Research Article

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Arithmetic convolution sums derived from eta quotients related to divisors of 6

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Abstract: The aim of this paper is to find arithmetic convolution sums of some restricted divisor functions. When divisors of a certain natural number satisfy a suitable condition for modulo 12, those restricted divisor functions are expressed by the coefficients of certain eta quotients. The coefficients of eta quotients are expressed by the sine function and cosine function, and this fact is used to derive formulas for the convolution sums of restricted divisor functions and of the number of divisors. In the sine function used to find the coefficients of eta quotients, the result is obtained by utilizing a feature with symmetry between the divisor and the corresponding divisor. Let N, r be positive integers and d be a positive divisor of N. Let $e_r(N;\ 12)$ denote the difference between the number of $\frac{2N}{d} - d$ congruent to r modulo 12 and the number of those congruent to $-r$ modulo 12. The main results of this article are to find the arithmetic convolution identities for $\sum_{a_1+\cdots+a_j=N}(\prod_{i=1}^{j}\hat{e}(a_i))$ with $\hat{e}(a_i) = e_1(a_i;\ 12) + 2e_3(a_i;\ 12) + e_5(a_i;\ 12)$ and $j = 1, 2, 3, 4$. All results are obtained using elementary number theory and modular form theory.

Keywords: restricted divisor functions, eta quotient, convolution sums, q-series

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1 Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ will denote the set of natural numbers, the set of non-negative integers, and the ring of integers, respectively. For $d, m, N \in \mathbb{N}$, $r, s \in \mathbb{N}_0$, and $\mathfrak{A}, \mathfrak{B} \subset \mathbb{N}_0$, we define some restricted divisor functions for our use in the sequel. Let

$$E_r(N;\ m) := \sum_{\substack{d|N \\ d\equiv r \pmod m}} 1 - \sum_{\substack{d|N \\ d\equiv -r \pmod m}} 1, \quad E_{r,\ldots,s}(N;\ m) := E_r(N;\ m) + \cdots + E_s(N;\ m),$$

$$e_r(N;\ m) := \sum_{\substack{d|N \\ \frac{2N}{d}-d\equiv r \pmod m}} 1 - \sum_{\substack{d|N \\ \frac{2N}{d}-d\equiv -r \pmod m}} 1, \quad e_{r,\ldots,s}(N;\ m) := e_r(N;\ m) + \cdots + e_s(N;\ m),$$

$$\hat{e}(N) := e_{1,5}(N;\ 12) + 2e_3(N;\ 12), \quad \mathfrak{A} \otimes \mathfrak{B} = \{a \cdot b | a \in \mathfrak{A}, b \in \mathfrak{B}\}, \quad \sigma_s(N) := \sum_{d|N} d^s,$$

$$\bar{\sigma}_2(N) := \sum_{\substack{d|N \\ d\equiv 1 \pmod 4}} d^2 - \sum_{\substack{d|N \\ d\equiv -1 \pmod 4}} d^2, \quad \tilde{\sigma}_2(N) := \sum_{\substack{d|N \\ \frac{N}{d}\equiv 1 \pmod 4}} d^2 - \sum_{\substack{d|N \\ \frac{N}{d}\equiv -1 \pmod 4}} d^2,$$

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and

$$\hat{\sigma}(N) := 2\left(\sum_{\substack{d|N\\ d\equiv\frac{N}{d}(\mathrm{mod}\ 6)}} d - \sum_{\substack{d|N\\ d-\frac{N}{d}\equiv 3(\mathrm{mod}\ 6)}} d\right) + \left(\sum_{\substack{d|N\\ d-\frac{N}{d}\equiv\pm1(\mathrm{mod}\ 6)}} d - \sum_{\substack{d|N\\ d-\frac{N}{d}\equiv\pm2(\mathrm{mod}\ 6)}} d\right).$$

Here, $d|N$ means that d is a divisor of N. We also make use of the following convention:

$$\sigma_s(N) = E_r(N;\ m) = e_r(N;\ m) = 0 \quad \text{if}\ \ N\notin\mathbb{Z} \quad \text{or} \quad N\le 0, \quad \sigma(N) := \sigma_1(N) = \sum_{d|N} d.$$

The exact evaluation of the basic convolution sum $\sum_{k=1}^{N-1}\sigma_1(k)\sigma_1(N-k)$ first appeared in a letter from Besge to Liouville in 1862 [1]. Much is known about the convolution sums of the divisor functions $\sum_{k=1}^{N-1}\sigma_s(k)\sigma_r(N-k)$ and $\sum_{k=1}^{N-1}\sigma_{s,r}(k;\ m)\sigma_{t,r}(N-k;\ m)$, where $\sigma_{s,r}(N;\ m) = \sum_{\substack{d|N\\ d\equiv r(\mathrm{mod}\ m)}} d^s$ with $r, s, t\in\mathbb{N}_0$ and m, $N\in\mathbb{N}$. Among them, the beautiful results were found by Ramanujan [2,3] and Glaisher [4–7] introduced interesting results. In recent years, related studies have been fulfilled in [8–11]. The results of Cangul for special convolution sums related to a new graph invariant Ω can also be found in [12,13]. In the convolution sum of the restricted divisor functions $\sum_{k=1}^{N-1}E_s(k;\ m)E_r(N-k;\ m)$, we introduce the results of Farkas [14,15], Williams [16], Guerzhoy and Raji [17], and Raji [18]. The convolution sum is characterized by having the same first and last term, and two symmetry structures in each term. Most of the aforementioned studies are the results of considering the divisors as modulo m. However, this paper attempts to calculate the convolution sums with a condition that considers the divisor of a given number and the corresponding divisor together. In other words, for a given natural number N and its divisor ω, we consider a convolution sum with the condition $\omega\equiv i(\mathrm{mod}\ m)$ instead of the condition $\frac{2N}{\omega}-\omega\equiv i(\mathrm{mod}\ m)$ with $i\in\mathbb{N}_0$ and $0\le i<m$. Specifically, in this paper, we would like to deal with $\sum_{a_1+\cdots+a_j=N}(\prod_{i=1}^{j}\hat{e}(a_i))$, $a_1,\ldots,a_j\in\mathbb{N}$, and $j=1,2,3,4$. Our results are different from those of Ramanujan and Farkas because the condition of the divisors of a given number is different. In order to derive these results, identities of infinite sums and infinite products given by eta quotients are needed. So we introduce eta quotients below.

Eta quotients are important subjects that are found in many fields of the theory of basic hyper-geometric series, partition functions, and modular forms [19]. An eta quotient is a function of the form $f(\tau) = \prod_{\omega|T}\eta^{b_\omega}(\omega\tau)$, where η is the Dedekind eta function defined by $\eta(\tau) := q^{\frac{1}{24}}\prod_{n\ge1}(1-q^n)$. Here, q will denote a fixed complex number with $|q|<1$ and b_ω, $T\in\mathbb{Z}$, so that we may write $q=e^{2\pi i\tau}$, where $Im(\tau)>0$.

From here, we introduce the basic identity for infinite sums and infinite products through the work of Fine [20]. Let us define

$$H(q)^k := \left(\frac{\eta(\tau)\eta(2\tau)\eta(3\tau)}{\eta(6\tau)}\right)^k = \left(\prod_{n\ge1}\frac{(1-q^n)(1-q^{2n})(1-q^{3n})}{(1-q^{6n})}\right)^k = \sum_{N=0}^{\infty}\mathfrak{a}_k(N)q^N \tag{1.1}$$

with $k\in\mathbb{N}$. Here, $\mathfrak{a}_k(0)=1$ and $\mathfrak{a}_k(N)$ is the coefficient of q^N in $H(q)^k$ with $N\ge1$. In this article, the coefficients of $H(q)^k$ related to positive divisors of 6 are studied.

More precisely, we prove the following theorems.

Theorem 1. *Let* $a_1,\ldots,a_k,k,N\in\mathbb{N}$. *If* $k-N\equiv 2(\mathrm{mod}\ 4)$, *then*

$$\sum_{\substack{a_1+\cdots+a_k=N\\ a_1,\ldots,a_k\ \text{odd}}}\hat{e}(a_1)\cdots\hat{e}(a_k) = \sum_{\substack{a_1+\cdots+a_k=N\\ a_1,\ldots,a_k:\ \text{odd}}}\mathfrak{a}_1(a_1)\cdots\mathfrak{a}_1(a_k) = 0.$$

For any N, does $\sigma(N)$ become odd? The answer to this is well-known to [21, p. 28]:

"$\sigma(N)$ is odd if and only if N is a perfect square."

Naturally, for other arithmetic functions, we can think of this question. Theorems 2 and 3 can give partial answers in terms of $\mathfrak{a}_i(N)$ with $i = 1, 2$.

Theorem 2. *Let N be an odd positive integer.* $\mathfrak{a}_1(N)$ *is odd if and only if* $N \equiv 1 \pmod{12}$ *is a perfect square. Furthermore,*

$$\mathfrak{a}_1(N) = -\hat{e}(N) = \begin{cases} -E_{1,5}(N;\,12) & \text{if } N \equiv 1 \pmod{12}, \\ 4E_1(N;\,12) & \text{if } N \equiv 5 \pmod{12}, \\ -4E_{1,5}(N;\,12) & \text{if } N \equiv 9 \pmod{12}, \\ 0 & \text{if } N \equiv 3 \pmod{4}. \end{cases}$$

Theorem 3. *There does not exist an odd positive integer N satisfying* $\mathfrak{a}_2(N) \equiv 1 \pmod 2$.

Theorem 4. *Let* $N(\geq 2)$ *be an integer with* $N = 2^n \cdot 3^m \cdot M'$ *with* $\gcd(6, M') = 1$ *and* $n, m \in \mathbb{N}_0$. *Then*

$$\sum_{k=1}^{N-1} \hat{e}(k)\hat{e}(N-k) = \begin{cases} -2(\chi(N;\,12) + 6\sigma(M')) & \text{if } N \equiv 0 \pmod{12}, \\ 2(E_{1,5}(N;\,12) - \sigma(N)) & \text{if } N \equiv 1 \pmod{12}, \\ 4E_{1,5}(M';\,12) - 3\sigma(M') & \text{if } N \equiv 2 \pmod{12}, \\ 4\sigma(M') & \text{if } N \equiv 3 \pmod{12}, \\ -2(E_{1,5}(M';\,12) - 3\sigma(M')) & \text{if } N \equiv 4 \pmod{12}, \\ -8E_1(N;\,12) + \sigma(N) & \text{if } N \equiv 5 \pmod{12}, \\ -2(\chi(N;\,12) + 6\sigma(M')) & \text{if } N \equiv 6 \pmod{12}, \\ -2\sigma(N) & \text{if } N \equiv 7 \pmod{12}, \\ 4E_{1,5}(M';\,12) - 3\sigma(M') & \text{if } N \equiv 8 \pmod{12}, \\ 4(2E_{1,5}(N;\,12) + \sigma(M')) & \text{if } N \equiv 9 \pmod{12}, \\ -2(E_{1,5}(M';\,12) - 3\sigma(M')) & \text{if } N \equiv 10 \pmod{12}, \\ \sigma(N) & \text{if } N \equiv 11 \pmod{12}. \end{cases}$$

Here,

$$\chi(N;\,12) := \begin{cases} 0 & \text{if } m \equiv 1 \pmod 2 \text{ or } M' \equiv 7, 11 \pmod{12}, \\ 4E_{1,5}(M';\,12) & \text{if } m \equiv 0 \pmod 2 \text{ and } M' \equiv 1, 5 \pmod{12}. \end{cases}$$

In particular, we obtain (Table 1).

Using Theorems 1 and 4, we obtain:

Corollary 1. *Let* $2N = 2^n \cdot 3^m \cdot M'$ *be a positive integer with* $\gcd(6, M') = 1$, $m \in \mathbb{N}_0$, *and* $n \in \mathbb{N}$. *Then*

$$\sum_{k=1}^{N-1} \hat{e}(2k)\hat{e}(2N-2k) = \begin{cases} -2(\chi(N;\,12) + 6\sigma(M')) & \text{if } 2N \equiv 0 \pmod{12}, \\ -2(E_{1,5}(M';\,12) - 3\sigma(M')) & \text{if } 2N \equiv 4 \pmod{12}, \\ 4E_{1,5}(M';\,12) - 3\sigma(M') & \text{if } 2N \equiv 8 \pmod{12}. \end{cases}$$

Table 1: Values of $\Sigma_{k=1}^{N-1}\hat{e}(k)\hat{e}(N-k)$ when $N = 2^n \cdot 3^m$

N	2^{2n+1}	2^{2n+2}	3^{2m+1}	3^{2m+2}	$2^{n+1} \cdot 3^{2m+1}$	$2^{n+1} \cdot 3^{2m}$
$\sum_{k=1}^{N-1}\hat{e}(k)\hat{e}(N-k)$	1	4	4	12	−12	−20

From now on, using modular form theory, we obtain the formulas $\sum_{a_1+\cdots+a_j=N}(\prod_{i=1}^{j}\hat{e}(a_i))$, $a_1,\ldots,a_j\in\mathbb{N}$ and $j=3,4$. Here, N is an odd integer. If N is an even integer, then the formula is very complex, so we will not deal with it in this article.

