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APPROXIMATION BY RATIONAL FUNCTIONS ON DOUBLY CONNECTED DOMAINS IN WEIGHTED GENERALIZED GRAND SMIRNOV CLASSES

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Let $G \subset \mathbb{C}$ be a doubly connected domain bounded by two rectifiable Carleson curves. We use the higher modulus of smoothness in order to investigate the approximation properties of $(p-\varepsilon)$-Faber–Laurent rational functions in the subclass of weighted generalized grand Smirnov classes $E^{p),\theta}(G,\omega)$ of analytic functions.

1. Introduction

Assume that B is a simply connected domain bounded by a rectifiable Jordan curve Γ. By $L^p(\Gamma)$ and $E^p(B)$, $1 \le p < \infty$, we denote the set of all measurable complex-valued functions such that $|f|^p$ is Lebesgue integrable with respect to arclength on Γ and the Smirnov class of analytic functions in B, respectively. Recall that if there exists a sequence (γ_n), $n = 1, 2, \ldots$, of rectifiable Jordan curves in B that converges to Γ as $n \to \infty$ such that

$$\sup_n \left\{ \int_{\gamma_n} \left|f(z)\right|^p |dz| \right\} < \infty,$$

then we say that f belongs to the Smirnov class $E^p(B)$ [24, p. 168]. Each function $f \in E^p(B)$ has a nontangential limit almost everywhere (a.e.) on Γ and if we use the same notation for the limit function of f, then $f \in L^p(\Gamma)$. Note that $L^p(\Gamma)$ and $E^p(B)$ are Banach spaces with respect to the norm

$$\|f\|_{E^p(B)} := \|f\|_{L^p(\Gamma)} := \left(\int_{\Gamma} \left|f(z)\right|^p |dz| \right)^{1/p}, \quad 1 \le p < \infty.$$

Let $G \subset \mathbb{C}$ be a doubly connected domain in the complex plane $\mathbb{C}$ bounded by rectifiable Jordan curves Γ_1 and Γ_2 such that Γ_2 is in Γ_1.

Let $G_1^- := \operatorname{Ext}\Gamma_1$, $G_1 := \operatorname{Int}\Gamma_1$, and $G_2^- := \operatorname{Ext}\Gamma_2$, $G_2 := \operatorname{Int}\Gamma_2$. Without loss of generality we can assume that $0 \in G_2$.

Also let $\mathbb{T} := \left\{ w \in \mathbb{C} \colon |w| = 1 \right\}$, $\mathbb{U} := \operatorname{Int}\mathbb{T}$, and $\mathbb{U}^- := \operatorname{Ext}\mathbb{T}$. We denote by φ and φ_1 the conformal mappings of G_1^- and G_2 onto $\mathbb{U}^-$, respectively, normalized by

$$\varphi(\infty) = \infty, \qquad \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0, \qquad \text{and} \qquad \varphi_1(0) = \infty, \qquad \lim_{z \to 0} z\varphi_1(z) > 0.$$

Let ψ and ψ_1 be the inverse mappings for φ and φ_1, respectively. The functions φ and ψ have continuous extensions to Γ_1 and $\mathbb{T}$, their derivatives φ' and ψ' have definite nontangential limit values a.e. on Γ_1 and $\mathbb{T}$.

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They are integrable with respect to the Lebesgue measure on Γ_1 and $\mathbb{T}$, respectively. Similarly, the functions φ_1 and ψ_1 have continuous extensions to Γ_2 and $\mathbb{T}$, their derivatives φ_1' and ψ_1' have definite nontangential limit values a.e. on Γ_2 and $\mathbb{T}$ that are integrable with respect to the Lebesgue measure on Γ_2 and $\mathbb{T}$ [16, p. 19–438].

We set

$$L_r := \{z \in G_1^- : |\varphi(z)| = r > 1\} \qquad \text{and} \qquad L_R := \{z \in G_2 : |\varphi(z)| = R > 1\}.$$

Let $G_r^- := \operatorname{Ext} L_r,\ G_r := \operatorname{Int} L_r$ and $G_R^- := \operatorname{Ext} L_R,\ G_R := \operatorname{Int} L_R$.

