Approximation in Weighted Orlicz Spaces with a generating Young function that might be non-convex

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Abstract

The aim of this paper is to investigate the order of approximation by some linear summation methods of trigonometric Fourier series in weighted Orlicz spaces which have generating Young functions not necessary to be convex. Obtained estimations base on the fractional modulus of smoothness and the best approximation. Furthermore, a convolution type operator is defined and its estimation by the best approximation is obtained.

Keywords: Linear summation processes, Fourier series, trigonometric approximation, weighted Orlicz spaces, Muckenhoupt weight.

Konveks olması gerekmeyen genelleştirilmiş Young fonksiyonu ile üretilen Ağırlıklı Orlicz Uzaylarında yaklaşım

Öz

Bu çalışmada, konveks olması gerekmeyen Young fonksiyonları ile üretilen ağırlıklı Orlicz uzaylarında trigonometrik Fourier serilerinin bazı lineer toplam metodları ile yaklaşım problemleri incelenmiştir. Elde edilen sonuçlar kesirli düzgünlük modülüne ve en iyi yaklaşım sayısına dayanmaktadır. Ayrıca, konvolüsyon tipli dönüşüm tanımlayıp, bu dönüşüm ile en iyi yaklaşım sayısı arasındaki ilişki değerlendirilmiştir.

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Anahtar kelimeler: *Lineer toplam metodları, Fourier serileri, trigonometrik yaklaşım, ağırlıklı Orlicz uzayları, Muckenhoupt ağırlık fonksiyonu.*

1.Introduction

The concept of Orlicz space expands the well-known concept of the space $L_p, p \geq 1$. The function t^p in the defnition of the space L_p is replaced by a more general convex function which is called a Young function. Further information about Orlicz space can be found in [11, 17]. Many applications of Orlicz spaces have emerged [16]. On the other hand, in the paper [3], a diferent approach to the Orlicz space appeared. In this study, Chen generalized the definition of Orlicz space saving almost all known properties of this space. In this generalization, Young function does not have to be convex. Later, in [1] this approach was developed with Muckenhoupt weights and was proved direct and inverse trigonometric approximation theorems in these spaces by Akgun. For the other papers about trigonometric approximation in these spaces, see [2, 12, 22, 4, 23]. We firstly introduce this space.

We denote by Φ the class of the strictly increasing functions $\phi \colon [0, \infty) \to [0, \infty)$ such that $\phi(\infty) = \infty$. Let $N[p, q]$ be denote the class of even functions $\varphi \in \Phi$ such that $\varphi(x)x^{-p}$ is non-decreasing and $\varphi(x).x^{-q}$ is non-increasing when $|x|$ is increasing $(0, \infty)$. By $N < p, q >$, we denote the class of function φ in $N[p + \varepsilon, q - \delta]$ for some small number $\varepsilon, \delta > 0$. Suppose that Φ_p denotes the class of function M in $N < p, q >$ for some $1 < p \leq q < \infty$. We say that the function M satifies the condition Δ_2 if there exists a constant $c > 0$ and $u_0 > 0$ such that $M(2u) \leq cM(u)$ for $u \geq u_0$. In this case, we denote $M \in \Delta_2$. The functions in Φ_p, p > 1, are continuous and satisfy the conditions M(0) = 0 and $M \in \Delta_2$. These functions may not be convex [3, p. 67-68].

Let $T = [-\pi, \pi]$. We say a nonnegative function ω is a weight function if it is measurable and positive almost everywhere on T.

Let $M \in \Phi_p$, $p > 1$ and ω be a weight function on T. We define $\varphi_M(t) := M(t)/t$. Since $1 < p < \infty$, we get $\varphi_M(t) \to \infty$ as $t \to \infty$. Let $\psi_M(t)$ be denote the inverse function of positive non-decreasing continuous function $\varphi_M(t)$. We set

$$\Phi_M(x) = \int_0^x \varphi_M(t)dt$$

and

$$\Psi_M(x) = \int_0^x \psi_M(t)dt.$$

Φ_M is a convex function and so $\Psi_M(x)$ is the complementary function of Φ_M, in the sense of Young. We define the weighted Orlicz space as

$$L_{M,\omega}(T) = \left\{ f \colon T \to R \colon f \text{ measureable function and } \int_T \Phi_M(|f(x)|)\omega(x)dx < \infty \right\}$$

On this space, we define Orlicz norm as

$$\|f\|_{M,\omega} := \sup_{g} \left\{ \int_T (|f(x)g(x)|)\omega(x)dx : \int_T \Psi_M(|g(x)|)\omega(x)dx \le 1 \right\}$$

and the Luxemburg norm

$$\|f\|_{(M),\omega} := \inf \left\{ k > 0 : \int_T \Phi_M(k^{-1}|f(x)|)\omega(x)dx \le 1 \right\}.$$

The equivalence

$$\|f\|_{(M),\omega} \sim \|f\|_{M,\omega}$$

is valid [1].

It is seen that $L_{M,\omega}(T) \subset L^1(T)$ and $L_{M,\omega}(T)$ is a Banach space with the above norms [1]. This space is called weighted Orlicz space. If we take $M(x,p) := x^p, 1 < p < \infty$, then this space becomes the weighted Lebesgue space $L^p(T,\omega)$.

A weight function $\omega : T \to [0,\infty]$ belongs to the Muckenhoupt class A_p[14], $1 < p < \infty$ if

$$\left(\frac{1}{|I|} \int_I \omega(x)dx \right) \left(\frac{1}{|I|} \int_I \omega^{\frac{1}{1-p}}(x)dx \right)^p \le C,$$

with a finite constant C independent of the interval I with lenght $\le 2\pi$, where $|I|$ denotes the length of I.