Let us define

$$\begin{aligned}
T_1(q) &:= \frac{\eta(4\tau)^2\eta(8\tau)^2\eta(12\tau)^4}{\eta(24\tau)^2} = q\prod_{n\geq1}\frac{(1-q^4)^2(1-q^8)^2(1-q^{12})^4}{(1-q^{24})^2},\\
T_2(q) &:= \frac{\eta(2\tau)\eta(6\tau)\eta(8\tau)^4\eta(12\tau)}{\eta(4\tau)} = q^2\prod_{n\geq1}\frac{(1-q^2)(1-q^6)(1-q^8)^4(1-q^{12})}{(1-q^4)},\\
T_3(q) &:= \frac{\eta(4\tau)^4\eta(12\tau)^2\eta(24\tau)^2}{\eta(8\tau)^2} = q^3\prod_{n\geq1}\frac{(1-q^4)^4(1-q^{12})^2(1-q^{24})^2}{(1-q^8)^2},\\
T_4(q) &:= \frac{\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(24\tau)^4}{\eta(12\tau)} = q^4\prod_{n\geq1}\frac{(1-q^2)(1-q^4)(1-q^6)(1-q^{24})^4}{(1-q^{12})},
\end{aligned}$$

and

$$T(q) := \frac{18}{7}(T_1(q)+2T_2(q)+T_3(q)+2T_4(q)) = \sum_{N\geq1}\mathfrak{t}(N)q^N.$$

Theorem 5. *Let $N(\geq3)$ be an odd positive integer with $N=3^m\cdot M'$ with $\gcd(3,M')=1$ and $m\in\mathbb{N}_0$. Then*

$$\sum_{\substack{a_1+a_2+a_3=N\\ a_1,a_2,a_3\geq1}}\hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3) = \begin{cases}
3E_{1,5}(N;\,12)-6\sigma(N)+\frac{3}{7}\bar{\sigma}_2(N)+\mathfrak{t}(N) & \text{if } N\equiv1\pmod{12},\\
12\sigma(M')-\frac{5}{7}\bar{\sigma}_2(N)-\frac{135}{7}\bar{\sigma}_2\left(\frac{N}{3}\right)+\mathfrak{t}(N) & \text{if } N\equiv3\pmod{12},\\
-12E_1(N;\,12)+3\sigma(N)+\frac{3}{7}\bar{\sigma}_2(N)+\mathfrak{t}(N) & \text{if } N\equiv5\pmod{12},\\
-6\sigma(N)-\frac{5}{7}\bar{\sigma}_2(N)+\mathfrak{t}(N) & \text{if } N\equiv7\pmod{12},\\
12E_{1,5}(N;\,12)+12\sigma(M')+\frac{3}{7}\bar{\sigma}_2(N)+\frac{81}{7}\bar{\sigma}_2\left(\frac{N}{3}\right)+\mathfrak{t}(N) & \text{if } N\equiv9\pmod{12},\\
3\sigma(N)-\frac{5}{7}\bar{\sigma}_2(N)+\mathfrak{t}(N) & \text{if } N\equiv11\pmod{12}.
\end{cases}$$

To find the formula of $\sum_{a_1+a_2+a_3+a_4=N}\hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3)\hat{e}(a_4)$, we need the following eta quotients:

$$S_i(q) := \frac{\eta(\tau)^{10-2i}\eta(3\tau)^{i-1}\eta(6\tau)^{5-i}\eta(18\tau)^{2i-2}}{\eta(2\tau)^{5-i}\eta(9\tau)^{i-1}} = q^i\prod_{n\geq1}\frac{(1-q)^{10-2i}(1-q^3)^{i-1}(1-q^6)^{5-i}(1-q^{18})^{2i-2}}{(1-q^2)^{5-i}(1-q^9)^{i-1}}$$

and

$$S(q) := -\frac{3}{5}(7S_1(q)+64S_2(q)+192S_3(q)+192S_4(q))+4T(q) = \sum_{N\geq1}\mathfrak{s}(N)q^N.$$

Theorem 6. *Let* $N(\geq 4)$ *be an odd positive integer with* $N = 3^m \cdot M'$ *with* $\gcd(3, M') = 1$ *and* $m \in \mathbb{N}_0$. *Then*

$$\sum_{\substack{a_1+a_2+a_3+a_4=N\\ a_1,a_2,a_3,a_4\geq 1}} \hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3)\hat{e}(a_4)$$

$$=\begin{cases} 4E_{1,5}(N;\,12) - 12\sigma(N) + \frac{12}{7}\bar{\sigma}_2(N) + \frac{1}{5}\sigma_3(N) + \mathfrak{s}(N) & \text{if } N \equiv 1(\text{mod } 12),\\ -16E_1(N;\,12) + 6\sigma(N) + \frac{12}{7}\bar{\sigma}_2(N) - \frac{2}{5}\sigma_3(N) + \mathfrak{s}(N) & \text{if } N \equiv 5(\text{mod } 12),\\ -12\sigma(N) - \frac{20}{7}\bar{\sigma}_2(N) + \frac{1}{5}\sigma_3(N) + \mathfrak{s}(N) & \text{if } N \equiv 7(\text{mod } 12),\\ 6\sigma(N) - \frac{20}{7}\bar{\sigma}_2(N) - \frac{2}{5}\sigma_3(N) + \mathfrak{s}(N) & \text{if } N \equiv 11(\text{mod } 12),\\ 24\sigma(M') - \frac{20}{7}\bar{\sigma}_2(N) - \frac{540}{7}\bar{\sigma}_2\left(\frac{N}{3}\right) + \frac{28}{5}\sigma_3\left(\frac{N}{3}\right) + \mathfrak{s}(N) & \text{if } N \equiv 3, 15(\text{mod } 36),\\ 24\sigma(M') - \frac{20}{7}\bar{\sigma}_2(N) - \frac{540}{7}\bar{\sigma}_2\left(\frac{N}{3}\right) - \frac{1}{10}\sigma_3(N) + \frac{42}{5}\sigma_3\left(\frac{N}{3}\right) & \\ \quad -\frac{243}{10}\sigma_3\left(\frac{N}{9}\right) + \mathfrak{s}(N) & \text{if } N \equiv 27(\text{mod } 36),\\ 16E_{1,5}(N;\,12) + 24\sigma(M') + \frac{12}{7}\bar{\sigma}_2(N) + \frac{324}{7}\bar{\sigma}_2\left(\frac{N}{3}\right) - \frac{1}{10}\sigma_3(N) & \\ \quad +\frac{42}{5}\sigma_3\left(\frac{N}{3}\right) - \frac{243}{10}\sigma_3\left(\frac{N}{9}\right) + \mathfrak{s}(N) & \text{if } N \equiv 9(\text{mod } 36),\\ 16E_{1,5}(N;\,12) + 24\sigma(M') + \frac{12}{7}\bar{\sigma}_2(N) + \frac{324}{7}\bar{\sigma}_2\left(\frac{N}{3}\right) + \frac{28}{5}\sigma_3\left(\frac{N}{3}\right) & \text{if } N \equiv 21, 33(\text{mod } 36).\\ \quad + \mathfrak{s}(N) & \end{cases}$$

This paper is organized as follows. In Section 2, some properties of certain infinite products and infinite sums are given. By using these equations, we derive computation formulas for the restricted divisor functions. In Section 3, we give values of $\mathfrak{a}_1(N)$ with $N \equiv 1(\text{mod } 2)$. Furthermore, we prove Theorems 1 and 2. In Section 4, we give values of $\mathfrak{a}_1(N)$ with $N \equiv 1 \pm 2, \pm 4(\text{mod } 12)$. In Section 5, we give values of $\mathfrak{a}_1(N)$ with $N \equiv 6(\text{mod } 12)$. In Section 6, we give values of $\mathfrak{a}_1(N)$ with $N \equiv 0(\text{mod } 12)$. In Section 7, we give values of $\mathfrak{a}_2(N)$ and prove Theorems 3 and 4. In Section 8, we prove Theorems 5 and 6 using the theory of modular forms.

2 Preliminary

In [20, p. 10, 21], we find two curious identities

$$\prod_{n\geq 1}\frac{(1-q^n)^2}{(1-2q^n\cos 2u+q^{2n})} = 1 - 4\sin u\sum_{N\geq 1}q^N\sum_{\omega|N}\sin\left(\frac{2N}{\omega}-\omega\right)u \tag{2.1}$$

and

$$\prod_{n\geq 1}\frac{(1-q^n)^4}{(1-2q^n\cos u+q^{2n})^2} = 1 - 8\sin^2\frac{u}{2}\sum_{N\geq 1}q^N\sum_{\substack{nk=N\\ n,k\geq 1}} n\cos(k-n)u. \tag{2.2}$$

Set $u = \frac{\pi}{6}$ in (2.1):

$$\prod_{n\geq 1}\frac{(1-q^n)(1-q^{2n})(1-q^{3n})}{(1-q^{6n})} = 1 - 2\sum_{N\geq 1}q^N\sum_{\omega|N}\sin\left(\frac{2N}{\omega}-\omega\right)\frac{\pi}{6}. \tag{2.3}$$

Now if we set $u = \frac{\pi}{3}$ in (2.2), we obtain

$$\left(\prod_{n\geq 1}\frac{(1-q^n)(1-q^{2n})(1-q^{3n})}{(1-q^{6n})}\right)^2 = 1 - 2\sum_{N\geq 1} q^N \sum_{\substack{nk=N\\ n,k\geq 1}} n\cos(k-n)\frac{\pi}{3}. \tag{2.4}$$

Let $(\bar{a}, \bar{b}) := (a, b)(\text{mod } 12)$, that is, $\bar{a} \equiv a(\text{mod } 12)$ and $\bar{b} \equiv b(\text{mod } 12)$. Table 2 shows values of $\sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6}$ for each divisor ω of N. Table 3 is the operation table for the result of calculating the value of $\frac{2N}{\omega} \cdot \omega$ (mod 12).

Lemma 1. *Let N be a positive integer. Then*

$$T_{\text{even}}(N) := \sum_{\substack{\omega|N\\ \frac{2N}{\omega}-\omega\equiv 0(\text{mod } 2)}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} = 0.$$

Proof. Let ω be a positive divisor of N. If N is an odd integer, then $\frac{2N}{\omega} \equiv 0(\text{mod } 2)$, $\omega \equiv 1(\text{mod } 2)$, $\frac{2N}{\omega} - \omega \equiv 1(\text{mod } 2)$, and thus $T_{\text{even}}(N) = 0$. So, we consider an even integer N and an even divisor ω satisfying $\frac{2N}{\omega} - \omega \equiv 0(\text{mod } 2)$. It is easily checked that

$$\sum_{\substack{\omega|N\\ \frac{2N}{\omega}-\omega\equiv 0(\text{mod } 6)}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} = 0. \tag{2.5}$$

Let $S_k := \left\{\left(\frac{2N}{\omega}, \omega\right) \in \mathbb{N}\times\mathbb{N} \,\middle|\, \frac{2N}{\omega} - \omega \equiv k(\text{mod } 12),\ \omega|N\right\}$ with $k = 2, 4, 8, 10$.

If $\frac{2N}{\omega} - \omega \equiv 2$ (resp., 4) (mod 12), then $N = \left(\frac{2N}{\omega}\right)\cdot\frac{\omega}{2}$ and thus, $\frac{2N}{\omega}|N$. Put $\omega' := \frac{2N}{\omega}$. Then

$$\frac{2N}{\omega} - \omega = \omega' - \frac{2N}{\omega'} = -\left(\frac{2N}{\omega'} - \omega'\right) \quad \text{and} \quad \omega \neq \frac{2N}{\omega}. \tag{2.6}$$

Table 2: Values of $\sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6}$ when $\frac{2N}{\omega} - \omega \equiv i$ (mod 12)

$\frac{2N}{\omega} - \omega \equiv i$ (mod 12)	0	1	2	3	4	5	6	7	8	9	10	11
$\sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

Table 3: Multiplicative operation table of $\left(\bar{\omega}, \frac{\bar{N}}{\omega}\right)$ satisfying $\omega|N$

$\frac{\bar{N}}{\omega}\bar{\omega}$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	♣1	2	♠3	4	⋆5	6	♡7	8	♢9	10	△11
2	0	†2	4	♮6	8	•10	0	†2	4	♮6	8	•10
3	0	♠3	6	♢9	0	♠3	6	♢9	0	♠3	6	♢9
4	0	□4	8	0	4	∇8	0	□4	8	0	4	∇8
5	0	⋆5	10	♠3	8	♣1	6	△11	4	♢9	2	♡7
6	0	♮6	0	♮6	0	♮6	0	♮6	0	♮6	0	♮6
7	0	♡7	2	♢9	4	△11	6	♣1	8	♠3	10	⋆5
8	0	∇8	4	0	8	□4	0	∇8	4	0	8	□4
9	0	♢9	6	♠3	0	♢9	6	♠3	0	♢9	6	♠3
10	0	•10	8	♮6	4	†2	0	•10	8	♮6	4	†2
11	0	△11	10	♢9	8	♡7	6	⋆5	4	♠3	2	♣1

By (2.6), we directly see that

$$\frac{2N}{\omega} - \omega \equiv 2(\text{resp.}, 4)(\text{mod } 12) \quad \text{if and only if } \frac{2N}{\omega'} - \omega' \equiv 10(\text{resp.}, 8)(\text{mod } 12). \tag{2.7}$$

Thus, by (2.6) and (2.7), $f_i : S_{2i} \to S_{12-2i}$ with $f_i\left(\left(\frac{2N}{\omega}, \omega\right)\right) = \left(\frac{2N}{\omega'}, \omega'\right)$ is a bijective map and

$$\#S_{2i} = \#S_{12-2i} \text{ with } i = 1, 2. \tag{2.8}$$

Combining (2.8) with (2.5), we find

$$\begin{aligned}
T_{\text{even}}(N) &= \sum_{k=0}^{5} \sum_{\substack{\omega \mid N \\ \frac{2N}{\omega} - \omega \equiv 2k(\text{mod } 12)}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} \\
&= \sum_{k=1}^{2} \sum_{\substack{\omega \mid N \\ \frac{2N}{\omega} - \omega \equiv \pm 2k(\text{mod } 12)}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} \\
&= (\#S_2 + \#S_4)\frac{\sqrt{3}}{2} + (\#S_8 + \#S_{10})\left(-\frac{\sqrt{3}}{2}\right) \\
&= 0.
\end{aligned}$$

This completes the proof of Lemma 1. □

Using (2.1) and Table 2 we obtain the following result.

Remark 1. Let ω be a divisor of N satisfying $\frac{2N}{\omega} - \omega \equiv 0(\text{mod } 2)$. It is easily verified that

$$\frac{2N}{\omega} - \omega \equiv 0(\text{mod } 2) \text{ if and only if } \omega \equiv 0(\text{mod } 2). \tag{2.9}$$

Combining Lemma 1 with (2.9), we obtain

$$T_{\text{even}}(N) = \sum_{\substack{\omega \mid N \\ \omega \text{ even}}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} = 0. \tag{2.10}$$

Lemma 2. *Let N be a positive integer. Then*

$$T_{\text{odd}}(N) := \sum_{\substack{\omega \mid N \\ \frac{2N}{\omega} - \omega \equiv 1(\text{mod } 2)}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} = \sum_{\substack{\omega \mid N \\ \omega \text{ odd}}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} = \frac{1}{2}\hat{e}(N).$$

Proof. By (2.9) and (2.10), we can obtain

$$T_{\text{odd}}(N) = \sum_{\substack{\omega \mid N \\ \frac{2N}{\omega} - \omega \equiv 1(\text{mod } 2)}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} = \sum_{\substack{\omega \mid N \\ \omega \text{ odd}}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6}.$$

Also by Table 2,

$$T_{\text{odd}}(N) = \sum_{k=1}^{6} \left(\sum_{\substack{\omega \mid N \\ \frac{2N}{\omega} - \omega \equiv 2k-1(\text{mod } 12)}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} \right) = \frac{1}{2}\hat{e}(N).$$

□

Theorem 7. *Let $N \in \mathbb{N}_0$. Then*

$$\mathfrak{a}_1(N) = \begin{cases} 1 & \text{if } N = 0, \\ -\hat{e}(N) & \text{if } N > 0. \end{cases}$$

Table 4: Infinite product forms of $\hat{E}(N)$ and $\hat{e}(N)$

	$\hat{E}(N)$	$\hat{e}(N)$
Infinite product forms	$\prod_{n\geq 1}\frac{(1-q^{2n})(1-q^{4n})^2(1-q^{6n})^3}{(1-q^n)(1-q^{3n})(1-q^{12n})^2}$	$\prod_{n\geq 1}\frac{(1-q^n)(1-q^{2n})(1-q^{3n})}{(1-q^{6n})}$
References	[20, p. 82]	(1.1), Theorem 7

Proof. Equation (2.3) yields $\mathfrak{a}_1(0) = 1$. In fact, we can see that $\mathfrak{a}_1(N) = -2(T_{\text{odd}}(N) + T_{\text{even}}(N))$. Comparing Lemmas 1 and 2 with (2.3), we obtain the proof of Theorem 7. □

Remark 2. Comparing the infinite products for $\hat{e}(N)$ and $\hat{E}(N) := E_{1,5}(N;\,12) + 2E_3(N;\,12)$, they have curious forms. $\hat{E}(N)$ is a simpler arithmetic function than $\hat{e}(N)$, but from a perspective of infinite products, the infinite product for $\hat{e}(N)$ seems simpler than the infinite product for $\hat{E}(N)$ (Table 4).