Note that φ is an analytic function in G_r^- and

$$[\varphi(z)]^k\,[\varphi'(z)]^{1/(p-\varepsilon)}$$

has a pole of kth degree at ∞, where $0 < \varepsilon < p-1$. In addition, φ_1 is analytic function in G_R and

$$[\varphi_1(z)]^{k-\frac{2}{p-\varepsilon}}\,[\varphi_1'(z)]^{1/(p-\varepsilon)}$$

has a pole of kth degree at 0, where $0 < \varepsilon < p-1$. For the construction of polynomials of the approximation process, we need some expansions. For this purpose, by applying the same technique as in [5], for $1 < p < \infty$ and $0 < \varepsilon < p-1$, we obtain

$$\frac{[\psi'(w)]^{1-\frac{1}{p-\varepsilon}}}{\psi(w)-z} = \sum_{k=0}^{\infty} \frac{F_{k,p,\varepsilon}(z)}{w^{k+1}}, \quad z \in G_r, \quad w \in \mathbb{U}^-,$$

$$\frac{w^{\frac{-2}{p-\varepsilon}}[\psi_1'(w)]^{1-\frac{1}{p-\varepsilon}}}{\psi_1(w)-z} = \sum_{k=0}^{\infty} -\frac{\widetilde{F}_{k,p,\varepsilon}(1/z)}{w^{k+1}}, \quad z \in G_R^-, \quad w \in \mathbb{U}^-,$$

where $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$ are polynomials with respect to z and $1/z$, respectively. Note that, for the first time, $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$ were considered in [14]. As in the classical case, $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$ have the following integral representations for every $k = 0, 1, 2, \ldots$:

$$F_{k,p,\varepsilon}(z) = \frac{1}{2\pi i}\int_{L_r} \frac{[\varphi(\zeta)]^k(\varphi'(\zeta))^{\frac{1}{p-\varepsilon}}}{\zeta - z}\,d\zeta, \qquad z \in G_r, \quad r > 1, \tag{1}$$

$$\widetilde{F}_{k,p,\varepsilon}(1/z) = -\frac{1}{2\pi i}\int_{L_R} \frac{[\varphi_1(\xi)]^{k-2/p-\varepsilon}(\varphi_1'(\xi))^{\frac{1}{p-\varepsilon}}}{\xi - z}\,d\xi, \qquad z \in G_R^-, \quad R > 1. \tag{2}$$

The polynomials $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$ are called the $(p-\varepsilon)$-Faber polynomials for G_r and G_R^-, respectively.

If f is an analytic function in doubly connected domain bounded by curves L_r and L_R, then, for $k = 0, 1, 2, \ldots,$ by using the Cauchy integral formula and the expansions given for $F_{k,p,\varepsilon}$ and $\widetilde{F}_{k,p,\varepsilon}$, we get the following $(p-\varepsilon)$-Faber–Laurent series expansion:

$$f(z) = \sum_{k=0}^{\infty} a_k(f)\,F_{k,p,\varepsilon}(z) + \sum_{k=1}^{\infty} \widetilde{a}_k(f)\,\widetilde{F}_{k,p,\varepsilon}(1/z),$$

where

$$a_k(f) := \frac{1}{2\pi i} \int\limits_{|w|=r_1} \frac{f\left[\psi(w)\right]\left(\psi'(w)\right)^{1/(p-\varepsilon)}}{w^{k+1}} dw, \quad 1 < r_1 < r,$$

and

$$\widetilde{a}_k(f) := \frac{1}{2\pi i} \int\limits_{|w|=R_1} \frac{f\left[\psi_1(w)\right]\left(\psi_1'(w)\right)^{1/(p-\varepsilon)} w^{2/(p-\varepsilon)}}{w^{k+1}} dw, \quad 1 < R_1 < R.$$

The rational function

$$R_n(f)(z) := \sum_{k=0}^{n} a_k(f) F_{k,p,\varepsilon}(z) + \sum_{k=1}^{n} \widetilde{a}_k(f) \widetilde{F}_{k,p,\varepsilon}(1/z)$$

is called the $(p-\varepsilon)$-Faber–Laurent rational function of f of degree n.

Definition 1. *A rectifiable Jordan curve Γ is called a Carleson curve if the condition*

$$\sup_{z\in\Gamma} \sup_{r>0} \frac{\left|\Gamma(z,r)\right|}{r} < \infty$$

is satisfied, where $\Gamma(z,r)$ is a portion of Γ in the open disk of radius r centered at z and $\left|\Gamma(z,r)\right|$ is its length. We denote the set of all Carleson curves by S.