Without losing generality, suppose that the Fourier series of f is

$$\sum_{k=1}^{\infty} c_k(f)e^{ikx} =: \sum_{k=1}^{\infty} A_k(f,x), \tag{1.1}$$

where $c_k(f)$, $a_v(f)$ and $b_v(f)$ is the Fourier coefficients of the function $f \in L_{M,\omega}(T)$.

Let $S_n(f,x), (n = 0,1,2,\dots)$ be the nth partial sum of the series (1.1) at the point x, that is,

$$S_n(f,x) := \sum_{k=1}^{n} A_k(f,x).$$

In [1], it was proved that the operator $S_n : L_{M,\omega}(T) \to L_{M,\omega}(T)$ is bounded in $L_{M,\omega}(T)$ if $M \in \Phi_p$, $p > 1$, $\omega \in A_p$ and $f \in L_{M,\omega}(T)$. Hence we have [1]

$$\|S_n(f)\|_{(M),\omega} \le C\|f\|_{(M),\omega} \qquad n = 0,1,2,\dots$$

and

$$\|f - S_n(f)\|_{(M),\omega} \le C . E_n(f)_{(M),\omega} \qquad n = 0,1,2, \dots$$

The set of trigonometric polynomials is a dense subset of $L_{M,\omega}(\mathrm{T})$ since the hypothesis of Lemma 3 of [10] are fulfilled for $M \in \Phi_p$, $p > 1$, $\omega \in A_p$.
The well-known Steklov's mean operator is defined as

$$(\sigma_t f)(x) := \frac{1}{2t} \int_{-t}^{t} f(x + u) du, \, 0 < t < \pi, x \in T.$$

If $M \in \Phi_p$, $p > 1$, $\omega \in A_p$, it was proved that the Hardy Littlewood Maximal function is bounded in $L_{M,\omega}(T)$ [1]. So, the operator σ_t is bounded in $L_{M,\omega}(T)$ under conditions $M \in \Phi_p$, $p > 1$, $\omega \in A_p$. For $x, t \in T, r > 0$ and $L_{M,\omega}(T)$ we define

$$\sigma_t^r f(x) := (I - \sigma_t)^r f(x) = \sum_{k=0}^{\infty} (-1)^k [C_k^r] \frac{1}{(2t)^k} \int_{-t}^{t} \dots \int_{-t}^{t} f(x + u_1 + u_{2+} \dots + u_k) du_1 \dots du_k,$$

where $[C_k^r] := \frac{r.(r-1)\dots(r-k+1)}{k!}$ for $k > 1$, $[C_1^r] := r$ and $[C_0^r] = 1$ are binomial coefficients. From the inequality

$$|[C_k^r]| \le \frac{c}{k^{r+1}}, \quad k \in Z^+$$

we get

$$\sum_{k=0}^{\infty} |[C_k^r]| < \infty$$

and we have

$$\|\sigma_t^r f(x)\|_{M,\omega} \le c \|f\|_{M,\omega} < \infty$$
if $M \in \Phi_p, p > 1$ and , $\omega \in A_p$.

The fractional modulus of smoothness of index $r > 0$ for $L_{M,\omega}(T)$ is defined as

$$\Omega_{M,\omega}^r(f, \delta) := \sup_{0 < h_i, t \le \delta} \left\| \prod_{i=1}^{[r]} (I - \sigma_{t_i})(I - \sigma_t)^{r - [r]} f \right\|_{M,\omega}$$

where $[r]$ denotes the integer part of r. Since the operator σ_t is bounded in $L_{M,\omega}(T)$ we have

$$\Omega_{M,\omega}^r(f, \delta) \le c \|f\|_{M,\omega}$$
if , $\omega \in A_p, 1 < p < \infty$.

The sequence of the best approximation number of $f \in L_{M,\omega}(T)$ by trigonometric polynomials is defined by

$$E_n(f)_{M,\omega} = \inf_{T_n \in \tau_n} \|f - T_n\|_{M,\omega}$$

where τ_n is the set of trigonometric polynomial of degree $\leq n$.

Let $\alpha > 0$ be given. We define fractional derivative (in the sense of Weyl) of a function $f \in L^1(T)$ as

$$f^{(\alpha)}(x) := \frac{a_0(f)}{2} + \sum_{v=1}^{\infty} v^\alpha A_v\left(f, x + \frac{\alpha\pi}{2v}\right) =: \sum_{v=0}^{\infty} A_v\left(f^{(\alpha)}, x\right).$$

Let $W_{M,\omega}^\alpha, \alpha > 0$, be the class of function $f \in L_{M,\omega}(T)$ such that $f^{(\alpha)} \in L_{M,\omega}(T)$. If $\alpha > 0, 1 < p < \infty$, then $W_{M,\omega}^\alpha$ becomes a Banach space with the norm

$$\|f\|_{W_{M,\omega}^\alpha} := \|f\|_{L_{M,\omega}} + \left\|f^{(\alpha)}\right\|_{L_{M,\omega}}.$$

Let $\left\{\lambda_v^{(n)}\right\}, n = 0,1,\dots, v = 0,1,\dots n$ be an arbitrary infinite triangular number matrix. We associate every function $f \in L^1(T)$, on the basis of its Fourier expansion, with a polynomial

$$R_n(f, \lambda) := \frac{a_0}{2} \lambda_0^{(n)} + \sum_{v=1}^{n} \lambda_v^{(n)} A_v(f, x).$$

Thus, any triangular matrix $\left\{\lambda_v^{(n)}\right\}$ determines a method for constructing the polynomials $R_n(f, \lambda)$ or, in other words, a specific sequence of polynomial operators $R_n(f, \lambda)$ defined on the space $L_1(T)$. In this case, it is also said that the matrix $\left\{\lambda_v^{(n)}\right\}$ determines a specific method for summation of Fourier series. It is clear that, for every fixed n, the operators $R_n(f, \lambda)$ are linear. So, these methods are called linear methods (processes) of summation of Fourier series.