3 Coefficient of $\mathfrak{a}_1(N)$ with odd N

Lemma 3. *If* $N \equiv 7(\text{mod } 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = 0$.

Proof. By (2.3), $\mathfrak{a}_1(N) = -2\sum_{\omega|N}\sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6}$. In Table 3, ♡ represents all cases $\left(\frac{N}{\omega}, \omega\right)$ satisfying $\frac{N}{\omega}\cdot\omega \equiv 7(\text{mod } 12)$ and $\omega|N$. We note that $\omega^2 \equiv 1(\text{mod } 12)$ when $\omega \equiv 1, 5, 7, 11(\text{mod } 12)$. Hence,

$$\frac{2N}{\omega} - \omega \equiv \frac{2}{\omega} - \omega \equiv \frac{1}{\omega^2}(2\omega - \omega^2\cdot\omega) \equiv 2\omega - \omega \equiv \omega(\text{mod } 12). \tag{3.1}$$

Here, $\frac{1}{\omega}$ means that $\omega\cdot\frac{1}{\omega} \equiv 1(\text{mod } 12)$ and $\frac{1}{\omega}\in\mathbb{Z}$.

Using Table 2 and (3.1),

$$\mathfrak{a}_1(N) = -2\left(\sum_{\substack{\omega|N\\ \omega\equiv1(12)}}\sin\frac{\pi}{6} + \sum_{\substack{\omega|N\\ \omega\equiv5(12)}}\sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\ \omega\equiv7(12)}}\sin\frac{7\pi}{6} + \sum_{\substack{\omega|N\\ \omega\equiv11(12)}}\sin\frac{11\pi}{6}\right). \tag{3.2}$$

Let $F_i := \{\omega|\omega \equiv i(\text{mod } 12), \omega|N\}$ with $i = 1, 5, 7, 11$. For each $\omega|N$, let $f_k : F_k \to F_{k+6}$ be the maps defined by $f_k(\omega) = \frac{N}{\omega}$ with $k = 1, 5$. It is easily verified that f_k are bijective maps and $\#F_i = \#F_{i+6}$.

Therefore, $\mathfrak{a}_1(N) = -E_{1,5}(N;\,12) = -2\left\{(\#F_1 - \#F_7)\left(\frac{1}{2}\right) + (\#F_5 - \#F_{11})\left(\frac{1}{2}\right)\right\} = 0.$ □

Lemma 4. *If* $N \equiv 11(\text{mod } 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = 0$.

Proof. In Table 3, △ represents all cases $\left(\frac{N}{\omega}, \omega\right)$ satisfying $\frac{N}{\omega}\cdot\omega \equiv 11(\text{mod } 12)$ and $\omega|N$. That is, possible ordered pairs $\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)$ are $(\bar{1}, \bar{11})$, $(\bar{11}, \bar{1})$, $(\bar{5}, \bar{7})$, and $(\bar{7}, \bar{5})$. Thus,

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\sum_{\omega|N}\sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6}\\ &= -2\left(\sum_{\substack{\omega|N\\ \omega\equiv1(12)}}\sin\frac{9\pi}{6} + \sum_{\substack{\omega|N\\ \omega\equiv5(12)}}\sin\frac{9\pi}{6} + \sum_{\substack{\omega|N\\ \omega\equiv7(12)}}\sin\frac{3\pi}{6} + \sum_{\substack{\omega|N\\ \omega\equiv11(12)}}\sin\frac{3\pi}{6}\right)\\ &= -2\left\{\left(\sum_{\substack{\omega|N\\ \omega\equiv1(12)}}(-1) + \sum_{\substack{\omega|N\\ \omega\equiv11(12)}}1\right) + \left(\sum_{\substack{\omega|N\\ \omega\equiv5(12)}}(-1) + \sum_{\substack{\omega|N\\ \omega\equiv7(12)}}1\right)\right\}.\end{aligned} \tag{3.3}$$

It is trivial that $\omega \neq \frac{N}{\omega}$. If $\left(\frac{N}{\omega}, \omega\right)$ exists, then $\left(\omega, \frac{N}{\omega}\right)$ always exists. So, $\#\{\omega|\omega \equiv 1(\bmod\ 12), \omega|N\} = \#\{\omega|\omega \equiv 11(\bmod\ 12), \omega|N\}$ and $\#\{\omega|\omega \equiv 5(\bmod\ 12), \omega|N\} = \#\{\omega|\omega \equiv 7(\bmod\ 12), \omega|N\}$. □

Lemma 5. *If* $N \equiv 3(\bmod 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = 0$.

Proof. In Table 3, ♠ represents all cases $\left(\frac{N}{\omega}, \omega\right)$ satisfying $\frac{N}{\omega} \cdot \omega \equiv 3(\bmod\ 12)$ and $\omega|N$. In equation (3.4), $\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{a}, \bar{b})$ is abbreviated as $(\bar{a}, \bar{b})$. Thus,

$$\begin{aligned}\mathfrak{a}_1(N) = -2\Bigg(&\sum_{\substack{\omega|N\\(\bar{1},\bar{3})}} \sin\frac{11\pi}{6} + \sum_{\substack{\omega|N\\(\bar{3},\bar{1})}} \sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\(\bar{3},\bar{5})}} \sin\frac{\pi}{6} + \sum_{\substack{\omega|N\\(\bar{5},\bar{3})}} \sin\frac{7\pi}{6} + \sum_{\substack{\omega|N\\(\bar{3},\bar{9})}} \sin\frac{9\pi}{6} \\ &+ \sum_{\substack{\omega|N\\(\bar{9},\bar{3})}} \sin\frac{3\pi}{6} + \sum_{\substack{\omega|N\\(\bar{7},\bar{9})}} \sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\(\bar{9},\bar{7})}} \sin\frac{11\pi}{6} + \sum_{\substack{\omega|N\\(\bar{9},\bar{11})}} \sin\frac{7\pi}{6} + \sum_{\substack{\omega|N\\(\bar{11},\bar{9})}} \sin\frac{\pi}{6}\Bigg) = 0.\end{aligned} \tag{3.4}$$

The last identity in (3.4) is derived from the following identities. $\#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{1}, \bar{3}), \omega|N\} = \#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{3}, \bar{1}), \omega|N\}$, $\#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{3}, \bar{5}), \omega|N\} = \#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{5}, \bar{3}), \omega|N\}$, $\#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)= (\bar{3}, \bar{9}), \omega|N\} = \#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{9}, \bar{3}), \omega|N\}$, $\#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{7}, \bar{9}), \omega|N\} = \#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{9}, \bar{7}), \omega|N\}$, and $\#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{9}, \bar{11})|, \omega|N\} = \#\{\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)|\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right) = (\bar{11}, \bar{9}), \omega|N\}$. □

Proof of Theorem 1. If $N \equiv 3(\bmod\ 4)$ is a positive integer, then $\mathfrak{a}_1(N) = 0$ by Lemmas 3–5. Similarly, using Lemmas 3–5, we obtain

$$\begin{cases} \sum_{k=1}^{\frac{N}{2}} \mathfrak{a}_1(2k-1)\mathfrak{a}_1(N-2k+1) = 0 & \text{if } N \equiv 0(\bmod\ 4), \\ \sum_{\substack{a+b+c=N\\ a,b,c:\ \text{odd}}} \mathfrak{a}_1(a)\mathfrak{a}_1(b)\mathfrak{a}_1(c) = 0 & \text{if } N \equiv 1(\bmod\ 4), \\ \sum_{\substack{a+b+c+d=N\\ a,b,c,d:\ \text{odd}}} \mathfrak{a}_1(a)\mathfrak{a}_1(b)\mathfrak{a}_1(c)\mathfrak{a}_1(d) = 0 & \text{if } N \equiv 2(\bmod\ 4). \end{cases}$$

It is easily seen that $k - N \equiv 0(\bmod\ 2)$ by $a_1 \equiv \ldots \equiv a_k \equiv 1(\bmod\ 2)$ and $\sum_{i=1}^k a_i = N$. If $a_1 \equiv \ldots \equiv a_k \equiv 1(\bmod\ 4)$ and $N \not\equiv k(\bmod\ 4)$ then $\sum_{i=1}^k a_i \equiv k \not\equiv N(\bmod\ 4)$. Thus, there is at least one positive integer, $a_i \equiv 3(\bmod\ 4)$. By Lemmas 3–5, $\mathfrak{a}_1(a_i) = 0$. This completes the proof of Theorem 1. □

Lemma 6. *If* $N \equiv 1(\bmod 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = -E_{1,5}(N;\ 12)$.

Proof. $(\bar{1}, \bar{1})$, $(\bar{5}, \bar{5})$, $(\bar{7}, \bar{7})$, and $(\bar{11}, \bar{11})$ are ordered pairs of $\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)$ satisfying $\frac{N}{\omega} \cdot \omega \equiv 1(\bmod\ 12)$ and $\omega|N$ (see ♣ in Table 3). Thus, we have $\frac{2N}{\omega} - \omega \equiv \omega(\bmod\ 12)$ and

$$\mathfrak{a}_1(N) = -2\sum_{\omega|N} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6} = -2\sum_{i=1,5,7,11}\left(\sum_{\substack{\omega|N\\ \omega\equiv i(12)}} \sin\frac{\omega\pi}{6}\right) = -E_{1,5}(N;\ 12).$$

This completes the proof of Lemma 6. □

Corollary 2. *Let $p_1, \dots, p_t$ be distinct primes and $e_1, \dots, e_t \in \mathbb{N}$. If $p_i (1 \le i \le t)$ are congruent to 1 or 5 modulo 12, then $\mathfrak{a}_1(\prod_{i=1}^t p_i^{e_i}) = -\prod_{i=1}^t (e_i + 1)$ with $\prod_{i=1}^t p_i^{e_i} \equiv 1 (\mathrm{mod}\ 12)$. In particular, $\prod_{i=1}^t p_i^{e_i}$ is square if and only if $\mathfrak{a}_1(\prod_{i=1}^t p_i^{e_i})$ is odd.*

Lemma 7. *Let $q_1, \dots, q_s$ be distinct primes, $f_1, \dots, f_s \in \mathbb{N}$ and $N = \prod_{j=1}^s q_j^{f_j}$. If $q_j (1 \le j \le s)$ are congruent to 7 or 11 modulo 12, then*

$$\mathfrak{a}_1(N) = \begin{cases} -1 & \text{if } f_1 \equiv f_2 \equiv \dots \equiv f_s \equiv 0 \ (\mathrm{mod}\ 2), \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let $N = \prod_{j=1}^s q_j^{f_j}$ with $\sum_{j=1}^s f_j \equiv 1 (\mathrm{mod}\ 2)$. Then $\mathfrak{a}_1(N) = 0$ by Lemmas 3 and 4. So, we consider $N = \prod_{j=1}^s q_j^{f_j}$ with $\sum_{j=1}^s f_j \equiv 0 (\mathrm{mod}\ 2)$. Let us first consider the case where N is a square number. That is, say $N = \prod_{j=1}^s q_j^{f_j} = \prod_{j=1}^s q_j^{2e_j}$. Several sets are defined below for proof. For $1 \le i \le s$, let

$X_i := \{1, q_i^2, q_i^4, \dots, q_i^{2e_i}\}$,
$Y_i := \{q_i, q_i^3, \dots, q_i^{2e_i - 1}\}$,
$U_i := \{d | d \equiv 1, 5 (\mathrm{mod}\ 12), d | \prod_{j=1}^i q_j^{2e_j}\}$,
$V_i := \{d | d \equiv 7, 11 (\mathrm{mod}\ 12), d | \prod_{j=1}^i q_j^{2e_j}\}$.

We want to use mathematical induction to prove $\mathfrak{a}_1(\prod_{j=1}^s q_j^{2e_j}) = -1$. First, consider the case of $q_1^{2e_1}$. Using $\#U_1 = \#X_1$, $\#V_1 = \#Y_1$ and $\#U_1 - \#V_1 = 1$, we obtain $\mathfrak{a}_1(q_1^{2e_1}) = -1$ by Lemma 6. For the ease of understanding, we show the case where $s = 2$. Using $1 \cdot 1 \equiv 1 (\mathrm{mod}\ 4)$, $3 \cdot 3 \equiv 1 (\mathrm{mod}\ 4)$, and $1 \cdot 3 \equiv 3 (\mathrm{mod}\ 4)$, we obtain

$$\begin{aligned} U_2 = (U_1 \otimes X_2) \cup (V_1 \otimes Y_2),\ V_2 = (U_1 \otimes Y_2) \cup (V_1 \otimes X_2), \text{ and} \\ \#U_2 - \#V_2 = (e_1 + 1)(e_2 + 1) + e_1 e_2 - (e_1 + 1)e_2 - e_1(e_2 + 1) = 1. \end{aligned} \tag{3.5}$$

For $1 \le i \le s - 1$, we assume that $\#U_i - \#V_i = 1$.

Similarly, with the same method as in (3.5), we obtain

$$\begin{aligned} U_s = (U_{s-1} \otimes X_s) \cup (V_{s-1} \otimes Y_s), \quad V_s = (U_{s-1} \otimes Y_s) \cup (V_{s-1} \otimes X_s), \text{ and} \\ \#U_s - \#V_s = (z)(e_s + 1) + (z - 1)e_s - (z)e_s - (z - 1)(e_s + 1) = 1. \end{aligned}$$

Here, $\#U_{s-1} := z$ and $\#V_{s-1} := z - 1$. Therefore, by induction, $\mathfrak{a}_1\left(\prod_{j=1}^s q_j^{2e_j}\right) = -1$.