The direct and converse theorems of the approximation theory in weighted and nonweighted Smirnov classes have been extensively investigated under various conditions imposed on the boundaries of simply connected domains. In the case where Γ is an analytic curve, some results were obtained by Walsh and Russel [19]. In the case where Γ is a Dini-smooth curve, the direct and inverse theorems were proved by Alper [27]. For the Smirnov classes in which Γ is a Carleson curve, these results were generalized in [18]. In weighted Smirnov classes, some similar results for Carleson curves were obtained in [5–9, 17]. Similar theorems of the approximation theory in Smirnov–Orlicz classes were studied in [26, 30, 31, 34].

For a Dini-smooth curve Γ, the direct and inverse theorems of approximation theory in the Smirnov classes with variable exponent were proved in [10, 12]. Earlier similar results were stated without proofs in [15, 25]. The approximation properties of the Faber–Laurent series in Lebesgue spaces with variable exponent were investigated in [11].

On a doubly connected domain bounded by two Carleson curves, the rate of approximation by the p-Faber–Laurent rational functions in Smirnov classes was studied in [29]. On doubly connected domain bounded by Dini-smooth curves, the rate of approximation by the Faber rational functions in Smirnov–Orlicz classes and Smirnov classes with variable exponent were investigated in [28] and [3], respectively.

The direct and inverse theorems of approximation theory in weighted generalized grand Lebesgue spaces were proved in [13]. After this, in weighted generalized grand Smirnov classes defined on a simply connected domain bounded by a Carleson curve, some approximation theorems were proved in [14]. In the present work, we investigate the approximation property of so-called $(p-\varepsilon)$-Faber–Laurent rational functions in the weighted generalized grand Smirnov classes defined on doubly connected domains.

The set of all measurable functions f such that

$$\sup_{0<\varepsilon<p-1} \left\{ \varepsilon^{\theta} \frac{1}{|\Gamma|} \int\limits_{\Gamma} |f(x)|^{p-\varepsilon} \omega(x) dx \right\}^{1/(p-\varepsilon)} < \infty$$

forms a weighted generalized grand Lebesgue space $L^{p),\theta}(\Gamma,\omega)$. It becomes a Banach space equipped with the norm

$$\|f\|_{L^{p),\theta}(\Gamma,\omega)} := \sup_{0<\varepsilon<p-1} \left\{ \varepsilon^{\theta} \frac{1}{|\Gamma|} \int\limits_{\Gamma} \left|f(x)\right|^{p-\varepsilon} \omega(x)dx \right\}^{1/(p-\varepsilon)}.$$

If $\theta = 0$, then $L^{p),\theta}(\Gamma)$ turns into a classical Lebesgue space $L^p(\Gamma)$. In the nonweighted case where $\theta = 1$, $L^{p),\theta}(\Gamma)$ is called a grand Lebesgue space and denoted by $L^{p)}(\Gamma)$. The spaces $L^{p),\theta}(\Gamma)$ were introduced for $\theta = 1$ in [32] and for $\theta > 1$ in [21]. The dual spaces of $L^{p)}(\Gamma)$ were characterized in [1]. In the same work, it was shown that $L^{p)}(\Gamma)$ is a rearrangement invariant and Banach function space but it is not reflexive. We can show that

$$L^p(\Gamma) \subset L^{p)}(\Gamma) \subset L^{p-\varepsilon}(\Gamma).$$

It is possible to say that similar embedding relations hold in case of weighted generalized grand Lebesgue spaces: If $\theta_1 < \theta_2$ and $1 < p < \infty$, then the embeddings

$$L^p(\Gamma,\omega) \subset L^{p),\theta_1}(\Gamma,\omega) \subset L^{p),\theta_2}(\Gamma,\omega) \subset L^{p-\varepsilon}(\Gamma,\omega)$$

are valid.