In this work, we investigate some problems about approximation to the derivatives of the functions in $f \in L_{M,\omega}(T)$ by these linear methods. We estimate the rate of convergence of $R_n\left(f^{(\alpha)}, \lambda\right) - f^{(\alpha)}$ to zero in the $L_{M,\omega}(T)$ norm by the fractional modulus of smoothness and the best approximation.

In this paper, we will use the following notation

$$A(x) \preccurlyeq B(x) \iff \exists c > 0 : A(x) \leq cB(x).$$

2. Main results and discussion

Theorem 2.1 Let $M \in \Phi_p, p > 1$, $\omega \in A_p(T)$, and $f \in L_{M,\omega}(\boldsymbol{T})$. We assume that

$$\sum_{v=1}^{\infty} v^{\alpha-1} E_v(f)_{M,\omega} < \infty$$

for some $\alpha \in (0, \infty)$. Then for an arbitrary triangular matrix of the numbers $\left\{\lambda_v^{(n)}\right\}$ $\left(\lambda_0^{(n)} = 1, \lambda_v^{(n)} = 0, v > n , n = 0,1,2, \dots \right)$, we have

$$\left\| R_n\left(f^{(\alpha)}, \lambda\right) - f^{(\alpha)} \right\|_{M,\omega} \le \Omega_{M,\omega}^r \left(f^{(\alpha)}, \frac{1}{n}\right).$$

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Theorem 2.2 Let $M \in \Phi_p, p > 1$, $\omega \in A_p(T)$, and $f \in L_{M,\omega}(\boldsymbol{T})$. We assume that

$$\sum_{v=1}^{\infty} v^{\alpha-1} E_v(f)_{M,\omega} < \infty$$

for some $\alpha \in (0, \infty)$. Then for an arbitrary triangular matrix of the numbers $\left\{\lambda_v^{(n)}\right\}$ $\left(\lambda_0^{(n)} = 1, \lambda_v^{(n)} = 0, v > n , n = 0,1,2, \dots \right)$, we have

$$\left\| R_n\left(f^{(\alpha)}, \lambda\right) - f^{(\alpha)} \right\|_{M,\omega} \le \left(E_{2^\mu-1}^\gamma\left(f^{(\alpha)}\right)_{M,\omega} \delta_{2^\mu,n}^\gamma \right)^{\frac{1}{\gamma}} + E_n(f^{(\alpha)})_{M,\omega}$$

where $\gamma = min(2, p + \varepsilon)$ with a small positive number ε and

$$\delta_{2^\mu,n}^\gamma := \sum_{v=2^\mu}^{2^{\mu+1}-1} \left| \lambda_{v+1}^{(n)} - \lambda_v^{(n)} \right| + \left| 1 - \lambda_{2^{\mu+1}}^{(n)} \right|$$
$$(2^m \le n < 2^{m+1}).$$

Corollary 2.1 Let $\lambda_v^{(n)} = 1 - \frac{v}{n+1}$ for $0 \le v \le n$ and $\lambda_v^{(n)} = 0$ for $v > n$, $n = 0,1,2, \dots$. Then, for the Fejer means, we have

$$\left\| R_n(f^{(\alpha)}, \lambda) - f^{(\alpha)} \right\|_{M,\omega} \lessgtr \frac{1}{n+1} \left(\sum_{\mu=0}^{m} (\mu+1)^{\gamma-1} E_\mu^\gamma\left(f^{(\alpha)}\right)_{M,\omega} \right)^{1/\gamma}.$$

Corollary 2.2 Let $\lambda_v^{(n)} = 1 - \frac{v^k}{(n+1)^k}$ for $0 \le v \le n$ and $\lambda_v^{(n)} = 0$ for $v > n$, $n = 0,1,2, \dots$. Then, for the Zygmund means, we have

$$\left\| R_n(f^{(\alpha)}, \lambda) - f^{(\alpha)} \right\|_{pq,\omega} \lessgtr \frac{1}{(n+1)^k} \left(\sum_{\mu=0}^{m} (\mu+1)^{\gamma k-1} E^\gamma{}_\mu\left(f^{(\alpha)}\right)_{pq,\omega} \right)^{1/\gamma}.$$

For $\lambda_v(r) = r^v$ ($0 \leq r < 1, v = 0,1,2, ...$) we define

$$R_r(f, \lambda) := \sum_{v=0}^{\infty} \lambda_v(r) A_v(f, x).$$

Theorem 2.3 Let $M \in \Phi_p, p > 1$, $\omega \in A_p(T)$ and $f \in L_{M,\omega}(\boldsymbol{T})$. We assume that

$\sum_{v=1}^{\infty} v^{\alpha-1} E_v(f)_{M,\omega} < \infty$

for some $\alpha \in (0, \infty)$. Then, for the Abel-Poission means, we have

$$\left\| R_r\left(f^{(\alpha)}, \lambda\right) - f^{(\alpha)} \right\|_{M,\omega} \preccurlyeq \left((1 - r) \sum_{v=0}^{\infty} r^v (v + 1)^{\gamma-1} E_{2^v-1}^{\gamma} \left(f^{(\alpha)}\right)_{M,\omega} \right)^{\frac{1}{\gamma}}$$

where $\gamma = min(2, p + \varepsilon)$ with a small positive number ε.

