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Fixed-Disc Results on Metric Spaces

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Abstract. In this paper, we deal with the geometric properties of non-unique fixed points for self-mappings of a metric space (resp. an S-metric space). The fixed-disc (resp. fixed-circle) problem has been investigated in this setting. To obtain new fixed-disc results, we modify some known fixed-point techniques. Illustrative examples are also given to show the validity of our main results.

1. Introduction

The metric fixed-point theory was started by Banach contraction principle as a new scientific area [5]. This theory is one of the most significant research fields in nonlinear analysis (for some examples, see [1, 3, 4, 9, 13, 20, 30]). The Banach contraction principle based on the distance of two pairwise points $d(u, v)$ and the distance of images of the corresponding points $d(fu, fv)$ for all $u, v \in X$, where (X, d) is a metric space and $f : X \to X$ is a self-mapping. This principle guarantees that the existence and uniqueness of a fixed point of a self-mapping. But there are some examples of self-mappings which have a unique fixed point and do not satisfy the conditions of this principle. Therefore, some generalizations of this principle has been extensively investigated with various aspects. One of these generalizations is to generalize the used metric spaces (for several interesting generalizations, see [6, 7, 10, 14–16]). Another generalization is to generalize the used contractive condition (see [29] for some examples).

A recent geometric approach to the fixed-point theory is the study on the geometric properties of the fixed-point set $Fix(f)$ of a self-mapping f. In this context, the fixed-circle problem appeared in the cases where the set $Fix(f)$ is not a singleton [22]. The fixed-circle problem is the investigation of some conditions to ensure the set $Fix(f)$ contains a circle (or the set $Fix(f)$ can be equal to a circle) for a self-mapping. Related to this problem, some effective results have been obtained using different geometric aspects on a metric

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space (for more details, see [18, 22, 25, 27, 28, 34]). As a result of some fixed-circle theorems, fixed-disc results have been provided consequently (for example, see [19, 26]). Hence, it is an interesting problem to give new fixed-disc results and their applications in the context of metric spaces.

In this present paper, we prove new fixed-disc results on both a metric space and an S-metric space using the different numbers and some known families of functions. To do this, we investigate some existence conditions of a self-mapping to possess a fixed disc and give a characterization to exclude the identity map from the obtained fixed-disc results. Also, we present necessary illustrative examples to our main theorems and some consequences of obtained results.

2. Main Results

Let $s \geq 1$ and Ψ_s be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:
(i) ψ is nondecreasing;
(ii) $\sum\limits_{n=1}^{\infty} s^n \psi^n(\omega) < \infty$ for all $\omega > 0$.

It is known that if $\psi \in \Psi_s$ then we have $\psi(\omega) < \omega$ for all $\omega > 0$ [2]. If $s = 1$ then the family Ψ_1 is given as Ψ in [31]. Throughout this paper, we fix $s = 1$.

From now on, unless otherwise stated, we suppose that $f : X \to X$ is a self-mapping on the metric space (X, d) and $D_{u_0,r} = \{u \in X : d(u, u_0) \leq r\}$ is a disc on X.

Now, we recall the following fixed-disc definition (for more details see [19, 26]).

Definition 2.1. *The disc $D_{u_0,r}$ is said to be the fixed disc of a self-mapping f if $fu = u$ for all $u \in D_{u_0,r}$.*

In the following, we introduce a new type of contractive mapping.

Definition 2.2. *A self-mapping f is said to be a generalized α-ψ_c-contractive mapping of type A if there are $\alpha : X^2 \to [1,\infty)$, $\psi \in \Psi_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that the condition $d(u, fu) > 0$ implies*
(1) $r \leq d(u, fu)$
and
(2) $\alpha(u, u_0)d(u, fu) \leq \psi\left(M_A(u, u_0)\right)$,
for all $u \in X$, where

$$M_A(u, v) = \max\left\{d(u, v), d(u, fu), d(v, fv), \frac{d(u, fv)}{2}\right\}.$$

Using these new definitions, we obtain the following fixed-disc theorem.

