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Relations between Extrinsic and Intrinsic Invariants of Statistical Submanifolds in Sasaki-Like Statistical Manifolds

Hülya Aytimur [1,†], Adela Mihai [2,3,*,†] and Cihan Özgür [4,†]

[1] Department of Mathematics, Faculty of Arts and Sciences, Balıkesir University, Balıkesir 10145, Turkey; hulya.aytimur@balikesir.edu.tr
[2] Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, 20396 Bucharest, Romania
[3] Interdisciplinary Doctoral School (Faculty of Mathematics and Computer Science), Transilvania University of Braşov, 500036 Braşov, Romania
[4] Department of Mathematics, Faculty of Arts and Sciences, İzmir Democracy University, İzmir 35140, Turkey; cihan.ozgur@idu.edu.tr
* Correspondence: adela.mihai@utcb.ro
† These authors contributed equally to this work.

Abstract: The Chen first inequality and a Chen inequality for the $\delta(2,2)$-invariant on statistical submanifolds of Sasaki-like statistical manifolds, under a curvature condition, are obtained.

Keywords: Sasaki-like statistical manifold; submanifold; Chen inequality

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1. Introduction

One of the main topics of Information Geometry, which is regarded as a combination of differential geometry and statistics, deals with families of probability distributions, more exactly with their invariant properties. Information Geometry has many applications in image processing, physics, computer science, machine learning, etc.

In [1], Amari defined a statistical manifolds and presented some applications in Information Geometry. Such a manifold deals with dual connections (or conjugate connections), and, consequently, is closely related to an affine manifold.

Let $\widetilde{\nabla}$ be an affine connection on a Riemannian manifold $(\widetilde{M},\widetilde{g})$. A pair $\left(\widetilde{\nabla},\widetilde{g}\right)$ is a *statistical structure* on $\widetilde{M}$ if

$$\left(\widetilde{\nabla}_X\widetilde{g}\right)(Y,Z)-\left(\widetilde{\nabla}_Y\widetilde{g}\right)(X,Z)=0, \tag{1}$$

for any $X,Y,Z\in T\widetilde{M}$ [2]. A Riemannian manifold $(\widetilde{M},\widetilde{g})$ on which a pair of torsion-free affine connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ satisfying

$$X\widetilde{g}(Y,Z)=\widetilde{g}\left(\widetilde{\nabla}_XY,Z\right)+\widetilde{g}\left(Y,\widetilde{\nabla}^*_XZ\right) \tag{2}$$

is defined for any X,Y and $Z\in T\widetilde{M}$ is called a *statistical manifold*; one says that the connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are *dual connections* (see [1,3]).

Any torsion-free affine connection $\widetilde{\nabla}$ always has a dual connection given by

$$\widetilde{\nabla}+\widetilde{\nabla}^*=2\widetilde{\nabla}^0, \tag{3}$$

where $\widetilde{\nabla}^0$ denotes the Levi–Civita connection of $\widetilde{M}$ [1].

One challenge in submanifold theory is to obtain relations between the intrinsic and extrinsic invariants of a submanifold. An important new step in this topic is due to B.-Y. Chen, starting from 1993 [4]; he established such inequalities in a real space form, known as Chen inequalities. Since then, many geometers have studied this problem for different kind of submanifolds in certain ambient spaces (for example, see [5–10]). For the collections of the results related to Chen inequalities see also [11] and references therein.

The squared mean curvature is the main extrinsic invariant; the classical curvature invariants, namely the scalar curvature and the Ricci curvature, represent the main intrinsic invariants. A relation between the Ricci curvature and the main extrinsic invariant squared mean curvature for a submanifold in a real space form was given in [7] by B.-Y. Chen and now known as the Chen–Ricci inequality. In [12,13], K. Matsumoto and I. Mihai found relations between Ricci curvature and the squared mean curvature for submanifolds in Sasakian space forms. In [14], A. Mihai and I. N. Rădulescu proved a Chen inequality involving the scalar curvature and a Chen–Ricci inequality for special contact slant submanifolds of Sasakian space forms.

Furthermore, in [15], M. E. Aydın, A. Mihai and I. Mihai established relations between the extrinsic and intrinsic invariants for submanifolds in statistical manifolds of constant curvature. In [16], A. Mihai and I. Mihai considered statistical submanifolds of Hessian manifolds of constant Hessian curvature. As generalizations of the results given in [15], H. Aytimur and C. Özgür studied same problems for submanifolds in statistical manifolds of quasi constant curvature [17].

Recently, in [18], B.-Y. Chen, A. Mihai and I. Mihai gave the Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature.

In [19], H. Aytimur, M. Kon, A. Mihai, C. Özgür and K. Takano established a Chen first inequality and a Chen inequality for the invariant $\delta(2,2)$ for statistical submanifolds of Kähler-like statistical manifolds, under a curvature condition. Very recently, in [20], A. Mihai and I. Mihai proved a Chen inequality for the $\delta(2,2)$-invariant; also, the $\delta(2,2)$-invariant was studied in other ambient spaces by G. Macsim, A. Mihai and I. Mihai (see [21]), for example for Lagrangian submanifolds in quaternionic space forms.

Motivated by the above mentioned studies, as a continuation of the results obtained in [19], in the present paper we prove Chen first inequality and a Chen inequality for the invariant $\delta(2,2)$ for statistical submanifolds of Sasaki-like statistical manifolds, under a natural curvature condition.

2. Sasaki-Like Statistical Manifolds and Their Submanifolds

Let $\widetilde{M}$ be an odd dimensional manifold and ϕ, ξ, η be a tensor field of type $(1,1)$, a vector field and a 1-form on $\widetilde{M}$, respectively. If ϕ, ξ and η satisfy the following conditions

$$\eta(\xi) = 1, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{4}$$

for $X \in T\widetilde{M}$, then $\widetilde{M}$ is said to have an almost contact structure (ϕ, ξ, η) and it is called an *almost contact manifold*.