The similar theorems have been proved in diferent function spaces for $\alpha = 0$ and $\gamma = 1$ [5, 7, 8, 9, 13, 18, 19, 20, 21].

For the function $f \in L_{M,\omega}(\boldsymbol{T})$, we defined a mean value operator $B_h f$

$$(B_h f)(x, u) := \frac{1}{2h} \int_{-h}^{h} f(x + tu) dt, \ 0 < h < \pi, \ x \in T, \quad -\infty < u < \infty.$$

For $M \in \Phi_p, p > 1$, $\omega \in A_p(T)$, this operator is a bounded linear operator [1]. By means of the operator B_h we define a convolution operator

$$\int_{-\infty}^{\infty} (B_h f)(., u) d\mu(u)$$

with a bounded variation function $\mu(u)$ on the real line and denote

$$D(f, \mu, h, M) := \left\| \int_{-\infty}^{\infty} (B_h f)(., u) d\mu(u) \right\|_{M,\omega}.$$

In the approximation theory, the convolution operators are widely used. This type operators have important applications in this theory. Especially, these operators are very useful in constructing approximating polynomials in trigonometric approximation. In different function spaces, evaluating these operators with the best approximation numbers is an important problem in the approximation theory. Now, we investigate the relations between these transforms and the best approximations in the weighted Orlicz spaces.

Theorem 2.4 Let $M \in \Phi_p, p > 1$, $\omega \in A_p(T)$, and $f \in L_{M,\omega}(\boldsymbol{T})$. We assume that

$$\sum_{v=1}^{\infty} v^{\alpha-1} E_v(f)_{M,\omega} < \infty$$

for some $\alpha \in (0, \infty)$. Then, for every natural number m

$$D\big(f^{(\alpha)}, \mu, h, M\big) \preccurlyeq \sum_{r=0}^{m} \Big(E_{2^r}^{\gamma}\big(f^{(\alpha)}\big)_{M,\omega} \delta_{2^r,h}^{\gamma}\Big)^{\frac{1}{\gamma}} + E_{2^{m+1}}(f^{(\alpha)})_{M,\omega}$$

where $\gamma = (2, p + \varepsilon)$ with a small positive number ε and

$$\delta_{2^r,h}^{\gamma} := \sum_{l=2^r}^{2^{r+1}} |\hat{\mu}(lh) - \hat{\mu}((l+1)h| + |\hat{\mu}(2^{rh})|,$$

$$\hat{\mu}(x) := \int_{-\infty}^{\infty} \frac{sinux}{ux} d\mu(u), \quad 0 < h \leq \pi.$$

Theorem 2.5 Let $M \in \Phi_p, p > 1, \ \omega \in A_p(T)$, and $f \in L_{M,\omega}(\boldsymbol{T})$. We assume that

$$\sum_{v=1}^{\infty} v^{\alpha-1} E_v(f)_{M,\omega} < \infty$$

for some $\alpha \in (0, \infty)$. Suppose that the function $F(x)$ satifies the conditions

$$\|F(x)\| \leq c_1, \ \sum_{k=2^{\mu}}^{2^{\mu+1}-1} |F(kh) - F((k+1)h| \leq c_2, \qquad h \leq 2^{-m-1}$$

with some constants c_1, c_2. If $\hat{\mu}_1, \hat{\mu}_2$ are the functions satisfying the condition

$$\hat{\mu}_1(x) = \hat{\mu}_2(x)F(x), \quad |x| < 1$$

then

$$D\big(f^{(\alpha)}, \mu_1, h, M\big) \preccurlyeq D\big(f^{(\alpha)}, \mu_2, h, M\big) + E_{2^{m+1}}\big(f^{(\alpha)}\big)_{M,\omega}.$$

In [15], the convolution operator was defined by means of the usual shift $f(x + t)$ and these theorems were proved in Orlicz spaces with $\alpha = 0, \ \gamma = 1$. Using the operator B_h these theorems were proved in weighted Orlicz spaces, in which the Young function is convex, in [24] with $\alpha = 0, \gamma = 1$. This problem was also investigated in weighted Lorentz spaces [23] and variable exponent Lebesgue spaces [6] with $\alpha = 0, \ \gamma = 1$. The similar problem was investigated in [4] with $\alpha = 0$.

3. Auxiliary results

We give multiplier theorem and Littlewood-Paley theorem in the space $L_{M,\omega}(T)$.

Theorem 3.1 [1] Let a sequence ξ_k satisfy the conditions

$$|\xi_k| \le A, \quad \sum_{k=2^{j-1}}^{2^j-1} |\xi_k - \xi_{k+1}| \le A$$

where $A > 0$ is independent of k and j. If $M \in \Phi_p, p > 1, \omega \in A_p(T)$, and $f \in L_{M,\omega}(T)$ then there exists a function $F \in L_{M,\omega}(T)$ such that the series

$$\frac{\lambda_0 a_0}{2} + \sum_{k=0}^{\infty} \lambda_k (a_k(f) \cos kx + b_k(f) \sin kx)$$

is Fourier series for F and the inequality

$$\|F\|_{M,\omega} \le CA\|f\|_{M,\omega}$$

holds with some constant C independent of f.