Theorem 2.3. *If a self-mapping f is a generalized α-ψ_c-contractive mapping of type A with $u_0 \in X$ and $r \in \mathbb{R}^+ \cup \{0\}$, then $D_{u_0,r}$ is a fixed disc of f. That is, the set $Fix(f)$ contains the disc $D_{u_0,r}$.*

Proof. Let $u \in D_{u_0,r}$ and $u \neq fu$. Then we have $d(u, fu) > 0$. By the definition of a generalized α-ψ_c-contractive mapping of type A, there exist $\alpha : X^2 \to [1,\infty)$, $\psi \in \Psi_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that

$$r \leq d(u, fu) \tag{1}$$

and

$$\alpha(u, u_0)d(u, fu) \leq \psi\left(M_A(u, u_0)\right). \tag{2}$$

From the inequality (2) and the property of ψ, we have

$$\begin{aligned} \alpha(u, u_0)d(u, fu) &\leq \psi\left(M_A(u, u_0)\right) < M_A(u, u_0) \\ &= \max\left\{d(u, u_0), d(u, fu), d(u_0, fu_0), \frac{d(u, fu_0)}{2}\right\}. \end{aligned} \tag{3}$$

Now, we show that $fu_0 = u_0$ and so $d(u_0, fu_0) = 0$. To do this, assume that $fu_0 \neq u_0$. Then we have $d(u_0, fu_0) > 0$. By the hypothesis, we get

$$\begin{aligned} \alpha(u_0,u_0)d(u_0,fu_0) &\leq \psi\left(M_A(u_0,u_0)\right) < M_A(u_0,u_0) \\ &= \max\left\{d(u_0,u_0), d(u_0,fu_0), d(u_0,fu_0), \frac{d(u_0,fu_0)}{2}\right\} \\ &= d(u_0,fu_0), \end{aligned}$$

a contradiction since $\alpha(u_0, u_0) \in [1, \infty)$. So we find

$$fu_0 = u_0. \tag{4}$$

Combining the conditions (1), (3) and (4), we get

$$\alpha(u, u_0)d(u, fu) < d(u, fu).$$

It is a contradiction since $\alpha(u, u_0) \geq 1$. Hence it should be $fu = u$, that is, all points of the disc $D_{u_0,r}$ are fixed by f. This means that the disc $D_{u_0,r}$ is contained in the set $Fix(f)$. □

Remark 2.4. (1) *To obtain the notion of a generalized α-ψ_c-contractive mapping of type A, we have modified the concept of a generalized α-ψ-Suzuki-contractive mapping of type A given in [2] for $s = 1$ on metric spaces.*

(2) *Theorem 2.3 can be also considered as a fixed-circle theorem because a self-mapping f, which fixes a disc $D_{u_0,r}$, fixes every circle $C_{u_0,\rho}$ with $\rho \leq r$.*

(3) *If we take the self-mapping $f : X \to X$ as an identity map, then f satisfies the conditions of Theorem 2.3 and so f fixes every disc $D_{u_0,r}$.*

(4) *If $r = 0$, then we have $D_{u_0,r} = \{u_0\}$. From Theorem 2.3, the disc $D_{u_0,r}$ is fixed by f. In this case, Theorem 2.3 can be also considered as a fixed-point result.*

Example 2.5. *Let $X = \mathbb{R}$ be the metric space with the usual metric $d(u, v) = |u - v|$ for all $u, v \in \mathbb{R}$. Let us define the self-mapping $f : X \to X$ as*

$$fu = \begin{cases} u & ; \quad u \in [-3,3] \\ u+1 & ; \quad u \in (-\infty,-3) \cup (3,\infty) \end{cases},$$

for all $u \in \mathbb{R}$. Then f is a generalized α-ψ_c-contractive mapping of type A with $\psi(t) = \frac{t}{2}$, $r = 1$, $u_0 = 0$ and $\alpha : X^2 \to [1, \infty)$ is a function defined as $\alpha(u, v) = 1$. Indeed, we get

$$d(u, fu) = 1 > 0,$$

for all $u \in (-\infty, -3) \cup (3, \infty)$ and

$$\begin{aligned} M_A(u,0) &= \max\left\{d(u,0), d(u,u+1), d(0,0), \frac{d(u,0)}{2}\right\} \\ &= \max\left\{|u|, 1, 0, \frac{|u|}{2}\right\} = |u|. \end{aligned}$$

Then we have

$$\alpha(u,0)d(u,fu) = 1 \leq \frac{|u|}{2} = \psi\left(M_A(u,0)\right).$$

As a consequence, f fixes the disc $D_{0,1} = [-1, 1]$. Notice that we have $Fix(f) = [-3, 3]$ and the fixed disc is not unique.