Finally, we consider the case $N' = \prod_{j=1}^u q_j^{2e_j - 1} \prod_{j=u+1}^s q_j^{2e_j}$ with $u \ge 1$. Then $\#U_i = \#V_i$ with $i = 1, \dots, u$. Similarly, with the same method as in (3.5), we obtain

$$\begin{aligned} U_{u+1} = (U_u \otimes X_{u+1}) \cup (V_u \otimes Y_{u+1}), \quad V_{u+1} = (U_u \otimes Y_{u+1}) \cup (V_u \otimes X_{u+1}), \text{ and} \\ \#U_{u+1} - \#V_{u+1} = z'(e_{u+1} + 1) + z' e_{u+1} - z' e_{u+1} - z'(e_{u+1} + 1) = 0. \end{aligned}$$

Here, $\#U_u = \#V_u = z'$. So, we obtain $\#U_{u+1} - \#V_{u+1} = 0$. If we do this repeated calculation up to $u + 2, \dots, s$, we obtain $\#U_s - \#V_s = 0$ and $\mathfrak{a}_1(\prod_{j=1}^u q_j^{2e_j - 1} \prod_{j=u+1}^s q_j^{2e_j}) = 0$. After all the proof of Lemma 7 is completed. □

Using the method used in (3.5), Lemma 6, Corollary 2, and Lemma 7, we obtain the following corollary for general numbers.

Corollary 3. *Let $p_1, \dots, p_r \equiv 1, 5 (\mathrm{mod}\ 12)$ and $q_1, \dots, q_s \equiv 7, 11 (\mathrm{mod}\ 12)$ be distinct primes and $e_1, \dots, e_r, f_1, \dots, f_s \in \mathbb{N}_0$. Then*

$$\mathfrak{a}_1\left(\prod_{j=1}^r p_j^{e_j} \prod_{i=1}^s q_i^{f_i}\right) = \begin{cases} -(e_1 + 1) \cdots (e_r + 1) & \text{if } f_1 \equiv \dots \equiv f_s \equiv 0 \ (\mathrm{mod}\ 2), \\ -1 & \text{if } f_1 \equiv \dots \equiv f_s \equiv 0 \ (\mathrm{mod}\ 2), \quad e_1 = \dots = e_r = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Here, $f_1 \equiv f_2 \equiv \dots \equiv f_s \equiv 0 \ (\mathrm{mod}\ 2)$ includes $f_1 = \dots = f_s = 0$.

Lemma 8. *If* $N \equiv 5 (\bmod\ 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = 4E_1(N;\ 12)$.

Proof. Possible ordered pairs of $\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)$ satisfying $\frac{N}{\omega} \cdot \omega \equiv 5 (\bmod\ 12)$ and $\omega | N$ are $(\bar{1}, \bar{5})$, $(\bar{5}, \bar{1})$, $(\bar{7}, \bar{11})$, and $(\bar{11}, \bar{7})$ (see ⋆ in Table 3). Thus, $\#\{\omega | \omega \equiv 1 (\bmod\ 12), \omega | N\} = \#\{\omega | \omega \equiv 5 (\bmod\ 12), \omega | N\}$, $\#\{\omega | \omega \equiv 7 (\bmod\ 12), \omega | N\} = \#\{\omega | \omega \equiv 11 (\bmod\ 12), \omega | N\}$, and

$$\begin{aligned}
\mathfrak{a}_1(N) &= -2\left(\sum_{\substack{\omega|N \\ (\bar{1},\bar{5})}} \sin\frac{9\pi}{6} + \sum_{\substack{\omega|N \\ (\bar{5},\bar{1})}} \sin\frac{9\pi}{6} + \sum_{\substack{\omega|N \\ (\bar{7},\bar{11})}} \sin\frac{3\pi}{6} + \sum_{\substack{\omega|N \\ (\bar{11},\bar{7})}} \sin\frac{3\pi}{6}\right) \\
&= -2\left(\sum_{\substack{\omega|N \\ \omega\equiv 1,5(12)}} (-1) + \sum_{\substack{\omega|N \\ \omega\equiv 7,11(12)}} 1\right) = 4\left(\sum_{\substack{\omega|N \\ \omega\equiv 1(12)}} 1 + \sum_{\substack{\omega|N \\ \omega\equiv 11(12)}} (-1)\right) \\
&= 4E_1(N;\ 12).
\end{aligned}$$

□

Lemma 9. *If* $N \equiv 9 (\bmod\ 12)$ *is a natural number, then* $E_3(N;\ 12) = 0$.

Proof. Write $N = 3^m \cdot M$ with $\gcd(3, M) = 1$ and $m \in \mathbb{N}$. Since $N \equiv 1 (\bmod\ 4)$, we obtain $M \equiv 1 (\bmod\ 4)$ if m is even and $M \equiv 3 (\bmod\ 4)$ if m is odd. Let $D_N := \{d | d \in \mathbb{N}, d | N\}$ and $D_M := \{d | d \in \mathbb{N}, d | M\}$. Then

$$D_N = D_M \sqcup 3D_M \sqcup \cdots \sqcup 3^m D_M.$$

Here, $3^k D_M$ denotes the set $\{3^k d | d \in \mathbb{N}, d | M\}$ and $\sqcup$ is a symbol for disjoint union. Let d_i be a positive divisor of M. Note that d_i is not divisible by 2 and 3. If $d_i \equiv 1 (\bmod\ 4)$, then $3^{2l-1} d_i \equiv 3 (\bmod\ 12)$ and $3^{2l} d_i \equiv 9 (\bmod\ 12)$ for $l \in \mathbb{N}$. If $d_i \equiv 3 (\bmod\ 4)$, then $3^{2l-1} d_i \equiv 9 (\bmod\ 12)$ and $3^{2l} d_i \equiv 3 (\bmod\ 12)$ for $l \in \mathbb{N}$. This implies that $\#\{d | d \equiv 3 (\bmod\ 12), d \in 3^{2l-1} D_M \sqcup 3^{2l} D_M\} = \#\{d | d \equiv 9 (\bmod\ 12), d \in 3^{2l-1} D_M \sqcup 3^{2l} D_M\} = \# D_M$.

We assume that m is even. By the above observation, we have $E_3(N;\ 12) = E_3(M;\ 12)$. Since no divisors of M are congruent to ± 3 modulo 12, we conclude $E_3(N;\ 12) = 0$. Now let m be odd. Then

$$E_3(N;\ 12) = E_3(M;\ 12) + \#\{d | d \equiv 3 (\bmod\ 12), d \in 3D_M\} - \#\{d | d \equiv 9 (\bmod\ 12), d \in 3D_M\}.$$

Since $M \equiv 3 (\bmod\ 4)$, M is not a perfect square. Thus, $\# D_M$ is an even number. Let $\# D_M = 2r$ and write $D_M = \left\{d_1, \cdots, d_r, \frac{M}{d_1}, \cdots, \frac{M}{d_r}\right\}$. We observe that $d_i \equiv \pm 1 (\bmod\ 4)$ if and only if $\frac{M}{d_i} \equiv \mp 1 (\bmod\ 4)$. Thus, $3d_i \equiv \pm 3 (\bmod\ 12)$ if and only if $\frac{3M}{d_i} \equiv \mp 3 (\bmod\ 12)$ and then $\#\{d | d \equiv 3 (\bmod\ 12),\ d \in 3D_M\} = \#\{d | d \equiv 9 (\bmod\ 12), d \in 3D_M\}$. Finally, we obtain $E_3(N;\ 12) = E_3(M;\ 12) = 0$. □

Lemma 10. *If* $N \equiv 9 (\bmod\ 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = -4E_{1,5}(N;\ 12)$.

Proof. Possible ordered pairs of $\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)$ satisfying $\frac{N}{\omega} \cdot \omega \equiv 9 (\bmod\ 12)$ and $\omega | N$ are $(\bar{1}, \bar{9})$, $(\bar{9}, \bar{1})$, $(\bar{3}, \bar{3})$, $(\bar{3}, \bar{7})$, $(\bar{7}, \bar{3})$, $(\bar{3}, \bar{11})$, $(\bar{11}, \bar{3})$, $(\bar{5}, \bar{9})$, $(\bar{9}, \bar{5})$, and $(\bar{9}, \bar{9})$ (see ♢ in Table 3). Thus,

$$\begin{aligned}
\mathfrak{a}_1(N) &= -2\left(\sum_{\substack{(\bar{1},\bar{9}) \\ (\bar{9},\bar{1})}} \sin\frac{5\pi}{6} + \sum_{\substack{(\bar{3},\bar{7}) \\ (\bar{7},\bar{3})}} \sin\frac{11\pi}{6} + \sum_{\substack{(\bar{3},\bar{11}) \\ (\bar{11},\bar{3})}} \sin\frac{7\pi}{6} + \sum_{\substack{(\bar{5},\bar{9}) \\ (\bar{9},\bar{5})}} \sin\frac{\pi}{6} + \sum_{\substack{\omega|N \\ (\bar{3},\bar{3})}} \sin\frac{3\pi}{6} + \sum_{\substack{\omega|N \\ (\bar{9},\bar{9})}} \sin\frac{9\pi}{6}\right) \\
&= -2\left(\sum_{\substack{\omega|N \\ (\bar{9},\bar{1})}} 1 + \sum_{\substack{\omega|N \\ (\bar{3},\bar{7})}} (-1) + \sum_{\substack{\omega|N \\ (\bar{3},\bar{11})}} (-1) + \sum_{\substack{\omega|N \\ (\bar{9},\bar{5})}} 1 + \sum_{\substack{\omega|N \\ (\bar{3},\bar{3})}} 1 + \sum_{\substack{\omega|N \\ (\bar{9},\bar{9})}} (-1)\right) \\
&= -2\left(\sum_{\substack{\omega|N \\ \omega\equiv 1(12)}} 1 + \sum_{\substack{\omega|N \\ \omega\equiv 7(12)}} (-1) + \sum_{\substack{\omega|N \\ \omega\equiv 11(12)}} (-1) + \sum_{\substack{\omega|N \\ \omega\equiv 5(12)}} 1 + \left(\sum_{\substack{\omega|N \\ \omega\equiv 3(12)}} 1 - \sum_{\substack{\omega|N \\ \omega\equiv 7(12)}} 1 - \sum_{\substack{\omega|N \\ \omega\equiv 11(12)}} 1\right.\right.
\end{aligned}$$

$$+\left(\sum_{\substack{\omega|N\\ \omega\equiv 9(12)}}(-1)-\sum_{\substack{\omega|N\\ \omega\equiv 1(12)}}(-1)-\sum_{\substack{\omega|N\\ \omega\equiv 5(12)}}(-1)\right)\Bigg)$$
$$=-2(2E_{1,5}(N;\ 12)+E_3(N;\ 12)).$$

Now from Lemma 9 we have the result. □

Proof of Theorem 2. It is easy to see that $\mathfrak{a}_1(1)=-1$. If $n(<-1)$ is a negative and odd integer, then we find a natural number N satisfying $N=p_1^{-(n+1)}$ with a prime $p_1\equiv 1 \pmod{12}$. By Corollary 3, we obtain $\mathfrak{a}_1(N)=n$. So, there is a natural number N where $\mathfrak{a}_1(N)=n$ for all negative and odd integers n. The remaining part of the proof of Theorem 2 is completed by Lemmas 3, 4, 5, 6, Corollary 2, Lemma 7, Corollary 3, and Lemmas 8, 10. □

4 Coefficient of $\mathfrak{a}_1(N)$ with $N\equiv\pm 2,\pm 4 (\mathrm{mod}\ 12)$

Lemma 11. *When n is a natural number,*

$$\mathfrak{a}_1(2^n)=\begin{cases}-2 & \text{if } n\equiv 1\ (\mathrm{mod}\ 2),\\ 1 & \text{if } n\equiv 0\ (\mathrm{mod}\ 2).\end{cases}$$

Proof. It is well-known that

$$2^{n+1}-1\equiv\begin{cases}3(\mathrm{mod}\ 12) & \text{if } n\equiv 1\ (\mathrm{mod}\ 2),\\ 7(\mathrm{mod}\ 12) & \text{if } n\equiv 0\ (\mathrm{mod}\ 2).\end{cases}\tag{4.1}$$

By (2.3) and (2.10), we obtain

$$\mathfrak{a}_1(2^n)=-2\sin(2^{n+1}-1)\frac{\pi}{6}.\tag{4.2}$$

This completes the proof of Lemma 11 by (4.1) and (4.2). □

Lemma 12. *If* $N\equiv 2(\mathrm{mod}\ 12)$ *is a natural number, then* $\mathfrak{a}_1(N)=-2E_{1,5}(N;\ 12)$.

Proof. Possible ordered pairs of $\left(\frac{\bar N}{\omega},\bar\omega\right)$ satisfying $\frac{N}{\omega}\cdot\omega\equiv 2(\mathrm{mod}\ 12)$ and $\omega|N$ are $(\bar 1,\bar 2)$, $(\bar 2,\bar 1)$, $(\bar 2,\bar 7)$, $(\bar 7,\bar 2)$, $(\bar 5,\overline{10})$, $(\overline{10},\bar 5)$, $(\overline{10},\overline{11})$, and $(\overline{11},\overline{10})$ (Table 3). By (2.10), only odd divisors of N are possible. That is, possible pairs of $\left(\frac{\bar N}{\omega},\bar\omega\right)$ are $(\bar 2,\bar 1)$, $(\bar 2,\bar 7)$, $(\overline{10},\bar 5)$, and $(\overline{10},\overline{11})$ (see † in Table 3). Therefore,

$$\mathfrak{a}_1(N)=-2\sum_{\omega|N}\sin\left(\frac{2N}{\omega}-\omega\right)\frac{\pi}{6}$$
$$=-2\left(\sum_{\substack{\omega|N\\(\bar 2,\bar 1)}}\sin\frac{3\pi}{6}+\sum_{\substack{\omega|N\\(\bar 2,\bar 7)}}\sin\frac{9\pi}{6}+\sum_{\substack{\omega|N\\(\overline{10},\bar 5)}}\sin\frac{3\pi}{6}+\sum_{\substack{\omega|N\\(\overline{10},\overline{11})}}\sin\frac{9\pi}{6}\right)$$
$$=-2\left(\sum_{\substack{\omega|N\\ \omega\equiv 1(12)}}1+\sum_{\substack{\omega|N\\ \omega\equiv 7(12)}}(-1)+\sum_{\substack{\omega|N\\ \omega\equiv 5(12)}}1+\sum_{\substack{\omega|N\\ \omega\equiv 11(12)}}(-1)\right)$$
$$=-2E_{1,5}(N;\ 12).$$
□

Lemma 13. *If* $N\equiv 10(\mathrm{mod}\ 12)$ *is a natural number, then* $\mathfrak{a}_1(N)=E_{1,5}(N;\ 12)\equiv 0(\mathrm{mod}\ 2)$.