Theorem 3.2 [1] $M \in \Phi_p, p > 1, \omega \in A_p(T)$, and $f \in L_{M,\omega}(T)$ there exist constants $C > 0$ and $c > 0$ depending only on M and ω such that

$$c\|f\|_{M,\omega} \le \left\|\left(\sum_{j=0}^{\infty} \left|\sum_{k=2^{j-1}}^{2^j-1} A_k(x,f)\right|^2\right)^{\frac{1}{2}}\right\|_{M,\omega} \le C\|f\|_{M,\omega}.$$

4. Proofs of main results

Proof of Theorem 2.1 Let $2^m \le h < 2^{m+1}$. Under the conditions of the theorem, we have $f \in \mathcal{W}_{M,\omega}^{(\alpha)}(T)$ [1]. Let

$$S_n(f^{(\alpha)}, x) := \sum_{v=1}^{n} A_v(f^{(\alpha)}, x)$$

be the partial some of Fourier series of the function $f^{(\alpha)}$. By the properties of the norm, we have

$$\left\|R_n(f^{(\alpha)}, \lambda) - f^{(\alpha)}\right\|_{M,\omega} = \left\|\sum_{v=0}^{n} \lambda_v^{(n)} A_v(f^{(\alpha)}, x) - f^{(\alpha)}(x)\right\|_{M,\omega}$$

$$\leq \left\|\sum_{v=0}^{n}(1 - \lambda_v^{(n)})A_v\big(f^{(\alpha)}, x\big)\right\|_{M,\omega} + \left\|\sum_{v=n+1}^{\infty} A_v f^{(\alpha)}(x)\right\|_{M,\omega} =: I_1 + I_2.$$

From [4, Lemma 2.1] and [1, Th 1.5 and Th.1.1] we get

$$I_2 = \left\|\sum_{v=n+1}^{\infty} A_v\big(f^{(\alpha)}, x\big)\right\|_{M,\omega} \lesssim E_n(f^{(\alpha)})_{M,\omega} \lesssim \Omega_{M,\omega}^r\big(f^{(\alpha)}, \frac{1}{n}\big).$$

Now we estimate the norm

$$I_1 = \left\|\sum_{v=0}^{n} \frac{1 - \lambda_v^{(n)}}{\left(1 - \frac{\sin\frac{v}{n}}{\frac{v}{n}}\right)^r} A_v\big(f^{(\alpha)}, x\big)\left(1 - \frac{\sin\frac{v}{n}}{\frac{v}{n}}\right)^r\right\|_{M,\omega}.$$

Let

$$\mu_{v,r}^{(n)} := \begin{cases} \dfrac{(1 - \lambda_v^{(n)})}{\left(1 - \frac{\sin\frac{v}{n}}{\frac{v}{n}}\right)^r}, & v \leq n, \\ \\ 0 & v > n. \end{cases}$$

For the sequence $\left\{\mu_{v,r}^{(n)}\right\}$ the conditions of Theorem 3.1 are satisfied. Applying Theorem 3.1 and the proof method in [1, Th. 1.9] we get

$$I_1 = \left\|\sum_{v=0}^{n} \mu_{v,r}^{(n)} A_v\big(f^{(\alpha)}, x\big)\left(1 - \frac{\sin\frac{v}{n}}{\frac{v}{n}}\right)^r\right\|_{M,\omega} \lesssim \left\|\sum_{v=0}^{n} A_v\big(f^{(\alpha)}, x\big)\left(1 - \frac{\sin\frac{v}{n}}{\frac{v}{n}}\right)^r\right\|_{M,\omega}$$

$$\leq \left\|(I - \sigma_{1/n})^r f^{(\alpha)}\right\|_{M,\omega} = \left\|\sum_{v=0}^{n}(I - \sigma_{1/n})^{[r]}(I - \sigma_{1/n})^{r-[r]} f^{(\alpha)}\right\|_{M,\omega}$$

$$\lesssim \sup_{0 < h_i, t \leq \frac{1}{n}}\left\|\prod_{i=1}^{[r]}(I - \sigma_{h_i})(I - \sigma_t)^{r-[r]} f^{(\alpha)}\right\|_{M,\omega} \lesssim \Omega_{M,\omega}^r\big(f^{(\alpha)}, \frac{1}{n}\big)$$

This proves Theorem 2.1.

Proof of Theorem 2.2 Let $2^m \leq h < 2^{m+1}$. We have

$$\left\|R_n(f^{(\alpha)}, \lambda) - f^{(\alpha)}\right\|_{M,\omega} = \left\|\sum_{v=0}^{n}\lambda_v^{(n)} A_v\big(f^{(\alpha)}, x\big) - (f^{(\alpha)}, x)\right\|_{M,\omega}$$

$$\leq \left\| \sum_{v=0}^{n} (1 - \lambda_v^{(n)}) A_v(f^{(\alpha)}, x) \right\|_{M,\omega} + \left\| \sum_{v=n+1}^{\infty} A_v(f^{(\alpha)}, x) \right\|_{M,\omega}$$

$$\lesssim \left\| \sum_{v=1}^{n} (1 - \lambda_v^{(n)}) A_v(f^{(\alpha)}, x) \right\|_{M,\omega} + E_n(f^{(\alpha)})_{M,\omega}.$$

From Theorem 3.2, we get

$$\left\| \sum_{v=1}^{n} (1 - \lambda_v^{(n)}) A_v(f^{(\alpha)}, x) \right\|_{M,\omega} \leq c \left\| \left(\sum_{\mu=0}^{m} \left| \sum_{v=2^\mu}^{2^{\mu+1}-1} (1 - \lambda_v^{(n)}) A_v(f^{(\alpha)}, x) \right|^2 \right)^{\frac{1}{2}} \right\|_{M,\omega}.$$