$L^p(\Gamma,\omega)$ is not dense in $L^{p),\theta}(\Gamma,\omega)$. We denote by $\mathcal{L}^{p),\theta}(\Gamma,\omega)$ the closure of $L^p(\Gamma,\omega)$ with respect to the norm of $L^{p),\theta}(\Gamma,\omega)$. We state that (see [20, 22]) $\mathcal{L}^{p)}(\Gamma)$ is the set of functions satisfying the condition

$$\lim_{\varepsilon\to 0} \left(\varepsilon^{\theta} \frac{1}{|\Gamma|} \int\limits_{\Gamma} \left|f(x)\right|^{p-\varepsilon} \omega(x)dx \right) = 0.$$

We now construct the Smirnov class defined on doubly connected domains. Let G^* be a doubly connected domain in $\mathbb{C}$ and let f be an analytic function in G^*. If there exists a sequence $(\Delta_\nu)_{\nu=1}^{\infty}$ of domains whose boundaries $(\Gamma_\nu)_{\nu=1}^{\infty}$ consist of two rectifiable Jordan curves, the lengths of $(\Gamma_\nu)_{\nu=1}^{\infty}$ are bounded and such that the domain Δ_n contains each compact subset of G^* for every $n \geq N$ for some $n \in \mathbb{N}$, and

$$\limsup_{\nu\to\infty} \left\{ \int\limits_{\Gamma_\nu} |f(z)|^p |dz| \right\} < \infty,$$

then it is said that f belongs to the Smirnov classes $E^p(G^*)$, $p \geq 1$ [24, p. 182].

Definition 2. *Let $\Gamma := \Gamma_1 \cup \Gamma_2^-$ and let G be a doubly connected domain bounded by Γ_1 and $\Gamma_2 \in S$, where Γ_2 is in Γ_1. Also let ω be a weight function on Γ. The set*

$$E^{p),\theta}(G,\omega) := \left\{ f \in E^1(G) :\ f \in L^{p),\theta}(\Gamma,\omega) \right\}$$

is called a weighted generalized grand Smirnov class of analytic functions in G.

For $f \in E^{p),\theta}(G,\omega)$, the norm is defined by

$$\|f\|_{E^{p),\theta}(G,\omega)} := \|f\|_{L^{p),\theta}(\Gamma,\omega)}.$$

We denote by $\mathcal{E}^{p),\theta}(G,\omega)$ the closure of the Smirnov class $E^p(G,\omega)$ of analytic function with respect to the norm $E^{p),\theta}(G,\omega)$.

For almost all $z_0 \in \Gamma$, the Cauchy singular integral $S_\Gamma(f)$ and the Hardy–Littlewood maximal function $M_\Gamma(f)$ with $f \in L^1(\Gamma)$, are defined as follows:

$$S_\Gamma(f)(z_0) := \lim_{r\to 0} \int\limits_{\Gamma\setminus\Gamma(z_0,r)} \frac{f(z)}{z-z_0}dz \qquad \text{and} \qquad M_\Gamma(f)(z_0) := \sup_{r>0} \frac{1}{r} \int\limits_{\Gamma(z_0,r)} |f(z)|\,|dz|.$$

Definition 3. *Let ω be weight function on Γ such that $\Gamma \in S$. Let $1 < p < \infty$ and $1/p + 1/q = 1$. We say that ω satisfies Muckenhoupt's A_p condition on Γ if*

$$\sup_{z_0\in\Gamma}\sup_{r>0} \left(\frac{1}{r} \int\limits_{\Gamma(z_0,r)} \omega(z)|dz| \right) \left(\frac{1}{r} \int\limits_{\Gamma(z_0,r)} \big[\omega(z)\big]^{-1/(p-1)}|dz| \right)^{p-1} < \infty.$$

Theorem A [33]. *Let $\Gamma \in \mathcal{S}$, $1<p<\infty$ and $\theta>0$. The operators $S_\Gamma : f \to S_\Gamma(f)$ and $M_\Gamma : f \to M_\Gamma(f)$ are bounded in $L^{p),\theta}(\Gamma,\omega)$ if and only if $\omega \in A_p(\Gamma)$.*

The norm in the space $L^{p),\theta}(\mathbb{T},\omega)$ of 2π-periodic functions f is defined as

$$\|f\|_{L^{p),\theta}(\mathbb{T},\omega)} := \sup_{0<\varepsilon<p-1} \left\{ \frac{\varepsilon^\theta}{2\pi} \int\limits_0^{2\pi} \left|f\left(e^{it}\right)\right|^{p-\varepsilon} \omega\left(e^{it}\right) dt \right\}^{1/(p-\varepsilon)}.$$

Let $f \in L^{p),\theta}(\mathbb{T},\omega)$, $1 < p < \infty$, $\theta > 0$, and, for $r = 1,2,3,\ldots,$

$$\Delta_t^r f(w) = \sum_{s=0}^{r} (-1)^{r+s+1} \binom{r}{s} f\big(we^{ist}\big), \quad t > 0.$$