Proof. By (2.10), possible ordered pairs of $\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)$ satisfying $\frac{N}{\omega} \cdot \omega \equiv 10(\bmod\ 12)$, $\omega \equiv 1(\bmod\ 2)$, and $\omega|N$ are $(\bar{10}, \bar{1})$, $(\bar{2}, \bar{5})$, $(\bar{2}, \bar{11})$, and $(\bar{10}, \bar{7})$ (see • in Table 3). Thus, we obtain

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\left(\sum_{\substack{\omega|N\\(\bar{10},\bar{1})}} \sin\frac{7\pi}{6} + \sum_{\substack{\omega|N\\(\bar{2},\bar{5})}} \sin\frac{11\pi}{6} + \sum_{\substack{\omega|N\\(\bar{2},\bar{11})}} \sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\(\bar{10},\bar{7})}} \sin\frac{\pi}{6}\right)\\ &= -2\left(\sum_{\substack{\omega|N\\\omega\equiv1(12)}} \left(-\frac{1}{2}\right) + \sum_{\substack{\omega|N\\\omega\equiv5(12)}} \left(-\frac{1}{2}\right) + \sum_{\substack{\omega|N\\\omega\equiv7(12)}} \frac{1}{2} + \sum_{\substack{\omega|N\\\omega\equiv11(12)}} \frac{1}{2}\right)\\ &= E_{1,5}(N;\ 12).\end{aligned}$$

Since $N \equiv 10(\bmod 12)$, there exists an integer $L \equiv 5(\bmod 6)$ satisfying $N = 2L$. If ω is an odd divisor of N, then $\omega|L$. If $\omega \equiv 1(\text{resp., } 5)(\bmod 6)$, then $\frac{L}{\omega} \equiv 5(\text{resp., } 1)(\bmod 6)$. So, we obtain

$$\begin{aligned}E_{1,5}(N;\ 12) &= E_{1,5}(L;\ 12)\\ &= \#\left\{\left(\frac{L}{\omega}, \omega\right) \Big| \left(\frac{\bar{L}}{\omega}, \bar{\omega}\right) = (\bar{1}, \bar{5}), \omega|L\right\} \cdot 2 + \#\left\{\left(\frac{L}{\omega}, \omega\right) \Big| \left(\frac{\bar{L}}{\omega}, \bar{\omega}\right) = (\bar{1}, \bar{11}), \omega|L\right\} \cdot 0\\ &\quad + \#\left\{\left(\frac{L}{\omega}, \omega\right) \Big| \left(\frac{\bar{L}}{\omega}, \bar{\omega}\right) = (\bar{7}, \bar{5}), \omega|L\right\} \cdot 0 + \#\left\{\left(\frac{L}{\omega}, \omega\right) \Big| \left(\frac{\bar{L}}{\omega}, \bar{\omega}\right) = (\bar{7}, \bar{11}), \omega|L\right\} \cdot (-2)\\ &\equiv 0(\bmod\ 2).\end{aligned}$$

□

Lemma 14. *If* $N \equiv 4(\bmod 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = E_{1,5}(N;\ 12)$.

Proof. By (2.10), possible ordered pairs of $\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)$ satisfying $\frac{N}{\omega} \cdot \omega \equiv 4(\bmod\ 12)$, $\omega \equiv 1(\bmod\ 2)$, and $\omega|N$ are $(\bar{4}, \bar{1})$, $(\bar{4}, \bar{7})$, $(\bar{8}, \bar{5})$, and $(\bar{8}, \bar{11})$ (see □ in Table 3). Thus, we obtain

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\left(\sum_{\substack{\omega|N\\(\bar{4},\bar{1})}} \sin\frac{7\pi}{6} + \sum_{\substack{\omega|N\\(\bar{4},\bar{7})}} \sin\frac{\pi}{6} + \sum_{\substack{\omega|N\\(\bar{8},\bar{5})}} \sin\frac{11\pi}{6} + \sum_{\substack{\omega|N\\(\bar{8},\bar{11})}} \sin\frac{5\pi}{6}\right)\\ &= -2\left(\sum_{\substack{\omega|N\\\omega\equiv1(12)}} \left(-\frac{1}{2}\right) + \sum_{\substack{\omega|N\\\omega\equiv5(12)}} \left(-\frac{1}{2}\right) + \sum_{\substack{\omega|N\\\omega\equiv7(12)}} \frac{1}{2} + \sum_{\substack{\omega|N\\\omega\equiv11(12)}} \frac{1}{2}\right)\\ &= E_{1,5}(N;\ 12).\end{aligned}$$

□

Lemma 15. *If* $N \equiv 8(\bmod 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = -2E_{1,5}(N;\ 12)$.

Proof. By (2.10), possible pairs of $\left(\frac{\bar{N}}{\omega}, \bar{\omega}\right)$ satisfying $\frac{N}{\omega} \cdot \omega \equiv 8(\bmod\ 12)$, $\omega \equiv 1(\bmod\ 2)$, and $\omega|N$ are $(\bar{4}, \bar{5})$, $(\bar{4}, \bar{11})$, $(\bar{8}, \bar{1})$, and $(\bar{8}, \bar{7})$ (see ∇ in Table 3). Thus, we obtain

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\left(\sum_{\substack{\omega|N\\(\bar{4},\bar{5})}} \sin\frac{3\pi}{6} + \sum_{\substack{\omega|N\\(\bar{4},\bar{11})}} \sin\frac{9\pi}{6} + \sum_{\substack{\omega|N\\(\bar{8},\bar{1})}} \sin\frac{3\pi}{6} + \sum_{\substack{\omega|N\\(\bar{8},\bar{7})}} \sin\frac{9\pi}{6}\right)\\ &= -2\left(\sum_{\substack{\omega|N\\\omega\equiv1(12)}} 1 + \sum_{\substack{\omega|N\\\omega\equiv5(12)}} 1 + \sum_{\substack{\omega|N\\\omega\equiv7(12)}} (-1) + \sum_{\substack{\omega|N\\\omega\equiv11(12)}} (-1)\right)\\ &= -2E_{1,5}(N;\ 12).\end{aligned}$$

□

5 Coefficient of $\mathfrak{a}_1(N)$ with $N \equiv 6 \pmod{12}$

In this section, it is assumed that N is a positive integer that satisfies $N \equiv 6 \pmod{12}$. Then there exist integers L and $M(=2L+1)$ such that $N = 12L + 6 = 6(2L+1) = 6M$. Now, we classify odd M into six cases and compute each $\mathfrak{a}_1(N)$ step by step. Then, by (2.10), possible pairs of $\left(\bar{\frac{N}{\omega}}, \bar{\omega}\right)$ satisfying $\frac{N}{\omega} \cdot \omega \equiv 6 \pmod{12}$, $\omega \equiv 1 \pmod 2$, and $\omega | N$ are $(\bar{2}, \bar{3})$, $(\bar{2}, \bar{9})$, $(\bar{6}, \bar{1})$, $(\bar{6}, \bar{3})$, $(\bar{6}, \bar{5})$, $(\bar{6}, \bar{7})$, $(\bar{6}, \bar{9})$, $(\bar{6}, \bar{11})$, $(\bar{10}, \bar{3})$, and $(\bar{10}, \bar{9})$ (see ♮ in Table 3).

Lemma 16. *Let* $N \equiv 6 \pmod{12}$ *and* $N = 6M$ *with* $\gcd(6, M) = 1$. *Then* $\mathfrak{a}_1(N) = 0$.

Proof. Let $\omega | N$ and $\omega' | M$ with $\omega \equiv 1 \pmod 2$. Then, by (2.10) and Lemma 2, possible ordered pairs of $\left(\bar{\frac{N}{\omega}}, \bar{\omega}\right) = (\text{even}, \text{odd}) = \left(6\frac{M}{\omega'}, \omega'\right)$ or $\left(2\frac{M}{\omega'}, 3\omega'\right)$.

First, we consider the case $M \equiv 1 \pmod{12}$. Possible ordered pairs $\left(\overline{6\frac{M}{\omega'}}, \overline{\omega'}\right)$ are $(\bar{6}, \bar{1})$, $(\bar{6}, \bar{5})$, $(\bar{6}, \bar{7})$, and $(\bar{6}, \bar{11})$ and possible ordered pairs $\left(\overline{2\frac{M}{\omega'}}, \overline{3\omega'}\right)$ are $(\overline{2\cdot 1}, \overline{3\cdot 1})$, $(\overline{2\cdot 5}, \overline{3\cdot 5})$, $(\overline{2\cdot 7}, \overline{3\cdot 7})$, and $(\overline{2\cdot 11}, \overline{3\cdot 11})$. Thus, we obtain

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\left(T_{\left(6\frac{M}{\omega'},\omega'\right)}(N) + \sum_{\substack{\omega|N\\(\overline{2\cdot1},\overline{3\cdot1})}} \sin\frac{\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot5},\overline{3\cdot5})}} \sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot7},\overline{3\cdot7})}} \sin\frac{7\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot11},\overline{3\cdot11})}} \sin\frac{11\pi}{6}\right)\\ &= -2\left(T_{\left(6\frac{M}{\omega'},\omega'\right)}(N) + \sum_{\substack{\omega|N\\\omega\equiv1(12)}} \frac{1}{2} + \sum_{\substack{\omega|N\\\omega\equiv5(12)}} \frac{1}{2} + \sum_{\substack{\omega|N\\\omega\equiv7(12)}} \left(-\frac{1}{2}\right) + \sum_{\substack{\omega|N\\\omega\equiv11(12)}} \left(-\frac{1}{2}\right)\right)\\ &= 0.\end{aligned}$$

Here,

$$\begin{aligned}T_{\left(6\frac{M}{\omega'},\omega'\right)}(N) &:= \sum_{\substack{\omega|N\\(\bar{6},\bar{1})}} \sin\frac{11\pi}{6} + \sum_{\substack{\omega|N\\(\bar{6},\bar{5})}} \sin\frac{7\pi}{6} + \sum_{\substack{\omega|N\\(\bar{6},\bar{7})}} \sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\(\bar{6},\bar{11})}} \sin\frac{\pi}{6}\\ &= \sum_{\substack{\omega|N\\\omega\equiv1(12)}} \left(-\frac{1}{2}\right) + \sum_{\substack{\omega|N\\\omega\equiv5(12)}} \left(-\frac{1}{2}\right) + \sum_{\substack{\omega|N\\\omega\equiv7(12)}} \frac{1}{2} + \sum_{\substack{\omega|N\\\omega\equiv11(12)}} \frac{1}{2}.\end{aligned} \tag{5.1}$$

Second, we consider the case $M \equiv 11 \pmod{12}$. By (2.10) and Lemma 2, possible ordered pairs $\left(\overline{6\frac{M}{\omega'}}, \overline{\omega'}\right)$ are the same with the case $M \equiv 1 \pmod{12}$ and possible ordered pairs $\left(\overline{2\frac{M}{\omega'}}, \overline{3\omega'}\right)$ are $(\overline{2\cdot 1}, \overline{3\cdot 11})$, $(\overline{2\cdot 5}, \overline{3\cdot 7})$, $(\overline{2\cdot 7}, \overline{3\cdot 5})$, and $(\overline{2\cdot 11}, \overline{3\cdot 1})$. Using (5.1), we obtain

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\left(T_{\left(6\frac{M}{\omega'},\omega'\right)}(N) + \sum_{\substack{\omega|N\\(\overline{2\cdot11},\overline{3\cdot1})}} \sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot7},\overline{3\cdot5})}} \sin\frac{\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot5},\overline{3\cdot7})}} \sin\frac{11\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot1},\overline{3\cdot11})}} \sin\frac{7\pi}{6}\right)\\ &= -2\left(T_{\left(6\frac{M}{\omega'},\omega'\right)}(N) + \sum_{\substack{\omega|N\\\omega\equiv1(12)}} \frac{1}{2} + \sum_{\substack{\omega|N\\\omega\equiv5(12)}} \frac{1}{2} + \sum_{\substack{\omega|N\\\omega\equiv7(12)}} \left(-\frac{1}{2}\right) + \sum_{\substack{\omega|N\\\omega\equiv11(12)}} \left(-\frac{1}{2}\right)\right)\\ &= 0.\end{aligned}$$

Let $M \equiv 5 \pmod{12}$ be a positive integer. In the same method as above, we obtain possible ordered pairs $(\overline{2\frac{M}{\omega'}}, \overline{3\omega'})$ are $(\overline{2\cdot 1}, \overline{3\cdot 5})$, $(\overline{2\cdot 5}, \overline{3\cdot 1})$, $(\overline{2\cdot 7}, \overline{3\cdot 11})$, $(\overline{2\cdot 11}, \overline{3\cdot 7})$, and

$$\mathfrak{a}_1(N) = -2\left(T_{\left(6\frac{M}{\omega'},\omega'\right)}(N) + \sum_{\substack{\omega|N\\(\overline{2\cdot 5},\overline{3\cdot 1})}} \sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot 1},\overline{3\cdot 5})}} \sin\frac{\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot 11},\overline{3\cdot 7})}} \sin\frac{11\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot 7},\overline{3\cdot 11})}} \sin\frac{7\pi}{6}\right) = 0.$$

Let $M \equiv 7(\bmod\ 12)$ be a positive integer. Then possible ordered pairs $\left(\overline{2\frac{M}{\omega'}}, \overline{3\omega'}\right)$ are $(\overline{2\cdot 1}, \overline{3\cdot 7})$, $(\overline{2\cdot 5}, \overline{3\cdot 11})$, $(\overline{2\cdot 7}, \overline{3\cdot 1})$, $(\overline{2\cdot 11}, \overline{3\cdot 5})$, and

$$\mathfrak{a}_1(N) = -2\left(T_{\left(6\frac{M}{\omega'},\omega'\right)}(N) + \sum_{\substack{\omega|N\\(\overline{2\cdot 7},\overline{3\cdot 1})}} \sin\frac{\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot 11},\overline{3\cdot 5})}} \sin\frac{5\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot 1},\overline{3\cdot 7})}} \sin\frac{7\pi}{6} + \sum_{\substack{\omega|N\\(\overline{2\cdot 5},\overline{3\cdot 11})}} \sin\frac{11\pi}{6}\right) = 0.$$

This completes the proof of Lemma 16. □

The case of $N = 6M$ with $\gcd(6, M) = 3$ will be shown in Remark 3.

