23, 2017.

[7] N. Y. Özgür and N. Taş, "Some fixed-circle theorems on metric spaces," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 4, pp. 1433–1449, 2019.

[8] N. Y. Özgür and N. Taş, "FIXED-CIRCLE problem on *S*-METRIC spaces with a geometric viewpoint," *Facta Universitatis*,

Series: Mathematics and Informatics, vol. 34, no. 3, pp. 459–472, 2019.

[9] N. Y. Özgür and N. Taş, "Some fixed-circle theorems and discontinuity at fixed circle," in *AIP conference proceedings*, vol. 1926, AIP publishing LLC, 2018no. 1, Article ID 020048.

[10] N. Y. Özgür and N. Tas, "On the geometry of fixed points of self-mappings on $S-metric$ spaces," *Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics*, vol. 69, no. 2, pp. 190–198, 2020.

[11] H. N. Saleh, S. Sessa, W. M. Alfaqih, M. Imdad, and N. Mlaiki, "Fixed Circle and Fixed Disc Results for New Types of Θc-Contractive Mappings in Metric Spaces," *Symmetry*, vol. 12, no. 11, p. 1825, 2020.

[12] N. Taş, N. Y. Özgür, and N. Mlaiki, "New types of FC-contractions and the fixed-circle problem," *Mathematics*, vol. 6, no. 10, p. 188, 2018.

[13] N. Taş, "Suzuki-Berinde type fixed-point and fixed-circle results on S-metric spaces," *Journal of Linear and Topological Algebra*, vol. 7, no. 3, pp. 233–244, 2018.

[14] N. Taş, "Various types of fixed-point theorems on S-metric spaces," *Journal of Balikesir University Institute of Science and Technology*, vol. 20, no. 2, pp. 211–223, 2018.

[15] A. Tomar, M. Joshi, and S. K. Padaliya, "Fixed point to fixed circle and activation function in partial metric space," *Journal of Applied Analysis*, 2021.

[16] M. Joshi, A. Tomar, and S. K. Padaliya, *Fixed point to fixed disc and application in partial metric spaces*, Chapter in a book Fixed Point Theory and its Applications to Real World Problem, Nova Science Publishers, New York, USA, 2021.

[17] M. Joshi and A. Tomar, "On Unique and Nonunique Fixed Points in Metric Spaces and Application to Chemical Sciences," *Journal of Function Spaces*, vol. 2021, Article ID 5525472, 11 pages, 2021.

[18] N. Y. Özgür and N. Taş, "New discontinuity results at fixed point on metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 23, no. 2, pp. 1–14, 2021.

[19] R. P. Pant, N. Y. Özgür, and N. Taş, "On discontinuity problem at fixed point," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 1, pp. 499–517, 2020.

[20] N. Taş and N. Y. Özgür, "A new contribution to discontinuity at fixed point," 2017, https://arxiv.org/abs/1705.03699.

[21] N. Taş, "Bilateral-type solutions to the fixed-circle problem with rectified linear units application," *Turkish Journal of Mathematics*, vol. 44, no. 4, pp. 1330–1344, 2020.

[22] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[23] E. Kaplan and N. Taş, "Non-Unique Fixed Points and Some Fixed-Circle Theorems on G-Metric Spaces," in *Submitted to Fixed Point Theory*, 2022.

[24] N. Van Dung, N. T. Hieu, and S. Radojević, "Fixed point theorems for $g-monotone$ maps on partially ordered $S-metric$ spaces," *Filomat*, vol. 28, no. 9, pp. 1885–1898, 2014.

[25] R. P. Agarwal, E. Karapınar, D. O'Regan, and A. F. Roldán-López-de-Hierro, *Fixed Point Theory in Metric Type Spaces*, Springer, Cham, 2015.

[26] A. Branciari, "A fixed point theorem for mappings satisfying a general contractive condition of integral type," *International Journal of Mathematics and Mathematical Sciences*, vol. 29, no. 9, 536 pages, 2002.

[27] S. K. Chatterjea, "Fixed point theorems," *Comptes rendus de l'Académie bulgare des Sciences*, C. R. Acad, Ed., vol. 25, pp. 727–730, 1972.

[28] K. He, X. Zhang, S. Ren, and J. Sun, "Delving deep into rectifiers: surpassing human-level performance on image net classification," 2015, https://arxiv.org/abs/1502.01852.