To obtain new fixed-disc results, we recall the following family of functions:
Let $s \geq 1$ and $\mathcal{F}_s$ be the set of functions $\psi \in \Psi_s$ verifying the following additional assertions [2]:
(*iii*) $\psi(\omega_1 + \omega_2) \leq \psi(\omega_1) + \psi(\omega_2)$ for all $\omega_1, \omega_2 \geq 0$;
(*iv*) $\psi(c\omega) \geq c\psi(\omega)$ for all $c, \omega \geq 0$.
Using the family $\mathcal{F}_1$, we introduce the following notions for $s = 1$.

Definition 2.6. *A self-mapping f is said to be a generalized α-ψ_c-contractive mapping of type B if there are $\alpha : X^2 \to [1,\infty)$, $\psi \in \mathcal{F}_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that the condition $d(u, fu) > 0$ implies*
(1) $r \leq d(u, fu)$
and
(2) $\alpha(u, u_0)d(u, fu) \leq \psi\left(M_B(u, u_0)\right)$,
for all $u \in X$, where

$$M_B(u, v) = \frac{d(u, fu) + d(v, fv)}{2}.$$

Definition 2.7. *A self-mapping f is said to be a generalized α-ψ_c-contractive mapping of type C if there are $\alpha : X^2 \to [1,\infty)$, $\psi \in \mathcal{F}_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that the condition $d(u, fu) > 0$ implies*
(1) $r \leq d(u, fu)$
and
(2) $\alpha(u, u_0)d(u, fu) \leq \psi\left(M_C(u, u_0)\right)$,
for all $u \in X$, where

$$M_C(u, u) = \frac{d(u, v) + d(u, fu) + d(v, fv)}{2}.$$

Definition 2.8. *A self-mapping f is said to be a generalized α-ψ_c-contractive mapping of type D if there exist $\alpha : X^2 \to [1,\infty)$, $\psi \in \mathcal{F}_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that the condition $d(u, fu) > 0$ implies*
(1) $r \leq d(u, fu)$
and
(2) $\alpha(u, u_0)d(u, fu) \leq \psi\left(M_D(u, u_0)\right)$,
for all $u \in X$, where

$$M_D(u, v) = \frac{d(u, v) + d(u, fu) + d(v, fv) + d(v, T^2v) + d(Tu, Tv)}{4}.$$

In the following theorem, we give a characterization for a self-mapping to be an identity mapping.

Theorem 2.9. *A self-mapping f is a generalized α-ψ_c-contractive mapping of type B with $u_0 \in X$ and $r \in \mathbb{R}^+ \cup \{0\}$ if and only if $f = I_X$, that is, $fu = I_X(u) = u$ for all $u \in X$.*

Proof. Let $u \in X$ be so that $u \neq fu$. By the definition of a generalized α-ψ_c-contractive mapping of type B, there are $\alpha : X^2 \to [1,\infty)$, $\psi \in \mathcal{F}_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that $r \leq d(u, fu)$ and

$$\alpha(u, u_0)d(u, fu) \leq \psi\left(M_B(u, u_0)\right). \tag{5}$$

From the inequality (5), we have $fu_0 = u_0$ and

$$\alpha(u, u_0)d(u, fu) < \frac{d(u, fu)}{2},$$

a contradiction since $\alpha(u, u_0) \geq 1$. Then we have $u = fu$ for all $u \in X$ and this means that $f = I_X$.
Conversely, it is clear that the identity map I_X is a generalized α-ψ_c-contractive mapping of type B. □

Corollary 2.10. *If a self-mapping $f : X \to X$ is a generalized α-ψ_c-contractive mapping of type A and is not a generalized α-ψ_c-contractive mapping of type B, then $f \neq I_X$.*

Now, we give another fixed-disc results.