In [22], K. Takano started with a semi-Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ with the almost contact structure (ϕ, ξ, η), on which another tensor field ϕ^* of type $(1,1)$ satisfying

$$\widetilde{g}(\phi X, Y) + \widetilde{g}(X, \phi^* Y) = 0, \tag{5}$$

for vector fields X and Y on $\left(\widetilde{M}, \widetilde{g}\right)$ is considered.

$\left(\widetilde{M}, \widetilde{g}, \phi, \xi, \eta\right)$ is called an *almost contact metric manifold of certain kind* [22,23].

One has $(\phi^*)^2 X = -X + \eta(X)\xi$ and the following important relation holds:

$$\widetilde{g}(\phi X, \phi^* Y) = \widetilde{g}(X, Y) - \eta(X)\eta(Y). \tag{6}$$

From (4), it follows that the tensor field ϕ is not symmetric with respect to g. This means that $\phi + \phi^*$ does not vanish everywhere. On the almost contact manifold, we have $\phi\xi = 0$ and $\eta(\phi X) = 0$; then, on the almost contact metric manifold of certain kind, one has $\phi^*\xi = 0$ and $\eta(\phi^* X) = 0$.

In [22], Takano defined a statistical manifold on the almost contact metric manifold of certain kind. $\left(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \phi, \xi, \eta\right)$ is called a *Sasaki-like statistical manifold* if

$$\widetilde{\nabla}_X \xi = -\phi X, \quad \left(\widetilde{\nabla}_X \phi\right) Y = \widetilde{g}(X, Y)\xi - \eta(Y)X. \tag{7}$$

Suppose that the curvature tensor $\widetilde{R}$ with respect to $\widetilde{\nabla}$ satisfies

$$\begin{aligned} \widetilde{R}(X, Y)Z = \frac{c+3}{4}[\widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y] \\ +\frac{c-1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \widetilde{g}(X, Z)\eta(Y)\xi - \widetilde{g}(Y, Z)\eta(X)\xi \\ +\widetilde{g}(X, \phi Z)\phi Y - \widetilde{g}(Y, \phi Z)\phi X \\ +(\widetilde{g}(X, \phi Y) - \widetilde{g}(\phi X, Y))\phi Z], \end{aligned} \tag{8}$$

where c is a constant (see [22]).

By interchanging ϕ and ϕ^* in (8), one obtains the similar condition for curvature tensor $\widetilde{R}^*$.

If $\widetilde{M}$ is a Sasaki manifold, then the previous relation represents the curvature condition of being a Sasakian space form (i.e., the ϕ-sectional curvature is constant, c).

On a statistical manifold, the curvature tensor fields of $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively, denoted by $\widetilde{R}$ and $\widetilde{R}^*$ satisfy the relation

$$\widetilde{g}\left(\widetilde{R}(X, Y)Z, W\right) = -\widetilde{g}\left(\widetilde{R}^*(X, Y)W, Z\right) \tag{9}$$

(see [2]).

Let $f : M \longrightarrow \widetilde{M}$ be an immersion, where $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}\right)$ is a statistical manifold. One considers a pair (g, ∇) on M, defined by

$$g = f^*\widetilde{g}, \quad g(\nabla_X Y, Z) = \widetilde{g}\left(\widetilde{\nabla}_{f_*X} f_*Y, f_*Z\right),$$

for any $X, Y, Z \in TM$, where the connection induced from $\widetilde{\nabla}$ by f on the induced bundle $f^* : T\widetilde{M} \longrightarrow M$ is denoted by the same symbol $\widetilde{\nabla}$. Then (∇, g) is a statistical structure on M, called the one *induced by* f from $\left(\widetilde{\nabla}, \widetilde{g}\right)$ [2].

Let (M, g, ∇) and $\left(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}\right)$ be two statistical manifolds. Then $f : M \longrightarrow \widetilde{M}$ is a *statistical immersion* if (∇, g) coincides with the induced statistical structure, i.e., if (1) holds [2]. Recall that, for M an n-dimensional submanifold of $\widetilde{M}$, the Gauss formulas are

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\widetilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y),$$

where h and h^* are symmetric and bilinear, called *the imbedding curvature tensors* of M in $\widetilde{M}$ for $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively. The connections ∇ and ∇^* are called the *induced connections* of $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively. Since h and h^* are symmetric and bilinear, we have the linear transformations A_v and A^*_v defined by

$$g(A_v X, Y) = \widetilde{g}(h(X, Y), v) \tag{10}$$

and

$$g(A^*_v X, Y) = \widetilde{g}(h^*(X, Y), v), \tag{11}$$

for any unit vector in the normal bundle $v \in T^{\perp}M$ and $X, Y \in TM$ [3]. It is known that when we use the Levi–Civita connection, h and A_v are called the *second fundamental form* and the *shape operator* with respect to the unit $v \in T^{\perp}M$, respectively, [24].

Let $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ be affine and dual connections on $\widetilde{M}$. We denote the induced connections ∇ and ∇^* of $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively, on M. Let $\widetilde{R}$, $\widetilde{R}^*, R$ and R^* be the Riemannian curvature tensors of $\widetilde{\nabla}, \widetilde{\nabla}^*, \nabla$ and ∇^*, respectively. Then the Gauss equations are given by

$$\widetilde{g}\Big(\widetilde{R}(X,Y)Z,W\Big) = g(R(X,Y)Z,W) \tag{12}$$

$$+\widetilde{g}(h(X,Z),h^*(Y,W)) - \widetilde{g}(h^*(X,W),h(Y,Z))$$

and

$$\widetilde{g}\Big(\widetilde{R}^*(X,Y)Z,W\Big) = g(R^*(X,Y)Z,W)$$

$$+\widetilde{g}(h^*(X,Z),h(Y,W)) - \widetilde{g}(h(X,W),h^*(Y,Z)),$$

where $X, Y, Z, W \in TM$ [3].