We define an operator

$$\sigma_h^r f(w) := \frac{1}{h} \int\limits_0^h \big|\Delta_t^r f(w)\big| dt.$$

Now let $0 < h < \infty$. For given $\omega \in A_p(\mathbb{T})$, $1 < p < \infty$, $\theta > 0$, by using Theorem A, we conclude that

$$\sup_{|h|\le\delta} \big\|\sigma_h^r f(w)\big\|_{L^{p),\theta}(\mathbb{T},\omega)} \le c\|f\|_{L^{p),\theta}(\mathbb{T},\omega)} < \infty,$$

which implies the correctness of the following definition:

Definition 4. *Let $1 < p < \infty$, $\theta > 0$, and let $f \in L^{p),\theta}(\mathbb{T},\omega)$, $\omega \in A_p(\mathbb{T})$, $\delta > 0$. A function $\Omega_r(f,.)_{p),\theta,\omega} \colon [0,\infty) \to [0,\infty)$ defined by*

$$\Omega_r(f,\delta)_{p),\theta,\omega} := \sup_{|h|\le\delta} \big\|\sigma_h^r f(w)\big\|_{L^{p),\theta}(\mathbb{T},\omega)}$$

is called the rth mean modulus of f.

Let $\Gamma_1, \Gamma_2 \in \mathcal{S}$ and let ω be a weight function on $\Gamma_1 \cup \Gamma_2^-$. We can consider ω as a weight on Γ_1 and Γ_2 separately.

For any $f \in L^{p),\theta}(\Gamma_1, \omega)$ and $\omega \in A_p(\Gamma_1)$, we set

$$f_0(w) := f\big[\psi(w)\big]\big(\psi'(w)\big)^{1/(p-\varepsilon)}, \qquad \omega_0(w) := \omega\big[\psi(w)\big], \tag{3}$$

and, for any $f \in L^{p),\theta}(\Gamma_2, \omega)$ and $\omega \in A_p(\Gamma_2)$, we set

$$f_1(w) := f\big[\psi_1(w)\big]\big(\psi_1'(w)\big)^{\frac{1}{p-\varepsilon}} w^{\frac{2}{p-\varepsilon}}, \qquad \omega_1(w) := \omega\big[\psi_1(w)\big]w^{-2}. \tag{4}$$

In this case, we obviously have $f_0 \in L^{p),\theta}(\mathbb{T}, \omega_0)$ and $f_1 \in L^{p),\theta}(\mathbb{T}, \omega_1)$.

Let $f \in E^1(B)$, where B is a simply connected domain bounded with a rectifiable Jordan curve Γ^*. Then f has a nontangential limit a.e. on Γ^* and the boundary function belongs to $L^1(\Gamma^*)$. For given $f \in L^{p),\theta}(\Gamma^*, \omega)$ the functions f^+ and f^- defined by

$$f^+(z) := \frac{1}{2\pi i}\int\limits_{\Gamma^*} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{\big[\psi'(w)\big]^{1-\frac{1}{p-\varepsilon}}}{\psi(w) - z} f_0(w)dw, \quad z \in B,$$

$$f^-(z) := \frac{1}{2\pi i}\int\limits_{\Gamma^*} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{w^{\frac{-2}{p-\varepsilon}}\big[\psi_1'(w)\big]^{1-\frac{1}{p-\varepsilon}}}{\psi_1(w) - z} f_1(w)dw, \quad z \in B^-,$$

are analytic in B and B^-, respectively, and $f^-(\infty) = 0$. The functions f^+ and f^- have nontangential limits a.e. on Γ and the formulas

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z) \qquad \text{and} \qquad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z) \tag{5}$$

hold. Hence,

$$f(z) = f^+(z) - f^-(z) \tag{6}$$

holds a.e. on Γ [16].