Theorem 2.11. *If a self-mapping f is a generalized α-ψ_c-contractive mapping of type C with $u_0 \in X$ and $r \in \mathbb{R}^+ \cup \{0\}$ then $D_{u_0,r}$ is a fixed disc of f. That is, the set $Fix(f)$ contains the disc $D_{u_0,r}$.*

Proof. Let $u \in D_{u_0,r}$ and $u \neq fu$. Then we have $d(u, fu) > 0$. By the definition of a generalized α-ψ_c-contractive mapping of type C, there are $\alpha : X^2 \to [1,\infty)$, $\psi \in \mathcal{F}_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that

$$r \leq d(u, fu) \tag{6}$$

and

$$\alpha(u,u_0)d(u, fu) \leq \psi\left(M_C(u,u_0)\right). \tag{7}$$

From the inequality (7) and the property of ψ, we have

$$\begin{aligned}\alpha(u,u_0)d(u,fu) &\leq \psi\left(M_C(u,u_0)\right)\\ &= \psi\left(\frac{d(u,u_0)+d(u,fu)+d(u_0,fu_0)}{2}\right)\\ &\leq \psi\left(\frac{d(u,u_0)}{2}\right)+\psi\left(\frac{d(u,fu)}{2}\right)+\psi\left(\frac{d(u_0,fu_0)}{2}\right)\\ &< \frac{d(u,u_0)}{2}+\frac{d(u,fu)}{2}+\frac{d(u_0,fu_0)}{2}.\end{aligned} \tag{8}$$

Using the hypothesis, it can be easily seen that u_0 is a fixed point of f, that is,

$$fu_0 = u_0. \tag{9}$$

Combining the conditions (6), (8) and (9), we get

$$\alpha(u,u_0)d(u,fu) < d(u,fu),$$

a contradiction since $\alpha(u,u_0) \geq 1$. Consequently, it should be $fu = u$. □

By considering the standard metric space and $f : X \to X$ given as in Example 2.5, f is a generalized α-ψ_c-contractive mapping of type C with $\psi(t) = \frac{t}{2}$, $r = 1$, $u_0 = 0$ and the function $\alpha : X^2 \to [1,\infty)$ defined as $\alpha(u,v) = 1$. Therefore, f fixes the disc $D_{0,1} = [-1,1]$.

Theorem 2.12. *Let a self-mapping f be a generalized α-ψ_c-contractive mapping of type D with $u_0 \in X$ and $r \in \mathbb{R}^+ \cup \{0\}$. If the following conditions hold for all $u \in D_{u_0,r}$*
(i) $d(fu,u_0) \leq r$,
(ii) $f^2u = u$,
then $D_{u_0,r}$ is a fixed disc of f. That is, the set Fix(f) contains the disc $D_{u_0,r}$.

Proof. Let $u \in D_{u_0,r}$ and $u \neq fu$. Then we have $d(u, fu) > 0$. By the definition of a generalized α-ψ_c-contractive mapping of type D, there exist $\alpha : X^2 \to [1,\infty)$, $\psi \in \mathcal{F}_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that

$$r \leq d(u, fu) \tag{10}$$

and

$$\alpha(u,u_0)d(u, fu) \leq \psi\left(M_D(u,u_0)\right). \tag{11}$$

From the inequality (11) and the property of ψ, we have

$$\begin{aligned}\alpha(u,u_0)d(u,fu) &\leq \psi\left(M_D(u,u_0)\right)\\ &< \frac{d(u,u_0)}{4}+\frac{d(u,fu)}{4}+\frac{d(u_0,fu_0)}{4}\\ &\quad +\frac{d(u_0,f^2u)}{4}+\frac{d(fu,fu_0)}{4}.\end{aligned} \tag{12}$$

Now, we show that $fu_0 = u_0$ and so $d(u_0, fu_0) = 0$. To do this, suppose that $fu_0 \neq u_0$. One writes $d(u_0, fu_0) > 0$. By the hypothesis, we have

$$\alpha(u_0, u_0)d(u_0, fu_0) < \frac{d(u_0, fu_0)}{2},$$

a contradiction since $\alpha(u_0, u_0) \geq 1$. So we get

$$fu_0 = u_0. \tag{13}$$

Combining the conditions (10), (12) and (13), we find

$$\alpha(u, u_0)d(u, fu) < d(u, fu),$$

a contradiction since $\alpha(u, u_0) \geq 1$. So it should be $fu = u$. □

Now, we give an illustrative example to Theorem 2.12.