Using the Abel transformation, we obtain

$$\sigma_{n,\mu}(x) := \sum_{v=2^\mu}^{2^{\mu+1}-1} (1 - \lambda_v^{(n)}) A_v(f^{(\alpha)}, x)$$

$$= \sum_{v=2^\mu}^{2^{\mu+1}-1} (S_v(f, x) - S_{2^\mu - 1}(f, x))(\lambda_{v+1}^{(n)} - \lambda_v^{(n)})$$

$$+ (S_{2^{\mu+1}-1}(f, x) - S_{2^\mu - 1}(f, x))(1 - \lambda_{2^{\mu+1}}^{(n)}).$$

From Minkowski's inequality and the monotonicity of the sequence of the best approximation numbers, we get

$$\|\sigma_{n,\mu}(x)\|_{M,\omega} \leq \sum_{v=2^\mu}^{2^{\mu+1}-1} \|S_v(f, x) - S_{2^\mu - 1}(f, x)\|_{M,\omega} |\lambda_{v+1}^{(n)} - \lambda_v^{(n)}|$$

$$+ \|S_{2^{\mu+1}-1}(f, x) - S_{2^\mu - 1}(f, x)\|_{M,\omega} |1 - \lambda_{2^{\mu+1}}^{(n)}|$$

$$\leq c E_{2^\mu - 1}(f)_{M,\omega} \left(\sum_{v=2^\mu}^{2^{\mu+1}-1} |\lambda_{v+1}^{(n)} - \lambda_v^{(n)}| + |1 - \lambda_{2^{\mu+1}}^{(n)}| \right)$$

$$\leq c E_{2^\mu - 1}(f)_{M,\omega} \delta_{2^\mu, n}.$$

On the other hand, we have [1]

$$\left\| \left(\sum_{\mu=0}^{m} |\sigma_{n,\mu}(x)|^2 \right)^{1/2} \right\|_{M,\omega} \leq c \left(\sum_{\mu=0}^{m} \|\sigma_{n,\mu}(x)\|^\gamma \right)^{1/\gamma}, \tag{4.1}$$

where $\gamma = min(2, p + \varepsilon)$ with a small positive number ε. Therefore, we obtain the required inequality

$$\left\|\sum_{v=0}^{n}(1-\lambda_v^{(n)})A_v(f^{(\alpha)},x)\right\|_{M,\omega} \leq c\left(\sum_{\mu=0}^{m}E^\gamma_{2^{\mu-1}}(f^{(\alpha)})_{M,\omega}\cdot\delta^\gamma_{2^\mu,n}\right)^{1/\gamma}.$$

Proof of Theorem 2.3 Let $2^m \leq n = \left[\frac{1}{1-r}\right] < 2^{m+1}$ and $\lambda_v(r) = r^v$, $0 \leq r < 1$, $v = 0,1,2,...$ We have

$$\left\|R_r(f^{(\alpha)},\lambda) - f^{(\alpha)}\right\|_{M,\omega} = \left\|\sum_{v=0}^{\infty}(1-r^v)A_v(f^{(\alpha)},x)\right\|_{M,\omega}$$

$$\left\|\sum_{v=0}^{2^{m+1}-1}(1-r^v)A_v(f^{(\alpha)},x)\right\|_{M,\omega} + \left\|\sum_{v=2^{m+1}-1}^{\infty}(1-r^v)A_v(f^{(\alpha)},x)\right\|_{M,\omega} =: I_1 + I_2.$$

It is easily seen that the conditions of Theorem 3.1 are satisfied for the sequence $\{1 - r^v\}$. Then, by Theorem 3.1, [4, Lemma 2.1] and [1, Th.1.5] we have

$$I_2 \leq c\left\|\sum_{v=2^{m+1}}^{\infty}A_v(f^{(\alpha)},x)\right\|_{M,\omega} \leq cE_n(f^{(\alpha)})_{M,\omega}.$$

From Theorem 3.2 and (4.1)

$$I_1 \leq \left\|\left(\sum_{v=0}^{m}\left|\sum_{\mu=2^v}^{2^{v+1}-1}(1-r^\mu)A_\mu(f^{(\alpha)},x)\right|^2\right)^{\frac{1}{2}}\right\|_{M,\omega}$$

$$\leq \left(\Sigma_{v=0}^{m}\left\|\sum_{\mu=2^v}^{2^{v+1}-1}(1-r^\mu)A_\mu(f^{(\alpha)},x)\right\|_{M,\omega}^\gamma\right)^{\frac{1}{\gamma}}.$$

Applying Abel's transform and Theorem 3.1, we get

$$\left\|\sum_{\mu=2^v}^{2^{v+1}-1}(1-r^\mu)A_\mu(f^{(\alpha)},x)\right\|_{M,\omega}$$

$$\leq \sum_{\mu=2^v}^{2^{v+1}-1}\left\|S_\mu(f^{(\alpha)},x) - S_{2^v-1}(f^{(\alpha)},x)\right\|_{M,\omega}|r^{\mu+1} - r^\mu|$$

$$+ \left\|S_{2^{v+1}-1}(f^{(\alpha)},x) - S_{2^v-1}(f^{(\alpha)},x)\right\|_{M,\omega}|1 - r^{2^{v+1}}|$$

$$\leq cE_{2^v-1}(f^{(\alpha)})_{M,\omega}2^{v+1}(1-r).$$

From the monotonicity of the best approximation, we have

$$\left\| \sum_{\mu=2^v}^{2^{v+1}-1} (1 - r^\mu) A_\mu\big(f^{(\alpha)}, x\big) \right\|_{M,\omega}^\gamma \leq c E_{2^v-1}^\gamma\big(f^{(\alpha)}\big)_{M,\omega} 2^{\gamma(v+1)}(1-r)^\gamma$$

$$\leq c(1-r)^\gamma \sum_{\mu=2^{v-1}}^{2^v-1} E_{2^v-1}^\gamma\big(f^{(\alpha)}\big)_{M,\omega}.$$