In the following, we recall an example of a Sasaki-like statistical manifold, for which the curvature tensor of $\mathbb{R}^{2m+1}_m$ satisfies the Equation (8) with $c = -3$.

Example 1 ([22])**.** *Let $\mathbb{R}^{2m+1}_m$ be a $(2m+1)$-dimensional affine space with the standard coordinates $\{x_1, \ldots, x_m, y_1, \ldots, y_m, z\}$. One defines a semi-Riemannian metric $\widetilde{g}$ on $\mathbb{R}^{2m+1}_m$ by*

$$\widetilde{g} = \begin{pmatrix} 2\delta_{ij} + y_i y_j & 0 & -y_i \\ 0 & -\delta_{ij} & 0 \\ -y_j & 0 & 1 \end{pmatrix}.$$

One considers the affine connection $\widetilde{\nabla}$, given by

$$\widetilde{\nabla}_{\partial x_i}\partial x_j = -y_j \partial y_i - y_i \partial y_j,$$

$$\widetilde{\nabla}_{\partial x_i}\partial y_j = \widetilde{\nabla}_{\partial y_j}\partial x_i = y_i \partial x_j + (y_i y_j - 2\delta_{ij})\partial z,$$

$$\widetilde{\nabla}_{\partial x_i}\partial z = \widetilde{\nabla}_{\partial z}\partial x_i = \partial y_i,$$

$$\widetilde{\nabla}_{\partial y_i}\partial z = \widetilde{\nabla}_{\partial z}\partial y_i = -\partial x_i - y_i \partial z,$$

$$\widetilde{\nabla}_{\partial y_i}\partial y_i = \widetilde{\nabla}_{\partial z}\partial z = 0,$$

where $\partial x_i = \frac{\partial}{\partial x_i}, \partial y_i = \frac{\partial}{\partial y_i}$ and $\partial z = \frac{\partial}{\partial z}$.

Its conjugate can be find by straightforward calculations.

One also defines ϕ, ξ and η by

$$\phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{pmatrix}, \qquad \xi = \partial z = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

and $\eta = (-y_1, 0, -y_2, \ldots, -y_m, 0, 1)$.

Then $\left(\mathbb{R}^{2m+1}_m, \widetilde{\nabla}, \widetilde{g}, \phi, \xi, \eta\right)$ represents a Sasaki-like statistical manifold with the curvature tensor of $\mathbb{R}^{2m+1}_m$ satisfying the Equation (8) with $c = -3$. From here, it can be easily found that

$$\phi^* = \frac{1}{2}\begin{pmatrix} 0 & -\delta_{ij} & 0 \\ 4\delta_{ij} & 0 & 0 \\ 0 & -y_j & 0 \end{pmatrix}.$$

Moreover, this manifold is not Sasaki with respect to the Levi–Civita connection.

For $X \in TM$, one decomposes

$$\phi X = PX + FX,$$

where PX and FX are the tangential and normal components of ϕX, respectively.

Similarly, we can write

$$\phi^* X = P^* X + F^* X,$$

where $P^* X$ and $F^* X$ are the tangential and normal components of $\phi^* X$, respectively.

Recall the following definitions from [25]:

Let $\widetilde{M}$ be a Sasaki-like statistical manifold and M a submanifold of $\widetilde{M}$. For $X \in TM$, if $\phi X \in T^{\perp}M$, then M is called an *anti-invariant submanifold* of $\widetilde{M}$. On the other hand, for a submanifold M, if $\phi X \in TM$, then M is called an *invariant submanifold* of $\widetilde{M}$.

Remark 1. *For some examples of invariant and anti-invariant submanifolds of Sasaki-like statistical manifolds $\mathbb{R}^5$ and $\mathbb{R}^9$ endowed with the structure from the previous example see [25].*

We will use the following standard notations (see also [19]):

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(\phi e_i, e_j),$$

$$\text{trace } P = \sum_{i,j=1}^{n} g(Pe_i, e_j)$$

and

$$\text{trace } P^2 = \sum_{i,j=1}^{n} g(P^2 e_i, e_j).$$

Let $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ be orthonormal tangent and normal frames, respectively, on M. The mean curvature vector fields are given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n}\sum_{\alpha=1}^{2m-n+1}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)e_{n+\alpha} \quad , \quad h_{ij}^{\alpha} = \widetilde{g}\big(h(e_i, e_j), e_{n+\alpha}\big)$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i) = \frac{1}{n}\sum_{\alpha=1}^{2m-n+1}\left(\sum_{i=1}^{n} h_{ii}^{*\alpha}\right)e_{n+\alpha} \quad , \quad h_{ij}^{*\alpha} = \widetilde{g}\big(h^*(e_i, e_j), e_{n+\alpha}\big).$$

In [26], B. Opozda introduced the *K-sectional curvature* of the statistical manifold in the following way: let π be a plane in $T\widetilde{M}$; for an orthonormal basis $\{X, Y\}$ of π, the *K-sectional curvature* was defined by

$$\widetilde{K}(\pi) = \frac{1}{2}\Big[\widetilde{R}(X, Y) + \widetilde{R}^*(X, Y) - 2\widetilde{R}^0(X, Y)\Big], \tag{13}$$

where $\widetilde{R}^0$ is the curvature tensor field of Levi–Civita connection $\widetilde{\nabla}^0$ on $T\widetilde{M}$.