Example 2.13. *Let $X = [-2, 2]$ be endowed with the usual metric. Let us consider the self-mapping $g : X \to X$ defined by*

$$gu = \begin{cases} u & ; \quad u \in X - \{2\} \\ u - 1 & ; \quad u = 2 \end{cases},$$

for all $u \in X$. Then g is a generalized α-ψ_c-contractive mapping of type D with $\psi(t) = \frac{8}{9}t$, $r = 1$, $u_0 = 0$ and $\alpha : X^2 \to [1, \infty)$ defined as $\alpha(u, v) = 1$. Indeed, we get

$$d(u, gu) = 1 > 0,$$

for $u = 2$ and $M_D(2, 0) = \frac{5}{4}$. Then we obtain

$$\alpha(2, 0)d(2, g2) = 1 \leq \frac{10}{9} = \psi(M_D(2, 0)).$$

Also, we have $d(gu, 0) \leq 1$ and $g^2u = u$ for all $u \in D_{0,1}$. Consequently, g fixes the disc $D_{0,1} = [-1, 1]$.

Remark 2.14. (1) *To obtain the notion of a generalized α-ψ_c-contractive mapping of type B* (*resp. type C and type D*)*, we have modified the concept of a generalized α-ψ-Suzuki-contractive mapping of type B* (*resp. type C and type D*) *given in [2] for $s = 1$ on metric spaces.*

(2) *Theorem 2.11* (*resp. Theorem 2.12*) *can be also considered as a fixed-circle theorem.*

(3) *If $r = 0$ then we have $D_{u_0,r} = \{u_0\}$. From Theorem 2.11* (*resp. Theorem 2.12*)*, the disc $D_{u_0,r}$ is fixed by f, that is, u_0 is a fixed point of f. In this case, these theorems can be also considered as fixed-point results.*

(4) *If a self-mapping $f : X \to X$ satisfies the condition of Theorem 2.11* (*resp. Theorem 2.12*) *and does not satisfy the condition of Theorem 2.9, then $f \neq I_X$.*

(5) *Let (X, d) be a metric space and $D_{u_0,\rho}$ with $\rho > 0$ any disc on X. Let the self-mapping $f_\rho : X \to X$ be defined as follows*:

$$f_\rho u = \begin{cases} u_0 & ; \quad u \notin D_{u_0,\rho} \\ u & ; \quad u \in D_{u_0,\rho} \end{cases}.$$

The self-mapping f_ρ does not satisfy the conditions of Theorem 2.3 (*resp. Theorem 2.11 and Theorem 2.12*) *with u_0 and ρ. But f_ρ fixes the disc $D_{u_0,\rho}$. Consequently, the converse statement of Theorem 2.3* (*resp. Theorem 2.11 and Theorem 2.12*) *is not true everywhen.*

In the literature, there are a lot of generalizations of metric spaces. One of these generalizations is an S-metric space given as follows:

Let X be a nonempty set and $\mathcal{S} : X^3 \to [0,\infty)$ a function satisfying the following conditions for all $u, u^*, u^{**}, a \in X$:

$(S1)$ $\mathcal{S}(u,u^*,u^{**}) = 0$ if and only if $u = u^* = u^{**}$;

$(S2)$ $\mathcal{S}(u,u^*,u^{**}) \leq \mathcal{S}(u,u,a) + \mathcal{S}(u^*,u^*,a) + \mathcal{S}(u^{**},u^{**},a)$.

Then $\mathcal{S}$ is called an S-metric on X and the pair $(X,\mathcal{S})$ is called an S-metric space [32]. If we pay attention, the symmetry condition is not included in the definition of an S-metric. But, it is known that

$$\mathcal{S}(u,u,u^*) = \mathcal{S}(u^*,u^*,u), \tag{14}$$

for all $u, u^* \in X$ [32]. The equality (14) can be considered as the symmetry condition for an S-metric. On the other hand, there are some examples of an S-metric which is not generated by any metric (see [8, 11, 17, 21, 24] for more details). In this context, some fixed-circle results have been obtained using various approaches (see [8, 17, 23, 24, 35, 36]). Therefore, the fixed-disc problem can be considered on an S-metric space using the above approaches.

From now on, $(X,\mathcal{S})$ is assumed to be an S-metric space.