The main result of the present paper is the following theorem:

Theorem 1. *Let $\Gamma_1, \Gamma_2 \in \mathcal{S}$ and let G be a finite doubly connected domain bounded by Γ_1 and Γ_2 and such that the curve Γ_2 lies inside Γ_1. Also let $\Gamma := \Gamma_1 \cup \Gamma_2^-$ and let $\omega \in A_p(\Gamma)$, $\omega_0 \in A_p(\mathbb{T})$, $\omega_1 \in A_p(\mathbb{T})$, $1 < p < \infty$, $\theta > 0$. If $f \in \mathcal{E}^{p),\theta}(G, \omega)$, then there is a positive constant c independent of n such that*

$$\big\|f - R_n(f)\big\|_{L^{p),\theta}(\Gamma,\omega)} \le c\left[\Omega_r\left(f_0, \frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1, \frac{1}{n}\right)_{p),\theta,\omega_1}\right]$$

for $r = 1, 2, 3, \ldots$, where $R_n(f)$ is the nth partial sum of the $(p-\varepsilon)$-Faber–Laurent series of f.

2. Auxiliary Results

We denote by $c,\ c_1, \ldots,$ various constants (in general, different in different relations) that depend only on the numbers that are of no interest for our presentation.

Some properties of Faber polynomials were investigated in [2, 4, 23]. By analogy with p-Faber polynomials (see [5]), we can write the integral representations for $F_{k,p,\varepsilon}(z)$ and $\widetilde{F}_{k,p,\varepsilon}(1/z)$:

If $z \in G_r^-$, then

$$F_{k,p,\varepsilon}(z) = \big[\varphi(z)\big]^k \big[\varphi'(z)\big]^{1/(p-\varepsilon)} + \frac{1}{2\pi i} \int\limits_{L_r} \frac{\big[\varphi(\zeta)\big]^k \big[\varphi'(\zeta)\big]^{1/(p-\varepsilon)}}{\zeta - z} d\zeta. \tag{7}$$

Further, if $z \in G_R$, then

$$\widetilde{F}_{k,p,\varepsilon}\left(\frac{1}{z}\right) = \big[\varphi_1(z)\big]^{k-\frac{2}{p-\varepsilon}} \big[\varphi_1'(z)\big]^{1/(p-\varepsilon)}$$

$$- \frac{1}{2\pi i} \int\limits_{L_R} \frac{\big[\varphi_1(\xi)\big]^{k-\frac{2}{p-\varepsilon}} \big[\varphi_1'(\xi)\big]^{1/(p-\varepsilon)}}{\xi - z} d\xi. \tag{8}$$

By using Cauchy integral formulas, we obtain

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z} d\xi, \quad z \in G.$$

If $z \in G_2$ or $z \in G_1^-$, then

$$\frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z} d\xi = 0. \tag{9}$$

We define

$$I_1(z) := \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad \text{and} \qquad I_2(z) := \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z} d\xi.$$

The function I_1 determines the analytic functions I_1^+ and I_1^- for $z \in G_1$ and $z \in G_1^-$, respectively, while the function I_2 determines the analytic functions I_2^+ and I_2^- for $z \in G_2$ and $z \in G_2^-$, respectively.

Lemma 1 [14]. *Let* $\Gamma \in \mathcal{S}$, $\omega \in A_p(\Gamma)$, $1 < p < \infty$, *and* $\theta > 0$. *If* $f \in L^{p),\theta}(\Gamma, \omega)$, *then* $f^+ \in E^{p),\theta}(G, \omega)$ *and* $f^- \in E^{p),\theta}(G^-, \omega)$.

For $f_0 \in L^{p),\theta}(\mathbb{T}, \omega)$ and $\omega_0 \in A_p(\mathbb{T})$, Lemma 1 implies that $f_0^+ \in E^{p),\theta}(\mathbb{U}, \omega_0)$ and $f_0^- \in E^{p),\theta}(\mathbb{U}^-, \omega_0)$ such that $f_0^-(\infty) = 0$. Similarly, for $f_1 \in L^{p),\theta}(\mathbb{T}, \omega)$ and $\omega_1 \in A_p(\mathbb{T})$, Lemma 1 implies that $f_1^+ \in E^{p),\theta}(\mathbb{U}, \omega_1)$ and $f_1^- \in E^{p),\theta}(\mathbb{U}^-, \omega_1)$ such that $f_0^-(\infty) = 0$. Thus, by (6), for $k = 0, 1, 2, 3, \ldots,$ we get

$$a_k(f) = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw - \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}} dw = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw$$

and

$$\widetilde{a}_k(f) = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_1^+(w)}{w^{k+1}} dw - \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_1^-(w)}{w^{k+1}} dw = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_1^+(w)}{w^{k+1}} dw.$$

Hence, a_k and $\widetilde{a}_k,\ k=1,2,\ldots,$ are the Taylor coefficients of $f_0^+\in E^{p),\theta}(\mathbb{U},\omega_0)$ and $f_1^+\in E^{p),\theta}(\mathbb{U},\omega_1)$, respectively.