In next sections, we will use the same notation g for the metric on the ambient space, for the simplicity of writing.

3. Chen First Inequality

In the present section, we recall the following algebraic lemma which will be used in the proof of the main theorem.

Lemma 1 ([18,19]). *Let $n \geq 3$ be an integer and $\{a_1, \ldots, a_n\}$ n real numbers. Then we have*

$$\sum_{1 \leq i < j \leq n}^{n} a_i a_j - a_1 a_2 \leq \frac{n-2}{2(n-1)} \left(\sum_{i=1}^{n} a_i \right)^2 .$$

The equality case of the above inequality holds if and only if $a_1 + a_2 = a_3 = \ldots = a_n$.

Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasaki-like statistical manifold satisfying (8), M an n-dimensional statistical submanifold of $\widetilde{M}$, $p \in M$ and π a plane section at p. We consider an orthonormal basis $\{e_1, e_2\}$ of π and $\{e_1, \ldots, e_n\}$, $\{e_{n+1}, \ldots, e_{2m+1}\}$ orthonormal basis of T_pM and $T_p^{\perp}M$, respectively.

We denote by K^0 the sectional curvature of the Levi–Civita connection ∇^0 on M and by h^0 the second fundamental form of M. From (13), the sectional curvature $K(\pi)$ of the plane section π is

$$K(\pi) = \frac{1}{2}\Big[g(R(e_1, e_2)e_2, e_1) + g(R^*(e_1, e_2)e_2, e_1) - 2g\Big(R^0(e_1, e_2)e_2, e_1\Big)\Big].$$

From (8), (9) and (12),

$$g(R(e_1, e_2)e_2, e_1) = \frac{c+3}{4} + \frac{c-1}{4}\Big\{2g^2(e_1, \phi e_2) - \eta(e_2)^2 - \eta(e_1)^2$$
$$-g(e_2, \phi e_2)g(e_1, \phi e_1) - g(\phi e_1, e_2)g(e_1, \phi e_2)\Big\} + \sum_{\alpha=1}^{2m-n+1} (h_{11}^{*\alpha} h_{22}^{\alpha} - h_{12}^{*\alpha} h_{12}^{\alpha}),$$

and

$$-g(R^*(e_1, e_2)e_2, e_1) = g(R(e_1, e_2)e_1, e_2) = -\frac{c+3}{4} + \frac{c-1}{4}\Big\{-2g^2(\phi e_1, e_2) + \eta(e_2)^2 + \eta(e_1)^2$$
$$+g(e_2, \phi e_2)g(e_1, \phi e_1) + g(\phi e_1, e_2)g(e_1, \phi e_2)\Big\} + \sum_{\alpha=1}^{2m-n+1} (h_{12}^{*\alpha} h_{12}^{\alpha} - h_{11}^{\alpha} h_{22}^{*\alpha}).$$

So, we obtain

$$K(\pi) = \frac{c+3}{4} + \frac{c-1}{4}\Big\{g^2(e_1, Pe_2) + g^2(Pe_1, e_2) - \eta(e_1)^2 - \eta(e_2)^2$$
$$-g(e_2, Pe_2)g(e_1, Pe_1) - g(Pe_1, e_2)g(e_1, Pe_2)\Big\} - K_0(\pi)$$
$$+\frac{1}{2}\sum_{\alpha=n+1}^{2m+1} [h_{11}^{\alpha} h_{22}^{*\alpha} + h_{11}^{*\alpha} h_{22}^{\alpha} - 2h_{12}^{*\alpha} h_{12}^{\alpha}].$$

The last equality can be written again as

$$K(\pi) = \frac{c+3}{4} + \frac{c-1}{4}\Big\{g^2(e_1, Pe_2) + g^2(Pe_1, e_2) - \eta(e_1)^2 - \eta(e_2)^2$$
$$-g(e_2, Pe_2)g(e_1, Pe_1) - g(Pe_1, e_2)g(e_1, Pe_2)\Big\} - K_0(\pi)$$
$$+ \sum_{\alpha=1}^{2m-n+1} 2\Big[h_{11}^{0\alpha} h_{22}^{0\alpha} - \Big(h_{12}^{0\alpha}\Big)^2\Big] - \frac{1}{2}\sum_{\alpha=1}^{2m-n+1} \Big\{\Big[h_{11}^{\alpha} h_{22}^{\alpha} - (h_{12}^{\alpha})^2\Big] + h_{11}^{*\alpha} h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\Big\}.$$

By using the Gauss equation with respect to Levi–Civita connection, we find

$$K(\pi) = K_0(\pi) + \frac{c+3}{4} + \frac{c-1}{4}\Big\{g^2(e_1, Pe_2) + g^2(Pe_1, e_2) - \eta(e_1)^2 - \eta(e_2)^2$$
$$-g(e_2, Pe_2)g(e_1, Pe_1) - g(Pe_1, e_2)g(e_1, Pe_2)\Big\} - 2\widetilde{K}_0(\pi)$$

$$-\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\left[h_{11}^{\alpha}h_{22}^{\alpha}-(h_{12}^{\alpha})^2\right]-\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\left[h_{11}^{*\alpha}h_{22}^{*\alpha}-(h_{12}^{*\alpha})^2\right], \tag{14}$$

where $\widetilde{K}_0$ the sectional curvature of the Levi–Civita connection $\widetilde{\nabla}^0$ on $\widetilde{M}$.