At first, we recall the definitions of a circle [23] and a disc [32], respectively:

$$C^S_{u_0,r} = \{u \in X : \mathcal{S}(u,u,u_0) = r\}$$

and

$$D^S_{u_0,r} = \{u \in X : \mathcal{S}(u,u,u_0) \leq r\}.$$

Now, we define the following numbers to obtain new contractive conditions on an S-metric space.

$$M^S_A(u,u^*) = \max\left\{\mathcal{S}(u,u,u^*), \mathcal{S}(u,u,Tu), \mathcal{S}(u^*,u^*,Tu^*), \frac{\mathcal{S}(u,u,Tu^*)}{2}\right\},$$

$$M^S_B(u,u^*) = \frac{\mathcal{S}(u,u,Tu) + \mathcal{S}(u^*,u^*,Tu^*)}{2},$$

$$M^S_C(u,u^*) = \frac{\mathcal{S}(u,u,u^*) + \mathcal{S}(u,u,Tu) + \mathcal{S}(u^*,u^*,Tu^*)}{2}$$

and

$$M^S_D(u,u^*) = \frac{\mathcal{S}(u,u,u^*) + \mathcal{S}(u,u,Tu) + \mathcal{S}(u^*,u^*,Tu^*) + \mathcal{S}(u^*,u^*,T^2u) + \mathcal{S}(Tu,Tu,Tu^*)}{4}.$$

Using these quantities, we introduce the following notions.

Definition 2.15. *A self-mapping f is said to be a generalized α-ψ^s_c-contractive mapping of type A if there are $\alpha : X^2 \to [1,\infty)$, $\psi \in \Psi_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that the condition $\mathcal{S}(u,u,fu) > 0$ implies*

(1) $r \leq \mathcal{S}(u,u,fu)$

and

(2) $\alpha(u,u_0)\mathcal{S}(u,u,fu) \leq \psi\left(M^S_A(u,u_0)\right)$,

for all $u \in X$.

Definition 2.16. *A self-mapping f is said to be a generalized α-ψ^s_c-contractive mapping of type B (resp. a generalized α-ψ^s_c-contractive mapping of type C and a generalized α-ψ^s_c-contractive mapping of type D) if there are $\alpha : X^2 \to [1,\infty)$, $\psi \in \Psi_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that the condition $\mathcal{S}(u,u,fu) > 0$ implies*

(1) $r \leq \mathcal{S}(u,u,fu)$

and

(2) $\alpha(u,u_0)\mathcal{S}(u,u,fu) \leq \psi\left(M^S_B(u,u_0)\right)$ *(resp.* $\alpha(u,u_0)\mathcal{S}(u,u,fu) \leq \psi\left(M^S_C(u,u_0)\right)$ *and* $\alpha(u,u_0)\mathcal{S}(u,u,fu) \leq \psi\left(M^S_D(u,u_0)\right)$*),*

for all $u \in X$.

Now, we recall the definition of a fixed disc on an S-metric space before we are going to establish some fixed-disc results (see [17] for more details).

Definition 2.17. *The disc $D^S_{u_0,r}$ is named as the fixed disc of a self-mapping f if $fu = u$ for all $u \in D^S_{u_0,r}$.*

Theorem 2.18. *If a self-mapping f is a generalized α-ψ^s_c-contractive mapping of type A or a generalized α-ψ^s_c-contractive mapping of type C with $u_0 \in X$ and $r \in \mathbb{R}^+ \cup \{0\}$, then $D^S_{u_0,r}$ is a fixed disc of f. That is, the set $Fix(f)$ contains the disc $D^S_{u_0,r}$.*

Proof. Assume that f is a generalized α-ψ^s_c-contractive mapping of type A. Let $u \in D^S_{u_0,r}$ and $u \neq fu$. Then we have $\mathcal{S}(u,u,fu) > 0$. By the definition of a generalized α-ψ^s_c-contractive mapping of type A, there exist $\alpha : X^2 \to [1,\infty)$, $\psi \in \Psi_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that

$$r \leq \mathcal{S}(u,u,fu) \tag{15}$$

and

$$\alpha(u,u_0)\mathcal{S}(u,u,fu) \leq \psi\left(M^S_A(u,u_0)\right). \tag{16}$$