6 Coefficient of $\mathfrak{a}_1(N)$ with $N \equiv 0(\bmod\ 12)$

In this section, it is assumed that N is a positive integer that satisfies $N \equiv 0(\bmod\ 12)$. Then there exists an integer M satisfying $N = 12M$. Now, we classify positive integers M into 12 cases.

Lemma 17. *If* $M \equiv \pm 1, \pm 5(\bmod\ 12)$ *is a natural number, then* $\mathfrak{a}_1(N) = 0$.

Proof. Let ω be a divisor of M. It is easily checked that $\omega^2 \equiv 1(\bmod\ 12)$ and $\omega \equiv 1$ or -1 or 5 or $-5(\bmod\ 12)$. Thus, we have

$$24\frac{M}{\omega} - \omega \equiv -\omega(\bmod\ 12) \tag{6.1}$$

and

$$8\frac{M}{\omega} - 3\omega \equiv \left(8\omega^2\frac{M}{\omega} - 3\omega\right) \equiv (8M-3)\omega \equiv \begin{cases} 5\omega(\bmod\ 12) & \text{if } M \equiv 1(\bmod\ 6),\\ \omega(\bmod\ 12) & \text{if } M \equiv 5(\bmod\ 6).\end{cases} \tag{6.2}$$

Now we have, by (6.1) and (6.2) and by $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$,

$$\sum_{\omega|N}\left(\sin\left(24\frac{M}{\omega} - \omega\right)\frac{\pi}{6} + \sin\left(8\frac{M}{\omega} - 3\omega\right)\frac{\pi}{6}\right) = \sum_{\omega|N}\left(\sin\frac{-\omega\pi}{6} + \sin\frac{\nu\omega\pi}{6}\right) = 0. \tag{6.3}$$

Here, $\nu = 1$ or 5. Furthermore, it is easy to see that $\left\{\left(\frac{N}{\omega}, \omega\right) = (\text{even}, \text{odd}) | \omega \equiv 1(\bmod\ 2), \omega|N\right\} = \left\{\left(12\frac{M}{\omega}, \omega\right) | \omega \equiv 1(\bmod\ 2), \omega|M\right\} \cup \left\{\left(4\frac{M}{\omega}, 3\omega\right) | \omega \equiv 1(\bmod\ 2), \omega|M\right\}$ and $\left\{\left(12\frac{M}{\omega}, \omega\right) | \omega \equiv 1(\bmod\ 2), \omega|M\right\} \cap \left\{\left(4\frac{M}{\omega}, 3\omega\right) | \omega \equiv 1(\bmod\ 2), \omega|M\right\} = \emptyset$.

By (2.9), (6.3), and Lemma 2,

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\left(\sum_{\substack{\omega|N\\ \frac{2N}{\omega}-\omega\equiv 1(\bmod\ 2)}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6}\right)\\ &= -2\left(\sum_{\substack{\omega|N\\ \omega\ \text{odd}}} \sin\left(\frac{2N}{\omega} - \omega\right)\frac{\pi}{6}\right)\end{aligned}$$

$$= -2\left(\sum_{\omega|N}\left(\sin\left(24\frac{M}{\omega} - \omega\right)\frac{\pi}{6} + \sin\left(8\frac{M}{\omega} - 3\omega\right)\frac{\pi}{6}\right)\right)$$
$$= 0. \qquad \square$$

Lemma 18. *Let* $N = 2^2 3^{2k-1} M$ *with* $\gcd(6, M) = 1$ *and* $k \in \mathbb{N}$. *Then* $\mathfrak{a}_1(N) = 0$.

Proof. Let $d|N$ and $\omega|M$. It is the only ordered pair $\left(\frac{N}{d}, d\right) = (\text{even}, \text{odd})$ that satisfy $\left(2^2 \cdot 3^t \cdot \frac{M}{\omega}, 3^{2k-t-1} \cdot \omega\right)$ with $0 \le t \le 2k - 1$. Thus, by Lemma 2,

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\sum_{\omega|M}\sum_{t=0}^{2k-1}\sin\left(2^3 \cdot 3^t \cdot \frac{M}{\omega} - 3^{2k-t-1}\cdot\omega\right)\frac{\pi}{6}\\ &= -2\sum_{\omega|M}\left(\sin\left(2^3 \cdot \frac{M}{\omega} - 3^{2k-1}\cdot\omega\right)\frac{\pi}{6} + \sum_{t=1}^{2k-2}\sin(-3^{2k-t-1}\cdot\omega)\frac{\pi}{6} + \sin\left(\frac{-\omega\pi}{6}\right)\right).\end{aligned} \tag{6.4}$$

Using $\omega^2 \equiv 1 (\bmod 12)$, $3^{2t} \equiv -3 (\bmod 12)$, $3^{2t-1} \equiv 3 (\bmod 12)$,

$$2^3\frac{M}{\omega} - 3^{2k-1}\omega \equiv (8M - 3)\omega \equiv \begin{cases} 5\omega & \text{if } M \equiv 1, 7 (\bmod 12),\\ \omega & \text{if } M \equiv 5, 11 (\bmod 12),\end{cases}$$

and $\sin\frac{\omega\pi}{2} + \sin\left(\frac{-\omega\pi}{2}\right) = 0$, equation (6.4) becomes

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\left(\sum_{\omega|M}\sin(8M - 3)\frac{\omega\pi}{6} + (k-1)\left(\sin\left(\frac{-\omega\pi}{2}\right) + \sin\frac{\omega\pi}{2}\right) + \sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= -2\left(\sum_{\omega|M}\sin(8M - 3)\frac{\omega\pi}{6} + \sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= -4\sum_{\omega|M}\sin(2M - 1)\frac{\omega\pi}{3}\cos(4M - 1)\frac{\omega\pi}{6}\\ &= 0.\end{aligned}$$

Here, if $M \equiv 1 (\bmod 6)$, then $\cos(4M - 1)\frac{\omega\pi}{6} = 0$ and $M \equiv 5 (\bmod 6)$, then $\sin(2M - 1)\frac{\omega\pi}{3} = 0$. Therefore, the proof of Lemma 18 is completed. $\square$

Lemma 19. *Let* $N = 2^2 3^{2k} M$ *with* $\gcd(6, M) = 1$ *and* $k \in \mathbb{N}$. *Then* $\mathfrak{a}_1(N) = 4E_{1,5}(M; 12)$. *In particular, if* $M \equiv 7, 11 (\bmod 12)$, *then* $\mathfrak{a}_1(N) = 0$.

Proof. It is easy to see that

$$2^3\frac{M}{\omega} - 3^{2k}\omega \equiv (8M - 9)\omega \equiv \begin{cases} 11\omega & \text{if } M \equiv 1, 7 (\bmod 12),\\ 7\omega & \text{if } M \equiv 5, 11 (\bmod 12).\end{cases} \tag{6.5}$$

In a similar way to Lemma 18, we obtain

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\sum_{\omega|M}\sum_{t=0}^{2k}\sin\left(2^3 \cdot 3^t \cdot \frac{M}{\omega} - 3^{2k-t}\cdot\omega\right)\frac{\pi}{6}\\ &= -2\sum_{\omega|M}\left(\sin\left(2^3 \cdot \frac{M}{\omega} - 3^{2k}\cdot\omega\right)\frac{\pi}{6} + \sum_{t=1}^{2k-1}\sin(-3^{2k-t}\cdot\omega)\frac{\pi}{6} + \sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= -2\left(\sum_{\omega|M}\left(\sin(8M - 9)\frac{\omega\pi}{6} + \sin(-3\omega)\frac{\pi}{6} + \sin\left(\frac{-\omega\pi}{6}\right)\right)\right)\\ &= 4E_{1,5}(M; 12)\end{aligned} \tag{6.6}$$

by (6.5) and definition of $E_{1,5}(M; 12)$. By Lemmas 3 and 4, the proof of Lemma 19 is completed. $\square$

Corollary 4. *Let* $N = 12M$ *be an integer with* $M \equiv 3 (\bmod\ 6)$. *Then*

$$\mathfrak{a}_1(N) = \begin{cases} 0 & \text{if } M \equiv 9 \pmod{12}, \\ 4E_{1,5}(M'; 12) & \text{if } M \equiv 3 \pmod{12}, \quad M = 3^t M', \text{ and } \gcd(6, M') = 1. \end{cases}$$

In particular, if $M' \equiv 7, 11 (\bmod\ 12)$, *then* $\mathfrak{a}_1(N) = 0$.

Proof. Let $M \equiv 9 \pmod{12}$. For $k \in \mathbb{N}$, we have

$$N = \begin{cases} 2^2 \cdot 3 \cdot 3^{2k-2} \cdot M' & \text{with } M' \equiv 1, 5 \pmod{12} \text{or} \\ 2^2 \cdot 3 \cdot 3^{2k-1} \cdot M' & \text{with } M' \equiv 7, 11 \pmod{12}. \end{cases}$$

By Lemmas 18 and 19, $\mathfrak{a}_1(N) = 0$.

Let $M \equiv 3 \pmod{12}$. For $k \in \mathbb{N}$, we have

$$N = \begin{cases} 2^2 \cdot 3 \cdot 3^{2k-2} \cdot M' & \text{with } M' \equiv 7, 11 \pmod{12} \text{ or} \\ 2^2 \cdot 3 \cdot 3^{2k-1} \cdot M' & \text{with } M' \equiv 1, 5 \pmod{12}. \end{cases}$$

If $N = 2^2 \cdot 3 \cdot 3^{2k-2} \cdot M'$ with $M' \equiv 7, 11 (\bmod\ 12)$, then $\mathfrak{a}_1(N) = 4E_{1,5}(M'; 12) = 0$ by Lemma 18. Finally, if $M' \equiv 1, 5 \pmod{12}$, then $\mathfrak{a}_1(2^2 \cdot 3 \cdot 3^{2k-1} \cdot M') = 4E_{1,5}(M'; 12)$ by Lemma 19. This completes the proof of Corollary 4. □

Lemma 20. *Let* $N = 2^k \cdot 3 \cdot M$ *with* $\gcd(6, M) = 1$ *and* $k(\geq 2) \in \mathbb{N}$. *Then* $\mathfrak{a}_1(N) = 0$.

Proof. Let $d|N$ and $\omega|M$. It is the only ordered pair $\left(\frac{N}{d}, d\right) = (\text{even}, \text{odd})$ that satisfy $\left(2^k \cdot 3 \cdot \frac{M}{\omega}, \omega\right)$ and $\left(2^k \cdot \frac{M}{\omega}, 3\omega\right)$. By elementary observation, we obtain

$$2^k \equiv \begin{cases} 2 (\bmod\ 12) & \text{if } k = 1, \\ 4 (\bmod\ 12) & \text{if } k \equiv 0 \pmod 2, \\ 8 (\bmod\ 12) & \text{if } k \equiv 1 \pmod 2, \text{ and } k \neq 1, \end{cases} \tag{6.7}$$

$$\sin(4M-3)\frac{\omega\pi}{6} = \begin{cases} \sin\frac{\omega\pi}{6} & \text{if } M \equiv 1 \pmod 6, \\ \sin\frac{5\omega\pi}{6} & \text{if } M \equiv 5 \pmod 6, \end{cases} \tag{6.8}$$

and

$$\sin(8M-3)\frac{\omega\pi}{6} = \begin{cases} \sin\frac{5\omega\pi}{6} & \text{if } M \equiv 1 \pmod 6, \\ \sin\frac{\omega\pi}{6} & \text{if } M \equiv 5 \pmod 6. \end{cases} \tag{6.9}$$

First, let $N = 2^{2n+1} \cdot 3 \cdot M$ with $n \in \mathbb{N}$. By Lemma 2, (6.7), and (6.8), we obtain

$$\begin{aligned} \mathfrak{a}_1(N) &= -2 \sum_{\omega|M} \left(\sin\left(2^{2n+2} \cdot \frac{3M}{\omega} - \omega\right)\frac{\pi}{6} + \sin\left(2^{2n+2} \cdot \frac{M}{\omega} - 3\omega\right)\frac{\pi}{6} \right) \\ &= -2 \sum_{\omega|M} \left(\sin\left(\frac{-\omega\pi}{6}\right) + \sin(4M-3)\frac{\omega\pi}{6} \right) = 0. \end{aligned} \tag{6.10}$$

Next, let $N = 2^{2n} \cdot 3 \cdot M$ with $n \in \mathbb{N}$. By Lemma 2, (6.7), and (6.9), we obtain

$$\begin{aligned} \mathfrak{a}_1(N) &= -2 \sum_{\omega|M} \left(\sin\left(2^{2n+1} \cdot \frac{3M}{\omega} - \omega\right)\frac{\pi}{6} + \sin\left(2^{2n+1} \cdot \frac{M}{\omega} - 3\omega\right)\frac{\pi}{6} \right) \\ &= -2 \sum_{\omega|M} \left(\sin\left(\frac{-\omega\pi}{6}\right) + \sin(8M-3)\frac{\omega\pi}{6} \right) = 0. \end{aligned} \tag{6.11}$$

□

The following corollary is obtained from Lemma 20.

Corollary 5. *Let $N = 12M$ be an integer with $M \equiv \pm 2, \pm 4 (\bmod\ 12)$. Then $\mathfrak{a}_1(N) = 0$.*

Lemma 21. *Let $N = 2^3 \cdot 3^m \cdot M'$ with $\gcd(6, M') = 1$ and $m(\geq 2) \in \mathbb{N}$. Then*

$$\mathfrak{a}_1(N) = \begin{cases} 0 & \text{if } m \equiv 1 (\bmod\ 2), \\ 4E_{1,5}(M'; 12) & \text{otherwise}. \end{cases}$$

In particular, if $M' \equiv 7, 11 (\bmod\ 12)$, then $\mathfrak{a}_1(N) = 0$. That is, if $N = 12M = 2^3 \cdot 3^m \cdot M'$ with $M \equiv 6 (\bmod\ 12)$, then $\mathfrak{a}_1(N) = 0$ or $4E_{1,5}(M'; 12)$.