On the other hand, let τ be the scalar curvature of M. Then, using (13) and (9), we get

$$\tau=\frac{1}{2}\sum_{1\le i<j\le n}\left[g(R(e_i,e_j)e_j,e_i)+g(R^*(e_i,e_j)e_j,e_i)-2g\left(R^0(e_i,e_j)e_j,e_i\right)\right]$$

$$=\frac{1}{2}\sum_{1\le i<j\le n}[g(R(e_i,e_j)e_j,e_i)-g(R(e_i,e_j)e_i,e_j)]-\tau_0, \tag{15}$$

where τ_0 is the scalar curvature of the Levi–Civita connection ∇^0 on M. By using (12) and (8), we obtain

$$\sum_{1\le i<j\le n}g(R(e_i,e_j)e_j,e_i)=\sum_{1\le i<j\le n}[\frac{c+3}{4}\{g(e_j,e_j)g(e_i,e_i)-g(e_i,e_j)g(e_i,e_j)\}$$
$$+\frac{c-1}{4}\{g(e_i,e_j)\eta(e_j)\eta(e_i)-\eta(e_j)\eta(e_j)g(e_i,e_i)$$
$$+g(e_i,e_j)\eta(e_j)\eta(e_i)-g(e_j,e_j)\eta(e_i)\eta(e_i)$$
$$+g(e_i,\phi e_j)g(e_i,\phi e_j)-g(e_j,\phi e_j)g(\phi e_i,e_i)$$
$$+[g(e_i,\phi e_j)-g(\phi e_i,e_j)]g(e_i,\phi e_j)\}$$
$$+\widetilde{g}(h^*(e_i,e_i),h(e_j,e_j))-\widetilde{g}(h(e_i,e_j),h^*(e_i,e_j))].$$

Then, we have

$$\sum_{1\le i<j\le n}g(R(e_i,e_j)e_j,e_i)=\frac{c+3}{8}n(n-1)-\frac{c-1}{4}(n-1)\left\|\xi^T\right\|^2$$
$$+\frac{c-1}{4}\sum_{1\le i<j\le n}\{g(e_i,Pe_j)g(Pe_j,e_i)-g(e_j,Pe_j)g(Pe_i,e_i)$$
$$+[g(e_i,Pe_j)-g(Pe_i,e_j)]g(e_i,Pe_j)\}$$
$$+\sum_{1\le i<j\le n}[\widetilde{g}(h^*(e_i,e_i),h(e_j,e_j))-\widetilde{g}(h(e_i,e_j),h^*(e_i,e_j))].$$

By similar calculations, we get

$$\sum_{1\le i<j\le n}g(R(e_i,e_j)e_i,e_j)=-\frac{c+3}{8}n(n-1)+\frac{c-1}{4}(n-1)\left\|\xi^T\right\|^2$$
$$+\frac{c-1}{4}\sum_{1\le i<j\le n}\{g(e_j,Pe_j)g(Pe_i,e_i)-g(Pe_i,e_j)g(e_j,Pe_i)$$
$$-[g(e_i,Pe_j)-g(Pe_i,e_j)]g(Pe_i,e_j)\}$$
$$+\sum_{1\le i<j\le n}[\widetilde{g}(h^*(e_i,e_j),h(e_i,e_j))-\widetilde{g}(h(e_i,e_i),h^*(e_j,e_j))].$$

If we consider the last equality in (15), we obtain

$$\tau=\frac{c+3}{8}n(n-1)-\frac{c-1}{4}(n-1)\left\|\xi^T\right\|^2$$

$$+\frac{c-1}{4}\sum_{1\le i<j\le n}\{g(e_i,Pe_j)g(Pe_j,e_i)-g(e_j,Pe_j)g(Pe_i,e_i)$$
$$-g(e_i,Pe_j)g(Pe_i,e_j)+g(Pe_i,e_j)g(Pe_i,e_j)\}-\tau_0$$
$$+\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}\left[h^{*\alpha}_{ii}h^{\alpha}_{jj}+h^{\alpha}_{ii}h^{*\alpha}_{jj}-2h^{*\alpha}_{ij}h^{\alpha}_{ij}\right]. \tag{16}$$

After straightforward calculations, we find

$$\sum_{1\le i<j\le n}\{g(e_i,Pe_j)g(Pe_j,e_i)-g(e_j,Pe_j)g(Pe_i,e_i)-g(e_i,Pe_j)g(Pe_i,e_j)$$
$$+g(Pe_i,e_j)g(Pe_i,e_j)\}=\|P\|^2-\frac{(\text{trace } P)^2}{2}+\frac{1}{2}\sum_{i=1}^{n}g(Pe_i,P^*e_i).$$

Using the last equality and (5) in (16), we get

$$\tau=\frac{c+3}{8}n(n-1)-\frac{c-1}{4}\left\{(n-1)\left\|\xi^T\right\|^2-\|P\|^2+\frac{(\text{trace } P)^2}{2}\right.$$
$$\left.-\frac{1}{2}(\text{trace } P^2)\right\}+\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}\left[h^{*\alpha}_{ii}h^{\alpha}_{jj}+h^{\alpha}_{ii}h^{*\alpha}_{jj}-2h^{*\alpha}_{ij}h^{\alpha}_{ij}\right]-\tau_0.$$

The above equality can be written as

$$\tau=\frac{c+3}{8}n(n-1)-\frac{c-1}{4}\left\{(n-1)\left\|\xi^T\right\|^2-\|P\|^2+\frac{(\text{trace } P)^2}{2}\right.$$
$$\left.-\frac{1}{2}(\text{trace } P^2)\right\}+\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}2\left[h^{0\alpha}_{ii}h^{0\alpha}_{jj}-\left(h^{0\alpha}_{ij}\right)^2\right]$$
$$-\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}\left[h^{\alpha}_{ii}h^{\alpha}_{jj}-\left(h^{\alpha}_{ij}\right)^2\right]-\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}\left[h^{*\alpha}_{ii}h^{*\alpha}_{jj}-\left(h^{*\alpha}_{ij}\right)^2\right]-\tau_0.$$