Lemma 2. *Let* $\omega\in A_p(\mathbb{T}),\ 1<p<\infty,$ *and* $\theta>0$. *If* $g\in\mathcal{L}^{p),\theta}(\mathbb{T},\omega)$, *then*

$$\Omega_r(g^+,\cdot)_{p),\theta,\omega}\le c\,\Omega_r(g,\cdot)_{p),\theta,\omega}$$

for $r=1,2,3,\ldots$.

Proof. Let $g\in\mathcal{L}^{p),\theta}(\mathbb{T},\omega)$. First, we show that

$$\Omega_r\big(S_{\mathbb{T}}(g),\cdot\big)_{p),\theta,\omega}\le c\,\Omega_r(g,\cdot)_{p),\theta,\omega}.$$

By using the change of variables $\zeta=ue^{ist}$ and the Fubini theorem, we get

$$\begin{aligned}
\sigma_h^r\big[S_{\mathbb{T}}(g)(w)\big]&=\frac{1}{h}\int\limits_0^h\Delta_t^rS_{\mathbb{T}}(g(w))dt\\
&=\frac{1}{h}\int\limits_0^h\sum_{s=0}^r(-1)^{r+s+1}\binom{r}{s}S_{\mathbb{T}}(g\left(we^{ist}\right)dt\\
&=\frac{1}{h}\int\limits_0^h\sum_{s=0}^r(-1)^{r+s+1}\binom{r}{s}\left\{\frac{1}{2\pi i}(P.V)\int\limits_{\mathbb{T}}\frac{g(\zeta)}{\zeta-we^{ist}}d\zeta\right\}dt\\
&=\frac{1}{h}\int\limits_0^h\sum_{s=0}^r(-1)^{r+s+1}\binom{r}{s}\left\{\frac{1}{2\pi i}(P.V)\int\limits_{\mathbb{T}}\frac{g(ue^{ist})}{ue^{ist}-we^{ist}}e^{ist}du\right\}dt\\
&=\frac{1}{h}\int\limits_0^h\sum_{s=0}^r(-1)^{r+s+1}\binom{r}{s}\left\{\frac{1}{2\pi i}(P.V)\int\limits_{\mathbb{T}}\frac{g(ue^{ist})}{u-w}du\right\}dt\\
&=\frac{1}{2\pi i}(P.V)\int\mathbb{T}\frac{\left\{\dfrac{1}{h}\displaystyle\int_0^h\sum\nolimits_{s=0}^r(-1)^{r+s+1}\binom{r}{s}g(ue^{ist})dt\right\}}{u-w}du\\
&=\frac{1}{2\pi i}(P.V)\int\limits_{\mathbb{T}}\frac{\left\{\dfrac{1}{h}\displaystyle\int_0^h\Delta_t^r(g(u)dt\right\}}{u-w}du=S_{\mathbb{T}}\big[\sigma_h^rg(w)\big].
\end{aligned}$$

Taking the norm and supremum over $h\le\delta$ and applying Theorem A, we find

$$\Omega_r\big(S_{\mathbb{T}}(g),\cdot\big)_{p),\theta,\omega}=\sup_{h\le\delta}\big\|\sigma_h^r[S_{\mathbb{T}}(g)(w)]\big\|_{L^{p),\theta}(\mathbb{T},\omega)}$$

$$= \sup_{h\le\delta} \left\| S_{\mathbb{T}}[\sigma_h^r g(w)] \right\|_{L^{p),\theta}(\mathbb{T},\omega)}$$

$$\le \sup_{h\le\delta} c \left\| \sigma_h^r g(w) \right\|_{L^{p),\theta}(\mathbb{T},\omega)}$$

$$\le c \sup_{h\le\delta} \left\| \sigma_h^r g(w) \right\|_{L^{p),\theta}(\mathbb{T},\omega)} = \Omega_r(g,\cdot)_{p),\theta,\omega}. \tag{10}$$

Hence, by (5) and (10), we obtain

$$\Omega_r\big(g^+,\cdot\big)_{p),\theta,\omega} \le c\left\{\Omega_r\big(g,\cdot\big)_{p),\theta,\omega} + \Omega_r\big(S_{\mathbb{T}}(g),\cdot\big)_{p),\theta,\omega}\right\} \le c\,\Omega_r\big(g,\cdot\big)_{p),\theta,\omega}.$$

Lemma 2 is proved.