From the inequality (16) and the property of ψ, we get

$$\begin{aligned} \alpha(u,u_0)\mathcal{S}(u,u,fu) &\leq \psi\left(M^S_A(u,u_0)\right) < M^S_A(u,u_0) \\ &= \max\left\{ \begin{array}{c} \mathcal{S}(u,u,u_0), \mathcal{S}(u,u,fu), \\ \mathcal{S}(u_0,u_0,fu_0), \frac{\mathcal{S}(u,u,fu_0)}{2} \end{array} \right\}. \end{aligned} \tag{17}$$

Now, we show that $fu_0 = u_0$ and so $\mathcal{S}(u_0,u_0,fu_0) = 0$. To do this, suppose that $fu_0 \neq u_0$. Then we have $\mathcal{S}(u_0,u_0,fu_0) > 0$. By the inequality (17), we find

$$\alpha(u_0,u_0)\mathcal{S}(u_0,u_0,fu_0) < \mathcal{S}(u_0,u_0,fu_0).$$

It is a contradiction since $\alpha(u_0,u_0) \geq 1$. Therefore it should be

$$fu_0 = u_0. \tag{18}$$

Combining the conditions (15), (17) and (18), we obtain

$$\alpha(u,u_0)\mathcal{S}(u,u,fu) < \mathcal{S}(u,u,fu),$$

a contradiction because of $1 \leq \alpha(u,u_0) < \infty$. Hence it should be $fu = u$.

If f is a generalized α-ψ^s_c-contractive mapping of type C, then by the above similar arguments, the proof can be easily seen. Consequently, f fixes the disc $D^S_{u_0,r}$. □

Example 2.19. *Let $X = \mathbb{R}$ and the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by*

$$\mathcal{S}(u,u^*,u^{**}) = |u - u^{**}| + |u + u^{**} - 2u^*|,$$

for all $u, u^*, u^{**} \in \mathbb{R}$ [21]. Then $(\mathbb{R}, \mathcal{S})$ is an S-metric space and the S-metric $\mathcal{S}$ is not generated by any metric d (see [21] for more details). We consider the self-mapping $f : X \to X$ given as

$$fu = \begin{cases} u & ; \quad u \in \left[-\frac{3}{2}, \frac{3}{2}\right] \\ u + \frac{1}{2} & ; \quad u \in \left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right) \end{cases},$$

for all $u \in \mathbb{R}$. Then f is a generalized α-ψ^s_c-contractive mapping of type A (resp. a generalized α-ψ^s_c-contractive mapping of type C) with $\psi(t) = \frac{t}{2}$, $r = 1$, $u_0 = 0$ and the function $\alpha : X^2 \to [1,\infty)$ given as $\alpha(u,u^*) = 1$. By Theorem 2.18, f fixes the disc $D^S_{0,1} = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Theorem 2.20. *A self-mapping f is a generalized α-ψ_c^s-contractive mapping of type B with $u_0 \in X$ and $r \in \mathbb{R}^+ \cup \{0\}$ if and only if $f = I_X$, that is, $fu = I_X(u) = u$ for all $u \in X$.*

Proof. Let $u \in X$ be an arbitrary point so that $u \neq fu$. By the definition of a generalized α-ψ_c^s-contractive mapping of type B, there are $\alpha : X^2 \to [1, \infty)$, $\psi \in \mathcal{F}_1$, $r \in \mathbb{R}^+ \cup \{0\}$ and $u_0 \in X$ such that $r \leq \mathcal{S}(u, u, fu)$ and

$$\alpha(u, u_0)\mathcal{S}(u, u, fu) \leq \psi\left(M_B^S(u, u_0)\right). \tag{19}$$

From the inequality (19), we have $fu_0 = u_0$ and

$$\alpha(u, u_0)\mathcal{S}(u, u, fu) < \frac{\mathcal{S}(u, u, fu)}{2},$$

a contradiction since $\alpha(u, u_0) \geq 1$. Then we have $fu = u$ and $f = I_X$.