By using the Gauss equation with respect to the Levi–Civita connection, we have

$$\tau=\tau_0+\frac{c+3}{8}n(n-1)-\frac{c-1}{4}\left\{(n-1)\left\|\xi^T\right\|^2-\|P\|^2+\frac{(\text{trace } P)^2}{2}\right.$$
$$\left.-\frac{1}{2}(\text{trace } P^2)\right\}-\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}\left[h^{\alpha}_{ii}h^{\alpha}_{jj}-\left(h^{\alpha}_{ij}\right)^2\right]$$
$$-\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}\left[h^{*\alpha}_{ii}h^{*\alpha}_{jj}-\left(h^{*\alpha}_{ij}\right)^2\right]-2\tilde{\tau}_0. \tag{17}$$

By subtracting (14) from (17), we get

$$(\tau-K(\pi))-(\tau_0-K_0(\pi))=\frac{c+3}{8}(n-2)(n+1)-\frac{c-1}{4}\left\{(n-1)\left\|\xi^T\right\|^2\right.$$
$$\left.+\frac{(\text{trace } P)^2}{2}-\|P\|^2-\frac{1}{2}(\text{trace } P^2)+g^2(e_1,Pe_2)+g^2(Pe_1,e_2)-\eta(e_1)^2-\eta(e_2)^2\right\}$$
$$-g(e_2,Pe_2)g(e_1,Pe_1)-g(Pe_1,e_2)g(e_1,Pe_2)\}$$

$$-\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}\left\{\left[h_{ii}^{\alpha}h_{jj}^{\alpha}-\left(h_{ij}^{\alpha}\right)^2\right]+\left[h_{ii}^{*\alpha}h_{jj}^{*\alpha}-\left(h_{ij}^{*\alpha}\right)^2\right]\right\}$$
$$+\frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\left\{\left[h_{11}^{\alpha}h_{22}^{\alpha}-(h_{12}^{\alpha})^2\right]+\left[h_{11}^{*\alpha}h_{22}^{*\alpha}-(h_{12}^{*\alpha})^2\right]\right\}+2\widetilde{K}_0(\pi)-2\widetilde{\tau}_0.$$

From Lemma 1, we have

$$\sum_{1\le i<j\le n}h_{ii}^{\alpha}h_{jj}^{\alpha}-h_{11}^{\alpha}h_{22}^{\alpha}\le\frac{(n-2)}{2(n-1)}\left(\sum_{i=1}^{n}h_{ii}^{\alpha}\right)^2=\frac{n^2(n-2)}{2(n-1)}(H^{\alpha})^2,$$

$$\sum_{1\le i<j\le n}h_{ii}^{*\alpha}h_{jj}^{*\alpha}-h_{11}^{*\alpha}h_{22}^{*\alpha}\le\frac{(n-2)}{2(n-1)}\left(\sum_{i=1}^{n}h_{ii}^{*\alpha}\right)^2=\frac{n^2(n-2)}{2(n-1)}(H^{*\alpha})^2.$$

Using the above inequality, we get

$$(\tau-K(\pi))-(\tau_0-K_0(\pi))\ge\frac{c+3}{8}(n-2)(n+1)-\frac{c-1}{4}\Big\{(n-1)\left\|\xi^T\right\|^2$$
$$+\frac{(\text{trace }P)^2}{2}-\|P\|^2-\frac{1}{2}(\text{trace }P^2)+g^2(e_1,Pe_2)+g^2(Pe_1,e_2)-g(e_2,Pe_2)g(e_1,Pe_1)$$
$$-g(Pe_1,e_2)g(e_1,Pe_2)\Big\}-\frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2+\|H^*\|^2\right]-2\Big(\widetilde{\tau}_0-\widetilde{K}_0(\pi)\Big).$$

Next, we can state the following main theorem.

Theorem 1. *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasaki-like statistical manifold satisfying (8) and M an n-dimensional statistical submanifold of $\widetilde{M}$.*

(i) Assume that ξ is tangent to M.

(a) If M is invariant, then

$$\tau_0-K_0(\pi)\le\tau-K(\pi)-\frac{c+3}{8}(n-2)(n+1)+\frac{c-1}{4}\Big\{(n-1)+\frac{(\text{trace }P)^2}{2}-\|P\|^2$$
$$-\frac{1}{2}(\text{trace }P^2)+g^2(e_1,Pe_2)+g^2(Pe_1,e_2)-g(e_2,Pe_2)g(e_1,Pe_1)\Big\}$$
$$-g(Pe_1,e_2)g(e_1,Pe_2)\}+\frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2+\|H^*\|^2\right]+2\Big(\widetilde{\tau}_0-\widetilde{K}_0(\pi)\Big).$$

(b) If M is anti-invariant, then

$$\tau_0-K_0(\pi)\le\tau-K(\pi)-\frac{c+3}{8}(n-2)(n+1)+\frac{c-1}{4}(n-1)$$
$$+\frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2+\|H^*\|^2\right]+2\Big(\widetilde{\tau}_0-\widetilde{K}_0(\pi)\Big).$$

(ii) If ξ is normal to M and M is anti-invariant, then

$$\tau_0-K_0(\pi)\le\tau-K(\pi)-\frac{c+3}{8}(n-2)(n+1)$$
$$+\frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2+\|H^*\|^2\right]+2\Big(\widetilde{\tau}_0-\widetilde{K}_0(\pi)\Big).$$

Moreover, one of the equality holds in the all cases if and only if for any $1 \le \alpha \le 2m - n + 1$ we have

$$h^{\alpha}_{11} + h^{\alpha}_{22} = h^{\alpha}_{33} = \ldots = h^{\alpha}_{nn},$$
$$h^{*\alpha}_{11} + h^{*\alpha}_{22} = h^{*\alpha}_{33} = \ldots = h^{*\alpha}_{nn},$$
$$h^{\alpha}_{ij} = h^{*\alpha}_{ij} = 0,\ \ i \ne j,\ (i,j) \notin \{(1,2),(2,1)\}.$$

4. A Chen $\delta(2,2)$ Inequality

In this section, the following lemma has the important role in the proof of our main result.