Proof. Let $d|N$ and $\omega|M'$. First, we set $N = 2^3 \cdot 3^{2k+1} \cdot M'$. It is the only ordered pair $\left(\frac{N}{d}, d\right) = (\text{even}, \text{odd})$ that satisfies $\left(2^3 \cdot 3^t \cdot \frac{M'}{\omega}, 3^{2k-t+1} \cdot \omega\right)$ with $0 \leq t \leq 2k+1$. Thus, by Lemma 2 and (6.8),

$$\begin{aligned} \mathfrak{a}_1(N) &= -2 \sum_{\omega|M'} \sum_{t=0}^{2k+1} \sin\left(2^4 \cdot 3^t \cdot \frac{M'}{\omega} - 3^{2k-t+1} \cdot \omega\right)\frac{\pi}{6} \\ &= -2 \sum_{\omega|M'} \left(\sin(4M' - 3)\frac{\omega\pi}{6} + \sin\left(\frac{-\omega\pi}{6}\right)\right) \\ &= 0. \end{aligned}$$

In the same way as before, we prove the case $N = 2^3 \cdot 3^{2k} \cdot M'$. It is the only ordered pair $\left(\frac{N}{d}, d\right) = (\text{even}, \text{odd})$ that satisfies $(2^3 \cdot 3^t \cdot \frac{M'}{\omega}, 3^{2k-t} \cdot \omega)$ with $0 \leq t \leq 2k$. Using

$$\sin(4M' - 9)\frac{\omega\pi}{6} = \begin{cases} \sin\frac{7\omega\pi}{6} & \text{if } M' \equiv 1 \ (\bmod\ 6) \\ \sin\frac{-\omega\pi}{6} & \text{if } M' \equiv 5 \ (\bmod\ 6), \end{cases}$$

we obtain

$$\begin{aligned} \mathfrak{a}_1(N) &= -2 \sum_{\omega|M'} \sum_{t=0}^{2k} \sin\left(2^4 \cdot 3^t \cdot \frac{M'}{\omega} - 3^{2k-t} \cdot \omega\right)\frac{\pi}{6} \\ &= -2 \sum_{\omega|M'} \left(\sin\left(4 \cdot \frac{M'}{\omega} - 3^{2k} \cdot \omega\right)\frac{\pi}{6} + \sum_{t=1}^{2k-1} \sin(-3^{2k-t} \cdot \omega)\frac{\pi}{6} + \sin\left(\frac{-\omega\pi}{6}\right)\right) \qquad (6.12) \\ &= -2 \sum_{\omega|M'} \left(\sin(4M' - 9)\frac{\omega\pi}{6} + \sin(-3\omega)\frac{\pi}{6} + \sin\left(\frac{-\omega\pi}{6}\right)\right) \\ &= 4E_{1,5}(M'; 12). \end{aligned}$$

If $M' \equiv 7, 11 (\bmod\ 12)$, then $E_{1,5}(M'; 12) = 0$ and $\mathfrak{a}_1(N) = 4E_{1,5}(M'; 12) = 0$. □

Lemma 22. *Let $N = 2^n \cdot 3^{2m+1} \cdot M'$ with $\gcd(6, M') = 1$ and $n(\geq 4), m(\geq 1) \in \mathbb{N}$. Then $\mathfrak{a}_1(N) = 0$.*

Proof. Let $d|N$ and $\omega|M'$. First, we let $N = 2^{2l} \cdot 3^{2m+1} \cdot M'$ with $n = 2l$. It is the only ordered pair $\left(\frac{N}{d}, d\right) = (\text{even}, \text{odd})$ that satisfy $\left(2^{2l} \cdot 3^t \cdot \frac{M'}{\omega}, 3^{2m-t+1} \cdot \omega\right)$ with $0 \leq t \leq 2m+1$. It is clear that

$$2^{2u} \equiv 4 (\bmod\ 12), \quad 2^{2u+1} \equiv 8 (\bmod\ 12), \quad 3^{2u} \equiv 9 (\bmod\ 12), \quad \text{and} \quad 3^{2u+1} \equiv 3 (\bmod\ 12) \qquad (6.13)$$

with $u \in \mathbb{N}$. By using (6.11) and (6.13), we see directly that

$$\mathfrak{a}_1(N) = -2 \sum_{\omega|M'} \sum_{t=0}^{2m+1} \sin\left(2^{2l+1} \cdot 3^t \cdot \frac{M'}{\omega} - 3^{2m-t+1} \cdot \omega\right)\frac{\pi}{6}$$

$$= -2\sum_{\omega|M'}\left(\sin\left(2^{2l+1}\cdot\frac{M'}{\omega}-3^{2m+1}\cdot\omega\right)\frac{\pi}{6}+\sum_{t=1}^{2m}\sin(-3^{2m-t+1}\cdot\omega)\frac{\pi}{6}+\sin\left(\frac{-\omega\pi}{6}\right)\right)$$
$$= -2\sum_{\omega|M'}\left(\sin(8M'-3)\frac{\omega\pi}{6}+\sin\left(\frac{-\omega\pi}{6}\right)\right)$$
$$= 0.$$

Second, let $N = 2^{2l+1}\cdot 3^{2m+1}\cdot M'$. Then, by (6.10), we have

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\sum_{\omega|M'}\sum_{t=0}^{2m+1}\sin\left(2^{2l+2}\cdot 3^t\cdot\frac{M'}{\omega}-3^{2m-t+1}\cdot\omega\right)\frac{\pi}{6}\\ &= -2\sum_{\omega|M'}\left(\sin\left(2^{2l+2}\cdot\frac{M'}{\omega}-3^{2m+1}\cdot\omega\right)\frac{\pi}{6}+\sum_{t=1}^{2m}\sin(-3^{2m-t+1}\cdot\omega)\frac{\pi}{6}+\sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= -2\sum_{\omega|M'}\left(\sin(4M'-3)\frac{\omega\pi}{6}+\sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= 0.\end{aligned} \tag{6.14}$$

□

Lemma 23. *Let* $N = 2^n\cdot 3^{2m}\cdot M'$ *with* $\gcd(6, M') = 1$ *and* $n(\geq 4), m(\geq 1)\in\mathbb{N}$. *Then* $\mathfrak{a}_1(N) = 4E_{1,5}(M'; 12)$. *In particular, if* $M'\equiv 7, 11 (\mathrm{mod}\ 12)$ *then* $\mathfrak{a}_1(N) = 0$.

Proof. Let $d|N$ and $\omega|M'$. First, we let $N = 2^{2l}\cdot 3^{2m}\cdot M'$ with $n = 2l$. It is the only ordered pair $\left(\frac{N}{d}, d\right) =$ (even, odd) that satisfy $\left(2^{2l}\cdot 3^t\cdot\frac{M'}{\omega}, 3^{2m-t}\cdot\omega\right)$ with $0\leq t\leq 2m$. Then, by (6.6), we obtain

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\sum_{\omega|M'}\sum_{t=0}^{2m}\sin\left(2^{2l+1}\cdot 3^t\cdot\frac{M'}{\omega}-3^{2m-t}\cdot\omega\right)\frac{\pi}{6}\\ &= -2\sum_{\omega|M'}\left(\sin\left(2^{2l+1}\cdot\frac{M'}{\omega}-3^{2m}\cdot\omega\right)\frac{\pi}{6}+\sum_{t=1}^{2m-1}\sin(-3^{2m-t}\cdot\omega)\frac{\pi}{6}+\sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= -2\sum_{\omega|M'}\left(\sin(8M'-9)\frac{\omega\pi}{6}+\sin\left(\frac{-3\omega\pi}{6}\right)+\sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= 4E_{1,5}(M'; 12).\end{aligned}$$

Second, let $N = 2^{2l+1}\cdot 3^{2m}\cdot M'$. Then, by (6.12), we have

$$\begin{aligned}\mathfrak{a}_1(N) &= -2\sum_{\omega|M'}\sum_{t=0}^{2m}\sin\left(2^{2l+2}\cdot 3^t\cdot\frac{M'}{\omega}-3^{2m-t}\cdot\omega\right)\frac{\pi}{6}\\ &= -2\sum_{\omega|M'}\left(\sin\left(2^{2l+2}\cdot\frac{M'}{\omega}-3^{2m}\cdot\omega\right)\frac{\pi}{6}+\sum_{t=1}^{2m-1}\sin(-3^{2m-t}\cdot\omega)\frac{\pi}{6}+\sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= -2\sum_{\omega|M'}\left(\sin(4M'-9)\frac{\omega\pi}{6}+\sin\left(\frac{-3\omega\pi}{6}\right)+\sin\left(\frac{-\omega\pi}{6}\right)\right)\\ &= 4E_{1,5}(M'; 12).\end{aligned} \tag{6.15}$$

If $M'\equiv 7, 11(\mathrm{mod}\ 12)$, then $E_{1,5}(M'; 12) = 0$ and $\mathfrak{a}_1(N) = 4E_{1,5}(M'; 12) = 0$. □

By Lemmas 22 and 23, we obtain the following corollary.

Corollary 6. *Let* $N = 12M$ *be an integer with* $M\equiv 0(\mathrm{mod}\ 12)$, *that is,* $N = 2^n\cdot 3^m\cdot M'$ *with* $\gcd(6, M') = 1$ *and* $n(\geq 4), m(\geq 2)\in\mathbb{N}$. *Then*

$$\mathfrak{a}_1(N) = \begin{cases}0 & \text{if } m\equiv 1(\mathrm{mod}\ 2) \text{ or } M'\equiv 7, 11(\mathrm{mod}\ 12),\\ 4E_{1,5}(M'; 12) & \text{if } m\equiv 0(\mathrm{mod}\ 2) \text{ and } M'\equiv 1, 5(\mathrm{mod}\ 12).\end{cases}$$

Remark 3. We can easily obtain Lemma 16 again using the method in Section 6. In other words, let $N = 2 \cdot 3^{2m+1} \cdot M'$(resp., $N = 2 \cdot 3^{2m+2} \cdot M'$) and $\gcd(6, M') = 1$. Set $n = 0$ in (6.14) (resp., (6.15)):

$$\mathfrak{a}_1(N) = \begin{cases} 0 & \text{if } N = 2 \cdot 3^{2m+1} \cdot M' \text{ or } M' \equiv 7, 11 (\text{mod } 12), \\ 4E_{1,5}(M'; 12) & \text{if } N = 2 \cdot 3^{2m+2} \cdot M' \text{ and } M' \equiv 1, 5 (\text{mod } 12). \end{cases}$$

Here, $m \in \mathbb{N}_0$.

7 Coefficient of $\mathfrak{a}_2(N)$

We obtain four cases in (2.4):

$$n \cos(k-n)\frac{\pi}{3} = \begin{cases} n & \textit{if } (k-n) \equiv 0 \ (\text{mod } 6), \\ \frac{1}{2}n & \textit{if } (k-n) \equiv \pm 1 \ (\text{mod } 6), \\ -\frac{1}{2}n & \textit{if } (k-n) \equiv \pm 2 \ (\text{mod } 6), \\ -n & \textit{if } (k-n) \equiv 3 \ (\text{mod } 6). \end{cases} \tag{7.1}$$

Here, $k, n \in \mathbb{N}$.

By (2.4) and (7.1), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \mathfrak{a}_2(n) q^n &= \prod_{n \geq 1} \frac{(1-q^n)^4}{(1-q^n+q^{2n})^2} \\ &= 1 - 2 \sum_{N \geq 1} \left(\sum_{\substack{nk=N \\ n,k \geq 1}} n \cos(k-n)\frac{\pi}{3} \right) q^N \\ &= 1 - 2 \sum_{N \geq 1} \left(\sum_{\substack{d \mid N \\ d \equiv \frac{N}{d} (\text{mod } 6)}} d - \sum_{\substack{d \mid N \\ d - \frac{N}{d} \equiv 3 (\text{mod } 6)}} d + \sum_{\substack{d \mid N \\ d - \frac{N}{d} \equiv \pm 1 (\text{mod } 6)}} \frac{d}{2} - \sum_{\substack{d \mid N \\ d - \frac{N}{d} \equiv \pm 2 (\text{mod } 6)}} \frac{d}{2} \right) q^N \\ &= 1 - \sum_{N \geq 1} \hat{\sigma}(N) q^N. \end{aligned}$$

It follows that $\mathfrak{a}_2(0) = 1$ and $\mathfrak{a}_2(N) = -\hat{\sigma}(N)$ for $N \geq 1$.

Lemma 24. *If N is a positive integer with $N \equiv 1 (\text{mod } 6)$, then $\hat{\sigma}(N) = 2\sigma(N)$.*

Proof. Let d be a positive divisor of N with $d \equiv 1(\text{resp.}, -1)(\text{mod } 6)$. Then we obtain $\frac{N}{d} \equiv 1(\text{resp.}, -1)(\text{mod } 6)$. Hence, by the definition of $\hat{\sigma}(N)$, we obtain that

$$\begin{aligned} \hat{\sigma}(N) &= 2 \left(\sum_{\substack{d \mid N \\ d \equiv \frac{N}{d} (\text{mod } 6)}} d - \sum_{\substack{d \mid N \\ d - \frac{N}{d} \equiv 3 (\text{mod } 6)}} d \right) + \left(\sum_{\substack{d \mid N \\ d - \frac{N}{d} \equiv \pm 1 (\text{mod } 6)}} d - \sum_{\substack{d \mid N \\ d - \frac{N}{d} \equiv \pm 2 (\text{mod } 6)}} d \right) \\ &= 2 \sum_{\substack{d \mid N \\ d \equiv \frac{N}{d} (\text{mod } 6)}} d = 2 \sum_{d \mid N} d = 2\sigma(N). \end{aligned}$$

□

Similar to Lemma 24's method, calculating $\hat{\sigma}(N)$ from $N \equiv 2(\bmod\ 6)$ to $N \equiv 6(\bmod\ 6)$ gives the following theorem.

Theorem 8. *Let $N, M' \in \mathbb{N}$ and $t, s \in \mathbb{N}_0$. If $N = 2^t 3^s M'$ is an integer with* $\gcd(6, M') = 1$, *then we have*

$$\mathfrak{a}_2(N) = -\hat{\sigma}(N) = \begin{cases} -2\sigma(M') & \text{if } N \equiv 1 \pmod 6, \\ -3\sigma(M') & \text{if } N \equiv 2 \pmod 6, \\ 4\sigma(M') & \text{if } N \equiv 3 \pmod 6, \\ 6\sigma(M') & \text{if } N \equiv 4 \pmod 6, \\ \sigma(M') & \text{if } N \equiv 5 \pmod 6, \\ -12\sigma(M') & \text{if } N \equiv 0 \pmod 6. \end{cases}$$

Proof of Theorem 3. If $N \equiv 1, 3(\bmod\ 6)$ is a positive integer, then $\mathfrak{a}_2(N)$ is even by Theorem 8. Let $N \equiv 5(\bmod\ 6)$ be a positive integer. It is well-known that $\sigma(N)$ is odd if and only if N is a perfect square in [21, p. 28]. But $N \equiv 5(\bmod\ 6)$ is not a perfect square. This completes the proof of Theorem 3. □

Proof of Theorem 4. We see from (1.1) and Theorem 7 that

$$\sum_{k=1}^{N-1} \hat{e}(k)\hat{e}(N-k) = \sum_{k=1}^{N-1} \mathfrak{a}_1(k)\mathfrak{a}_1(N-k) = -2\mathfrak{a}_1(N) + \mathfrak{a}_2(N) \tag{7.2}$$

with $N \geq 2$.