Then we get

$$I_1 \leq c \left((1-r)^\gamma \big\| A_1\big(f^{(\alpha)}, x\big) \big\|_{M,\omega}^\gamma (1-r)^\gamma \sum_{\mu=2^{v-1}}^{2^v-1} \mu^{\gamma-1} E_\mu^\gamma\big(f^{(\alpha)}\big)_{M,\omega} \right)^{\frac{1}{\gamma}}$$

$$\leq c \left((1-r) \sum_{\mu=0}^n (\mu+1)^{\gamma-1} E_\mu^\gamma\big(f^{(\alpha)}\big)_{M,\omega} \right)^{1/\gamma}.$$

Therefore, we have

$$\big\| R_r\big(f^{(\alpha)}, \lambda\big) - f^{(\alpha)} \big\|_{M,\omega} \leq c(1-r) \left(\sum_{\mu=0}^n (\mu+1)^{\gamma-1} E_\mu^\gamma\big(f^{(\alpha)}\big)_{M,\omega} \right)^{1/\gamma} + E_n(f^{(\alpha)})_{M,\omega}$$

$$\leq c(1-r) \left(\sum_{\mu=0}^n r^\mu (\mu+1)^{\gamma-1} E_\mu^\gamma\big(f^{(\alpha)}\big)_{M,\omega} \right)^{1/\gamma}.$$

Proof of Theorem 2.4 Let $m \in \mathbb{N}$, $2^m \leq h < 2^{m+1}$. Under the conditions of the theorem, we have $f \in \mathcal{W}_{M,\omega}^{(\alpha)}(T)$ [1]. We suppose that

$$S_{2^{m+1}} f^{(\alpha)} := \sum_{k=1}^{2^{m+1}} c_k(f^{(\alpha)}) e^{ikx}$$

is the partial sum of Fourier series of the function $f^{(\alpha)}$. From the definition of the quantity $D\big(f^{(\alpha)}, \mu, h, M\big)$ and the properties of the norm we have

$$D\big(f^{(\alpha)}, \mu, h, M\big) = \left\| \int_{-\infty}^\infty (B_h f^{(\alpha)})(x, u) d\mu(u) \right\|_{M,\omega}$$

$$\leq \left\| \int_{-\infty}^\infty \big[(B_h f^{(\alpha)})(x, u) - \big(B_h S_{2^{m+1}} f^{(\alpha)}\big)(x, u)\big] d\mu(u) \right\|_{M,\omega}$$

$$+ \left\| \int_{-\infty}^\infty \big(B_h S_{2^{m+1}} f^{(\alpha)}\big)(x, u) d\mu(u) \right\|_{M,\omega}.$$

By the boundedness of the operator B_h and [1,Th.1.5], we obtain

$$\left\| \int_{-\infty}^{\infty} \left[(B_h f^{(\alpha)})(x,u) - (B_h S_{2^{m+1}} f^{(\alpha)})(x,u) \right] d\mu(u) \right\|_{M,\omega}$$

$$\leq c \int_{-\infty}^{\infty} \left\| (B_h f^{(\alpha)})(x,u) - (B_h S_{2^{m+1}} f^{(\alpha)})(x,u) \right\|_{M,\omega} d\mu(u)$$

$$= \int_{-\infty}^{\infty} \left\| \left(B_h \left(f^{(\alpha)} - \sigma_h S_{2^{m+1}} f^{(\alpha)} \right)(x,u) \right) \right\|_{M,\omega} d\mu(u)$$

$$\leq c \int_{-\infty}^{\infty} \left\| f^{(\alpha)} - S_{2^{m+1}} f^{(\alpha)} \right\|_{M,\omega} d\mu(u)$$

$$= \int_{-\infty}^{\infty} \left\| f^{(\alpha)} - S_{2^{m+1}} f^{(\alpha)} \right\|_{M,\omega} d\mu(u)$$

$$= \int_{-\infty}^{\infty} \left\| f^{(\alpha)} - S_{2^{m+1}}^{(\alpha)} \right\|_{M,\omega} d\mu(u)$$

$$\leq c E_{2^{m+1}} \left(f^{(\alpha)} \right)_{M,\omega}.$$

So, we have

$$D\left(f^{(\alpha)}, \mu, h, M \right) \leq \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}} f^{(\alpha)})(x,u) d\mu(u) \right\|_{M,\omega} + c E_{2^{m+1}} \left(f^{(\alpha)} \right)_{M,\omega}.$$