Lemma 2 ([19,20]). *Let $n \ge 4$ be an integer and $\{a_1, \ldots, a_n\}$ n real numbers. Then we have*

$$\sum_{1 \le i < j \le n}^{n} a_i a_j - a_1 a_2 - a_3 a_4 \le \frac{n-3}{2(n-2)} \left(\sum_{i=1}^{n} a_i \right)^2.$$

Equality holds if and only if $a_1 + a_2 = a_3 + a_4 = a_5 = \ldots = a_n$.

Consider $\widetilde{M}$ a Sasaki-like statistical manifold satisfying (8). For $p \in M$, we take $\pi_1, \pi_2 \subset T_pM$, mutually orthogonal, spanned, respectively, by $sp\{e_1, e_2\} = \pi_1$, $sp\{e_3, e_4\} = \pi_2$. Consider orthonormal bases $\{e_1, \ldots, e_n\} \subset T_pM$, $\{e_{n+1}, \ldots, e_{2m+1}\} \subset T_p^{\perp}M$. Then, from (14), for the planes π_1 and π_2 we have

$$\begin{aligned} K(\pi_1) = K_0(\pi_1) + \frac{c+3}{4} + \frac{c-1}{4}\Big\{ & g^2(e_1, Pe_2) + g^2(Pe_1, e_2) - \eta(e_1)^2 - \eta(e_2)^2 \\ & - g(e_2, Pe_2) g(e_1, Pe_1) - g(Pe_1, e_2) g(e_1, Pe_2)\Big\} - 2\widetilde{K}_0(\pi_1) \\ & - \frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\Big[h^{\alpha}_{11}h^{\alpha}_{22} - (h^{\alpha}_{12})^2\Big] - \frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\Big[h^{*\alpha}_{11}h^{*\alpha}_{22} - (h^{*\alpha}_{12})^2\Big] \end{aligned} \tag{18}$$

and

$$\begin{aligned} K(\pi_2) = K_0(\pi_2) + \frac{c+3}{4} + \frac{c-1}{4}\Big\{ & g^2(e_3, Pe_4) + g^2(Pe_3, e_4) - \eta(e_3)^2 - \eta(e_4)^2 \\ & - g(e_4, Pe_4) g(e_3, Pe_3) - g(Pe_3, e_4) g(e_3, Pe_4)\Big\} - 2\widetilde{K}_0(\pi_2) \\ & - \frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\Big[h^{\alpha}_{33}h^{\alpha}_{44} - (h^{\alpha}_{34})^2\Big] - \frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\Big[h^{*\alpha}_{33}h^{*\alpha}_{44} - (h^{*\alpha}_{34})^2\Big]. \end{aligned} \tag{19}$$

From (17)–(19), it follows

$$\begin{aligned} & (\tau - \tau_0) - (K(\pi_1) - K_0(\pi_1)) - (K(\pi_2) - K_0(\pi_2)) \ge \\ & \frac{c+3}{8}\Big(n^2 - n - 4\Big) - \frac{c-1}{4}\Big\{(n-1)\left\|\xi^T\right\|^2 + \frac{(\text{trace } P)^2}{2} - \|P\|^2 - \frac{1}{2}(\text{trace } P^2) \\ & + g^2(e_1, Pe_2) + g^2(Pe_1, e_2) - \eta(e_1)^2 - \eta(e_2)^2 - g(e_2, Pe_2) g(e_1, Pe_1) \\ & - g(Pe_1, e_2) g(e_1, Pe_2) + g^2(e_3, Pe_4) + g^2(Pe_3, e_4) - \eta(e_3)^2 - \eta(e_4)^2 \\ & - g(e_4, Pe_4) g(e_3, Pe_3) - g(Pe_3, e_4) g(e_3, Pe_4)\Big\} \\ & - \frac{1}{2}\sum_{\alpha=1}^{2m-n+1}\sum_{1 \le i < j \le n}\Big\{\Big[h^{\alpha}_{ii}h^{\alpha}_{jj} - h^{\alpha}_{11}h^{\alpha}_{22} - h^{\alpha}_{33}h^{\alpha}_{44}\Big] + \Big[h^{*\alpha}_{ii}h^{*\alpha}_{jj} - h^{*\alpha}_{11}h^{*\alpha}_{22} - h^{*\alpha}_{33}h^{*\alpha}_{44}\Big]\Big\} \\ & - 2\Big(\widetilde{\tau}_0 - \widetilde{K}_0(\pi_1) - \widetilde{K}_0(\pi_2)\Big). \end{aligned}$$

Lemma 2 gives

$$\sum_{1\le i<j\le n}\left[h^{\alpha}_{ii}h^{\alpha}_{jj}-h^{\alpha}_{11}h^{\alpha}_{22}-h^{\alpha}_{33}h^{\alpha}_{44}\right]$$

$$\le\frac{n-3}{2(n-2)}\left(\sum_{i=1}^{n}h^{\alpha}_{ii}\right)^2=\frac{n^2(n-3)}{2(n-2)}(H^{\alpha})^2,$$

and similarly for h^*.

Summing, we get

$$\sum_{\alpha=1}^{2m-n+1}\sum_{1\le i<j\le n}\left[h^{\alpha}_{ii}h^{\alpha}_{jj}-h^{\alpha}_{11}h^{\alpha}_{22}-h^{\alpha}_{33}h^{\alpha}_{44}\right]\le\frac{n^2(n-3)}{2(n-2)}\|H\|^2$$

and similarly for H^*.