Using (7.2), lemmas, corollaries, and Theorem 8 in this article, the proof of Theorem 4 is completed. □

8 Theory of modular forms

To prove Theorems 5 and 6, we use the theory of modular forms in [22,23]. For $N \in \mathbb{N}$, we define the level N congruence subgroup $\Gamma_0(N)$ by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | c \equiv 0(\bmod\ N) \right\}.$$

Let $\mathbb{H} = \mathrm{x}\{\tau = u + iv | v > 0\}$ be the complex upper half plane, and let $k \in \mathbb{Z}$. Suppose that g is a holomorphic function on $\mathbb{H}$ and ρ is a Dirichlet character modulo N. Then g is called a modular form of weight k and level N with Nebentypus character ρ if the following hold:

(1) For any $\tau \in \mathbb{H}$ and any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, g satisfies

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = \rho(d)(c\tau + d)^k g(\tau).$$

(2) If $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then $(C\tau + D)^{-k} g\left(\frac{A\tau + B}{C\tau + D}\right)$ has a Fourier expansion of the form

$$(C\tau + D)^{-k} g\left(\frac{A\tau + B}{C\tau + D}\right) = \sum_{n \geq 0} a_\gamma(n) e^{\pi i n\tau / N}.$$

We denote by $M_k(N, \rho)$ the space of modular forms of weight k and level N with Nebentypus character ρ. If $a_\gamma(0) = 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then g is called a cusp form.

We denote by $S_k(N,\rho)$ the space of cusp forms of weight k and level N with Nebentypus character ρ. If ρ is trivial, then we simply write $M_k(N,\rho)=M_k(N)$ and $S_k(N,\rho)=S_k(N)$.

Before we prove the main theorems, we verify a property for $\bar{\sigma}_2(N)$ and $\tilde{\sigma}_2(N)$.

Lemma 25. *Let N be an odd positive integer. Then $\bar{\sigma}_2(N)=\tilde{\sigma}_2(N)$ if $N\equiv 1 \pmod 4$ and $\bar{\sigma}_2(N)=-\tilde{\sigma}_2(N)$ if $N\equiv 3 \pmod 4$.*

Proof. Let d be any positive divisor of N. If $N\equiv 1(\bmod\ 4)$, then $d\equiv \pm1(\bmod\ 4)$ if and only if $\frac{N}{d}\equiv \pm1(\bmod\ 4)$. If $N\equiv 3(\bmod\ 4)$, then $d\equiv \pm1(\bmod\ 4)$ if and only if $\frac{N}{d}\equiv \mp1(\bmod\ 4)$. By the definitions of $\bar{\sigma}_2(N)$ and $\tilde{\sigma}_2(N)$, we have the result. □

We denote by $L(s,\rho)$ a Dirichlet L-function.

Proof of Theorem 5. Let ρ be the Dirichlet character modulo 24 with conductor 4. That is, $\rho(1)=\rho(5)=\rho(13)=\rho(17)=1$ and $\rho(7)=\rho(11)=\rho(19)=\rho(23)=-1$. Then $H(q)^3$ is a modular form in the space $M_3(24,\rho)$ by [23, Theorem 1.64, 1.65]. Using the dimension formula [23, Theorem 1.34], we obtain that the dimension of $M_3(24,\rho)$ is 12 and the dimension of $S_3(24,\rho)$ is 4. Also we can check that T_1, T_2, T_3, and T_4 are in $S_3(24,\rho)$ and are linearly independent. Now let

$$\bar{E}_3(q) := \frac{1}{2}L(-2,\bar{\rho}) + \sum_{n\ge1}\bar{\sigma}_2(n)q^n, \quad \tilde{E}_3(q) := \sum_{n\ge1}\tilde{\sigma}_2(n)q^n,$$

where $\bar{\rho}$ is the Dirichlet character modulo 4 such that $\bar{\rho}(3)=-1$. In fact, $L(-2,\bar{\rho})=-\frac{1}{2}$ which can be calculated by [23, Proposition 1.51]. By arguments in [22, §4.5], the set

$$\{T_1(q), T_2(q), T_3(q), T_4(q), \bar{E}_3(q), \bar{E}_3(q^2), \bar{E}_3(q^3), \bar{E}_3(q^6), \tilde{E}_3(q), \tilde{E}_3(q^2), \tilde{E}_3(q^3), \tilde{E}_3(q^6)\}$$

forms a basis for $M_3(24,\rho)$. Hence, the modular form $H(q)^3$ can be expressed as a linear sum of the elements of the above basis. We have

$$\begin{aligned} H(q)^3 = \frac{1}{7}\big(&\bar{E}_3(q) - 2\bar{E}_3(q^2) + 27\bar{E}_3(q^3) - 54\bar{E}_3(q^6) - 4\tilde{E}_3(q) + 32\tilde{E}_3(q^2) + 108\tilde{E}_3(q^3) - 864\tilde{E}_3(q^6) - 18T_1(q) \\ &- 36T_2(q) - 18T_3(q) - 36T_4(q)\big) \end{aligned}$$

and then we obtain, for $N\in\mathbb{N}$,

$$\begin{aligned} \mathfrak{a}_3(N) = \frac{1}{7}\bigg(&\bar{\sigma}_2(N) - 2\bar{\sigma}_2\Big(\frac{N}{2}\Big) + 27\bar{\sigma}_2\Big(\frac{N}{3}\Big) - 54\bar{\sigma}_2\Big(\frac{N}{6}\Big) - 4\tilde{\sigma}_2(N) + 32\tilde{\sigma}_2\Big(\frac{N}{2}\Big) + 108\tilde{\sigma}_2\Big(\frac{N}{3}\Big) - 864\tilde{\sigma}_2\Big(\frac{N}{6}\Big)\bigg) \\ &- \mathfrak{t}(N). \end{aligned} \tag{8.1}$$

Meanwhile, $H(q)^3$ can be expressed as follows:

$$H(q)^3 = \Big(1 - \sum_{N\ge1}\hat{e}(N)q^N\Big) = 1 - 3\sum_{N\ge1}\hat{e}(N)q^N + 3\sum_{N\ge2}\sum_{k=1}^{N-1}\hat{e}(k)\hat{e}(N-k)q^N - \sum_{N\ge3}\sum_{\substack{a_1+a_2+a_3=N\\ a_1,a_2,a_3\ge1}}\hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3)q^N.$$

Then, for $N\ge3$, we have

$$\sum_{\substack{a_1+a_2+a_3=N\\ a_1,a_2,a_3\ge1}}\hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3) = -3\hat{e}(N) + 3\sum_{k=1}^{N-1}\hat{e}(k)\hat{e}(N-k) - \mathfrak{a}_3(N).$$

From Theorem, 2, 4, and (8.1), we obtain

$$\sum_{\substack{a_1+a_2+a_3=N\\ a_1,a_2,a_3\geq 1}} \hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3)$$

$$= \begin{cases} 3E_{1,5}(N;\ 12) - 6\sigma(N) - \dfrac{1}{7}(\bar{\sigma}_2(N) - 4\tilde{\sigma}_2(N)) + \mathfrak{t}(N) & \text{if } N \equiv 1(\text{mod } 12),\\ 12\sigma(M') - \dfrac{1}{7}\left(\bar{\sigma}_2(N) + 27\bar{\sigma}_2\left(\dfrac{N}{3}\right) - 4\tilde{\sigma}_2(N) + 108\tilde{\sigma}_2\left(\dfrac{N}{3}\right)\right) + \mathfrak{t}(N) & \text{if } N \equiv 3(\text{mod } 12),\\ -12E_1(N;\ 12) + 3\sigma(N) - \dfrac{1}{7}(\bar{\sigma}_2(N) - 4\tilde{\sigma}_2(N)) + \mathfrak{t}(N) & \text{if } N \equiv 5(\text{mod } 12),\\ -6\sigma(N) - \dfrac{1}{7}(\bar{\sigma}_2(N) - 4\tilde{\sigma}_2(N)) + \mathfrak{t}(N) & \text{if } N \equiv 7(\text{mod } 12),\\ 12E_{1,5}(N;\ 12) + 12\sigma(M') - \dfrac{1}{7}\left(\bar{\sigma}_2(N) + 27\bar{\sigma}_2\left(\dfrac{N}{3}\right) - 4\tilde{\sigma}_2(N)\right. & \text{if } N \equiv 9(\text{mod } 12),\\ \quad \left. + 108\tilde{\sigma}_2\left(\dfrac{N}{3}\right)\right) + \mathfrak{t}(N) & \\ 3\sigma(N) - \dfrac{1}{7}(\bar{\sigma}_2(N) - 4\tilde{\sigma}_2(N)) + \mathfrak{t}(N) & \text{if } N \equiv 11(\text{mod } 12). \end{cases}$$

Finally, by applying Lemma 25, the proof is completed. □

Proof of Theorem 6. The proof is similar to that of Theorem 5. $H(q)^4$ is in the 13-dimensional space $M_4(18)$, which is spanned by the set

$$\{S_1(q), S_2(q), S_3(q), S_4(q), S_5(q), E_4(q), E_4(q^2), E_4(q^3), E_4(q^6), E_4(q^9), E_4(q^{18}), F_4(q), F_4(q^2)\},$$

where

$$E_4(q) := \frac{1}{240} + \sum_{n\geq 1}\sigma_3(n)q^n, \quad F_4(q) := \sum_{\substack{n\geq 1\\ n\equiv 1(3)}}\sigma_3(n)q^n - \sum_{\substack{n\geq 1\\ n\equiv 2(3)}}\sigma_3(n)q^n.$$

Then $H(q)^4$ can be expressed as

$$\begin{aligned} H(q)^4 = \frac{1}{10}(&-E_4(q) + 16E_4(q^2) + 84E_4(q^3) - 1344E_4(q^6) - 243E_4(q^9) + 3888E_4(q^{18}) + 3F_4(q)\\ &+ 48F_4(q^2) - 42S_1(q) - 384S_2(q) - 1152S_3(q) - 1152S_4(q)). \end{aligned} \tag{8.2}$$

Meanwhile,

$$\begin{aligned} H(q)^4 &= \left(1 - \sum_{N\geq 1}\hat{e}(N)q^N\right)^4\\ &= 1 - 4\sum_{N\geq 1}\hat{e}(N)q^N + 6\sum_{N\geq 2}\sum_{k=1}^{N-1}\hat{e}(k)\hat{e}(N-k)q^N - 4\sum_{N\geq 3}\sum_{\substack{a_1+a_2+a_3=N\\ a_1,a_2,a_3\geq 1}}\hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3)q^N\\ &\quad + \sum_{N\geq 4}\sum_{\substack{a_1+a_2+a_3+a_4=N\\ a_1,a_2,a_3,a_4\geq 1}}\hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3)\hat{e}(a_4)q^N, \end{aligned}$$

so we obtain, for $N \geq 4$,

$$\sum_{\substack{a_1+a_2+a_3+a_4=N\\ a_1,a_2,a_3,a_4\geq 1}}\hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3)\hat{e}(a_4) = 4\hat{e}(N) - 6\sum_{k=1}^{N-1}\hat{e}(k)\hat{e}(N-k) + 4\sum_{\substack{a_1+a_2+a_3=N\\ a_1,a_2,a_3\geq 1}}\hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3) + \mathfrak{a}_4(N).$$

From Theorems 2, 4, 5, and (8.2), we obtain

$$\sum_{\substack{a_1+a_2+a_3+a_4=N\\ a_1,a_2,a_3,a_4\geq 1}} \hat{e}(a_1)\hat{e}(a_2)\hat{e}(a_3)\hat{e}(a_4)$$

$$=\begin{cases} 4E_{1,5}(N;\,12) - 12\sigma(N) + \dfrac{12}{7}\bar{\sigma}_2(N) + \dfrac{1}{5}\sigma_3(N) + \mathfrak{s}(N) & \text{if } N \equiv 1(\text{mod } 12),\\ 24\sigma(M') - \dfrac{20}{7}\bar{\sigma}_2(N) - \dfrac{540}{7}\bar{\sigma}_2\left(\dfrac{N}{3}\right) - \dfrac{1}{10}\sigma_3(N) + \dfrac{42}{5}\sigma_3\left(\dfrac{N}{3}\right) & \\ \quad -\dfrac{243}{10}\sigma_3\left(\dfrac{N}{9}\right) + \mathfrak{s}(N) & \text{if } N \equiv 3(\text{mod } 12),\\ -16E_1(N;\,12) + 6\sigma(N) + \dfrac{12}{7}\bar{\sigma}_2(N) - \dfrac{2}{5}\sigma_3(N) + \mathfrak{s}(N) & \text{if } N \equiv 5(\text{mod } 12),\\ -12\sigma(N) - \dfrac{20}{7}\bar{\sigma}_2(N) + \dfrac{1}{5}\sigma_3(N) + \mathfrak{s}(N) & \text{if } N \equiv 7(\text{mod } 12),\\ 16E_{1,5}(N;\,12) + 24\sigma(M') + \dfrac{12}{7}\bar{\sigma}_2(N) + \dfrac{324}{7}\bar{\sigma}_2\left(\dfrac{N}{3}\right) - \dfrac{1}{10}\sigma_3(N) & \\ \quad +\dfrac{42}{5}\sigma_3\left(\dfrac{N}{3}\right) - \dfrac{243}{10}\sigma_3\left(\dfrac{N}{9}\right) + \mathfrak{s}(N) & \text{if } N \equiv 9(\text{mod } 12),\\ 6\sigma(N) - \dfrac{20}{7}\bar{\sigma}_2(N) - \dfrac{2}{5}\sigma_3(N) + \mathfrak{s}(N) & \text{if } N \equiv 11(\text{mod } 12). \end{cases}$$

We note that if $N \equiv 3, 15, 21, 33(\text{mod } 36)$, then N is not divisible by 9. In these cases, $\sigma_3(N) = \sigma_3(3)\sigma_3\left(\frac{N}{3}\right) = 28\sigma_3\left(\frac{N}{3}\right)$. Hence, we have $-\frac{1}{10}\sigma_3(N) + \frac{42}{5}\sigma_3\left(\frac{N}{3}\right) - \frac{243}{10}\sigma_3\left(\frac{N}{9}\right) = \frac{28}{5}\sigma_3\left(\frac{N}{3}\right)$. Then we have the desired result. ☐

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