Conversely, it is clear that the identity map I_X is a generalized α-ψ_c^s-contraction of type B. □

Theorem 2.21. *Let a self-mapping f a generalized α-ψ_c^s-contractive mapping of type D with $u_0 \in X$ and $r \in \mathbb{R}^+ \cup \{0\}$. If the following conditions hold for all $u \in D_{u_0,r}^S$*

(i) $\mathcal{S}(fu, fu, u_0) \leq r$,

(ii) $f^2u = u$,

then $D_{u_0,r}^S$ is a fixed disc of f.

Proof. It suffices to proceed similarly as in the proof of Theorem 2.18. □

Example 2.22. *Let $X = [-1, 1]$ be the S-metric space with the S-metric given in Example 2.19. Let us define the self-mapping $f : X \to X$ as*

$$fu = \begin{cases} u & ; \quad u \in X - \{1\} \\ \frac{1}{2} & ; \quad u = 1 \end{cases},$$

for all $u \in X$. Then f is a generalized α-ψ_c^s-contractive mapping of type D with $\psi(t) = \frac{8}{9}t$, $r = 1$, $u_0 = 0$ and the function $\alpha : X^2 \to [1, \infty)$ defined as $\alpha(u, u^*) = 1$. Also, we have $\mathcal{S}(fu, fu, 0) \leq 1$ and $f^2u = u$ for all $u \in D_{0,1}^S$. From Theorem 2.21, f fixes the disc $D_{0,1}^S = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Remark 2.23. (1) *Theorem 2.18 (resp. Theorem 2.21) corresponds to a fixed-circle theorem since a self-mapping f, which has a fixed-disc $D_{u_0,r}^S$, fixes every circle $C_{u_0,\rho}^S$ with $\rho \leq r$.*

(2) *If $r = 0$, then we have $D_{u_0,r}^S = \{u_0\}$. From Theorem 2.18 (or Theorem 2.21), the disc $D_{u_0,r}^S$ is fixed by f, that is, u_0 is a fixed point of f. Under this case, Theorem 2.18 and Theorem 2.21 can be also considered as fixed-point results.*

(3) *If a self-mapping $f : X \to X$ satisfies the conditions of Theorem 2.18 (or Theorem 2.21) and does not satisfy the conditions of Theorem 2.20, then $f \neq I_X$.*

(4) *If an S-metric is generated by any metric d, then any circle $C_{u_0,r}^S$ on the S-metric space $(X, \mathcal{S})$ is the circle $C_{u_0,\frac{r}{2}}$ on the metric space (X, d) [24]. Therefore, Theorem 2.3 (resp. Theorem 2.11 and Theorem 2.12) and Theorem 2.18 (or Theorem 2.21) can be considered as the same in this case.*

(5) *It is well known that every S-metric generates a b-metric [33]. Hence Theorem 2.18 (resp. Theorem 2.20 and Theorem 2.21) can be also considered on a b-metric space.*

(6) *Let $(X, \mathcal{S})$ be an S-metric space and $D_{u_0,\rho}^S$ with $\rho > 0$ be any disc on X. Let $f : X \to X$ be defined as follows:*

$$fu = \begin{cases} u_0 & ; \quad u \notin D_{u_0,\rho}^S \\ u & ; \quad u \in D_{u_0,\rho}^S \end{cases}.$$

f does not verify the conditions of Theorem 2.18 (resp. Theorem 2.21) with u_0 and ρ. But f fixes the disc $D_{u_0,\rho}^S$. Consequently, the converse statement of Theorem 2.18 (resp. Theorem 2.21) is not always true.

3. Conclusion and Future Work

In these days, the fixed circle problem continues to be studied with various applications as well as theoretical studies (see [22, 25, 27] for some examples). As new solutions to the fixed-circle problem, in this manuscript, we established new fixed-disc results on both a metric space and an S-metric space. It is known that some mappings having fixed discs have been used in neural networks as activation functions. For example, the complex valued function $zReLU(z)$ defined by

$$zReLU(z) = \begin{cases} z = u + iv & ; \quad u > 0, v > 0 \\ 0 & ; \quad otherwise \end{cases}$$

and the real valued function $LReLU(u)$ defined by

$$LReLU(u) = \begin{cases} \alpha u & ; \quad u < 0 \\ u & ; \quad u \geq 0 \end{cases}$$

are popular complex and real valued activation functions, respectively (see [12] and the references therein). As perspectives, it is an interesting problem to present some real life applications of fixed-disc results.

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