In this way, we obtain the following inequality:

$$\begin{aligned}&(\tau-\tau_0)-(K(\pi_1)-K_0(\pi_1))-(K(\pi_2)-K_0(\pi_2))\ge\\ &\frac{c+3}{8}\left(n^2-n-4\right)-\frac{c-1}{4}\Big\{(n-1)\left\|\xi^T\right\|^2+\frac{(\text{trace }P)^2}{2}-\|P\|^2-\frac{1}{2}(\text{trace }P^2)\\ &+g^2(e_1,Pe_2)+g^2(Pe_1,e_2)-\eta(e_1)^2-\eta(e_2)^2-g(e_2,Pe_2)g(e_1,Pe_1)\\ &-g(Pe_1,e_2)g(e_1,Pe_2)+g^2(e_3,Pe_4)+g^2(Pe_3,e_4)-\eta(e_3)^2-\eta(e_4)^2\\ &-g(e_4,Pe_4)g(e_3,Pe_3)-g(Pe_3,e_4)g(e_3,Pe_4)\}\\ &-\frac{n^2(n-3)}{4(n-2)}\left(\|H\|^2+\|H^*\|^2\right)-2\left(\widetilde{\tau}_0-\widetilde{K}_0(\pi_1)-\widetilde{K}_0(\pi_2)\right).\end{aligned}$$

So we state the following theorem.

Theorem 2. *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasaki-like statistical manifold satisfying (8) and M an n-dimensional statistical submanifold of $\widetilde{M}$.*

(i) Assume that ξ is tangent to M.

(a) If M is invariant, then

$$\begin{aligned}&(\tau-\tau_0)-(K(\pi_1)-K_0(\pi_1))-(K(\pi_2)-K_0(\pi_2))\ge\\ &\frac{c+3}{8}\left(n^2-n-4\right)-\frac{c-1}{4}\Big\{(n-1)+\frac{(\text{trace }P)^2}{2}-\|P\|^2-\frac{1}{2}(\text{trace }P^2)\\ &+g^2(e_1,Pe_2)+g^2(Pe_1,e_2)-\eta(e_1)^2-\eta(e_2)^2-g(e_2,Pe_2)g(e_1,Pe_1)\\ &-g(Pe_1,e_2)g(e_1,Pe_2)+g^2(e_3,Pe_4)+g^2(Pe_3,e_4)-\eta(e_3)^2-\eta(e_4)^2\\ &-g(e_4,Pe_4)g(e_3,Pe_3)-g(Pe_3,e_4)g(e_3,Pe_4)\}\\ &-\frac{n^2(n-3)}{4(n-2)}\left(\|H\|^2+\|H^*\|^2\right)-2\left(\widetilde{\tau}_0-\widetilde{K}_0(\pi_1)-\widetilde{K}_0(\pi_2)\right).\end{aligned}$$

(b) If M is anti-invariant, then

$$\begin{aligned}&(\tau-\tau_0)-(K(\pi_1)-K_0(\pi_1))-(K(\pi_2)-K_0(\pi_2))\ge\\ &\frac{c+3}{8}\left(n^2-n-4\right)-\frac{c-1}{4}\left\{(n-1)-\eta(e_1)^2-\eta(e_2)^2-\eta(e_3)^2-\eta(e_4)^2\right\}\\ &-\frac{n^2(n-3)}{4(n-2)}\left(\|H\|^2+\|H^*\|^2\right)-2\left(\widetilde{\tau}_0-\widetilde{K}_0(\pi_1)-\widetilde{K}_0(\pi_2)\right).\end{aligned}$$

(ii) If ξ is normal to M and M is anti-invariant, then

$$(\tau - \tau_0) - (K(\pi_1) - K_0(\pi_1)) - (K(\pi_2) - K_0(\pi_2)) \geq$$
$$\frac{c+3}{8}\left(n^2 - n - 4\right) - \frac{n^2(n-3)}{4(n-2)}\left(\|H\|^2 + \|H^*\|^2\right)$$
$$-2\left(\widetilde{\tau}_0 - \widetilde{K}_0(\pi_1) - \widetilde{K}_0(\pi_2)\right).$$

Moreover, one of the equality holds in the all cases if and only if for any $1 \leq \alpha \leq 2m - n + 1$ we have

$$h^{\alpha}_{11} + h^{\alpha}_{22} = h^{\alpha}_{33} = \ldots = h^{\alpha}_{nn},$$
$$h^{*\alpha}_{11} + h^{*\alpha}_{22} = h^{*\alpha}_{33} = \ldots = h^{*\alpha}_{nn},$$
$$h^{\alpha}_{ij} = h^{*\alpha}_{ij} = 0,\ \ i \neq j,\ \ (i,j) \notin \{(1,2), (2,1), (3,4), (4,3)\}\,.$$

5. Conclusions

In Information Geometry, which is regarded as a combination of Differential Geometry and Statistics, one of the main topics and a modern one, at the same time, deals with families of probability distributions, more exactly with their invariant properties.

A challenge in submanifold theory is to obtain relations between extrinsic and intrinsic invariants of a submanifold. An important new step in this topic is due to B. Y. Chen, starting from 1993; new intrinsic invariants were introduced and such inequalities, known as Chen inequalities, were first established in a real space form. The introduction of Chen invariants was considered in the literature as one of the main contributions in classical Riemannian Geometry in the last decade of the 20-th century.

In this article, relations between extrinsic and intrinsic invariants of a submanifold, more precisely the Chen first inequality and a Chen inequality for the $\delta(2,2)$-invariant on statistical submanifolds of Sasaki-like statistical manifolds, under a curvature condition, are obtained.

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