Then we get

$$\int_{-\infty}^{\infty} (B_h S_{2^{m+1}} f^{(\alpha)})(x,u) d\mu(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2h} \int_{-h}^{h} S_{2^{m+1}} f^{(\alpha)}(x+tu) dt \right) d\mu(u)$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2h} \int_{-h}^{h} \sum_{r=1}^{2^{m+1}} c_r(f^{(\alpha)}) e^{ir(x+tu)} dt \right) d\mu(u)$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2h} \sum_{r=1}^{2^{m+1}} c_r(f^{(\alpha)}) e^{irx} \int_{-h}^{h} e^{irtu} dt \right) d\mu(u)$$

$$= \sum_{r=1}^{2^{m+1}} c_r(f^{(\alpha)}) e^{irx} \int_{-\infty}^{\infty} \frac{e^{irhu} - e^{-irhu}}{2irhu} d\mu(u)$$

$$= \sum_{r=1}^{2^{m+1}} A_r(f^{(\alpha)}, x) \hat{\mu}(rh).$$

Therefore, we have

$$D\left(f^{(\alpha)}, \mu, h, M\right) \preccurlyeq \left\|\sum_{r=1}^{2^{m+1}} A_r(f^{(\alpha)}, x)\hat{\mu}(rh)\right\|_{M,\omega} + E_{2^{m+1}}\left(f^{(\alpha)}\right)_{M,\omega}.$$

From Theorem 3.2 and [1], we obtain

$$\left\|\sum_{r=1}^{2^{m+1}} A_r(f^{(\alpha)}, x)\hat{\mu}(rh)\right\|_{M,\omega} \preccurlyeq \left\|\left(\sum_{r=0}^{m}\left|\sum_{l=2^r}^{2^{r+1}} A_l(x)\hat{\mu}(lh)\right|^2\right)^{\frac{1}{2}}\right\|_{M,\omega}$$

$$:= c\left\|\left(\sum_{r=0}^{m}\Delta_{r,\mu}^2\right)^{\frac{1}{2}}\right\|_{M,\omega} \preccurlyeq c\left\|\sum_{r=0}^{m}(\Delta_{r,\mu}^2)^{\frac{1}{2}}\right\|_{M,\omega}$$

$$\left\|\left(\sum_{r=0}^{m}\Delta_{r,\mu}^2\right)^{\frac{1}{2}}\right\|_{M,\omega} \leq c\sum_{r=0}^{m}(\|\Delta_{r,\mu}^{\gamma}\|)_{M,\omega}^{1/\gamma}.$$

Applying the Abel transform to $\Delta_{r,\mu}$, we get

$$\Delta_{r,\mu}= \sum_{l=2^r}^{2^{r+1}}\left[S_l f^{(\alpha)}(x) - S_{2^{r+1}}f^{(\alpha)}(x)\right](\hat{\mu}(lh) - \hat{\mu}((l+1)h)$$

$$+ \left[S_{2^{r+1}}f^{(\alpha)}(x) - S_{2^r}f^{(\alpha)}(x)\right]\hat{\mu}(2^r h).$$

From [1, Th. 1.5],

$$\|\Delta_{r,\mu}\|_{M,\omega} = \sum_{l=2^r}^{2^{r+1}}\left\|S_l f^{(\alpha)}(x) - S_{2^{r+1}}f^{(\alpha)}(x)\right\|_{M,\omega}(\hat{\mu}(lh) - \hat{\mu}((l+1)h)$$

$$+ \left\|S_{2^{r+1}}f^{(\alpha)}(x) - S_{2^r}f^{(\alpha)}(x)\right\|_{M,\omega}\hat{\mu}(2^r h)$$

$$\leq E^{\gamma}_{2^r}\left(f^{(\alpha)}\right)_{M,\omega} \cdot \delta^{\gamma}_{2^r,h}.$$

Then

$$\left\|\sum_{r=1}^{2^{m+1}-1} A_r(f^{(\alpha)})e^{irx}\hat{\mu}(rh)\right\|_{M,\omega} \leq c\left(\sum_{r=0}^{m} E^{\gamma}_{2^{r-1}}\left(f^{(\alpha)}\right)_{M,\omega} \cdot \delta^{\gamma}_{2^r,h}\right)^{\frac{1}{\gamma}}.$$

This completes the proof.

Proof of Theorem 2.5 Under the conditions of the theorem, we have $f \in \mathcal{W}_{M,\omega}^{(\alpha)}(T)$ [1].We have

$$D\big(f^{(\alpha)},\mu_1,h,M\big) \leq \left\|\int_{-\infty}^{\infty}(B_h S_{2^{m+1}}f^{(\alpha)})(x,u)d\mu(u)\right\|_{M,\omega} + E_{2^{m+1}}\big(f^{(\alpha)}\big)_{M,\omega}$$

From the the properties of the function and Theorem 3.1, we obtain

$$\left\|\int_{-\infty}^{\infty}(B_h S_{2^{m+1}}f^{(\alpha)})(x,u)d\mu_1(u)\right\|_{M,\omega} = \left\|\sum_{r=1}^{2^{m+1}} c_r(f^{(\alpha)})e^{irx}\mu_1(rh)\right\|_{M,\omega}$$

$$= \left\|\sum_{r=1}^{2^{m+1}} c_r(f^{(\alpha)})e^{irx}\mu_2(rh)F(rh)\right\|_{M,\omega}$$

$$\leq c\left\|\sum_{r=1}^{2^{m+1}} c_r(f^{(\alpha)})e^{irx}\mu_2(rh)\right\|_{M,\omega} = \left\|\int_{-\infty}^{\infty}(B_h S_{2^{m+1}}f^{(\alpha)})(x,u)d\mu_2(u)\right\|_{M,\omega}$$

$$= \left\|S_{2^{m+1}}\int_{-\infty}^{\infty}(B_h f^{(\alpha)})(x,u)d\mu_2(u)\right\|_{M,\omega} \leq c\left\|\int_{-\infty}^{\infty}(B_h f^{(\alpha)})(x,u)d\mu_2(u)\right\|_{M,\omega}.$$

This completes the proof.

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