Lemma 3 [13]. *Let $g \in \mathcal{E}^{p),\theta}(\mathbb{U},\omega),\ \omega \in A_p(\mathbb{T}),\ 1<p<\infty,$ and $\theta > 0.$ If $\sum_{k=0}^{n} \gamma_k(g) w^k$ is the nth partial sum of the Taylor series of g at the origin, then there exists a positive constant c independent of $n = 1, 2, \ldots$ and such that*

$$\left\| g(w) - \sum_{k=0}^{n} \gamma_k(g) w^k \right\|_{L^{p),\theta}(\mathbb{T},\omega)} \le c\,\Omega_r\left(g, \frac{1}{n}\right)_{p),\theta,\omega}, \quad r = 1, 2, 3, \ldots .$$

3. Proof of Theorem 1

Let $\omega \in A_p(\Gamma),\ \omega_0 \in A_p(\mathbb{T}),\ \omega_1 \in A_p(\mathbb{T}),\ 1 < p < \infty,\ \theta > 0.$ Also let

$$\Gamma := \Gamma_1 \cup \Gamma_2^-,$$

where $\Gamma_1, \Gamma_2 \in \mathcal{S}$ and $f \in \mathcal{E}^{p),\theta}(G,\omega)$. We get

$$\left\| f - R_n(f) \right\|_{L^{p),\theta}(\Gamma,\omega)} \le \left\| f - R_n(f) \right\|_{L^{p),\theta}\left(\Gamma_1,\omega\right)} + \left\| f - R_n(f) \right\|_{L^{p),\theta}(\Gamma_2,\omega)}.$$

Since $f \in \mathcal{E}^{p),\theta}(G,\omega)$, we have $f_0 \in \mathcal{L}^{p),\theta}\big(\Gamma_1,\omega\big)$ and $f_1 \in \mathcal{L}^{p),\theta}(\Gamma_2,\omega)$. For $\zeta \in \Gamma_1$ and $\xi \in \Gamma_2$, by means of (3), (4), and (6), we obtain

$$f(\zeta) = \big[f_0^+\big(\varphi(\zeta)\big) - f_0^-\big(\varphi(\zeta)\big)\big]\big(\varphi'(\zeta)\big)^{1/(p-\varepsilon)} \tag{11}$$

and

$$f(\xi) = \Big[f_1^+\big(\varphi_1(\xi)\big) - f_1^-\big(\varphi_1(\xi)\big)\Big]\big(\varphi_1(\xi)\big)^{-2/(p-\varepsilon)}\big(\varphi_1'(\xi)\big)^{1/(p-\varepsilon)}. \tag{12}$$

It suffices to prove the validity of the inequalities

$$\left\| f - R_n(f) \right\|_{L^{p),\theta}\left(\Gamma_1,\omega\right)} \le c\left\{\Omega_r\left(f_0, \frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1, \frac{1}{n}\right)_{p),\theta,\omega_1}\right\} \tag{13}$$

and

$$\|f-R_n(f)\|_{L^{p),\theta}(\Gamma_2,\omega)}\le c\left\{\Omega_r\left(f_0,\frac{1}{n}\right)_{p),\theta,\omega_0}+\ \Omega_r\left(f_1,\frac{1}{n}\right)_{p),\theta,\omega_1}\right\}. \tag{14}$$

First, we prove estimation (13). We take $z'\in G_1^-$. Thus, by relations (7) and (11), we obtain

$$\begin{aligned}\sum_{k=0}^{n}a_kF_{k,p,\varepsilon}(z')&=\left[\varphi'(z')\right]^{1/(p-\varepsilon)}\sum_{k=0}^{n}a_k\left[\varphi(z')\right]^k\\&\quad+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}\sum_{k=0}^{n}a_k\left[\varphi(\zeta)\right]^k}{\zeta-z'}d\zeta\\&=\left[\varphi'(z')\right]^{1/(p-\varepsilon)}\sum_{k=0}^{n}a_k\left[\varphi\left(z'\right)\right]^k\\&\quad+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}\sum_{k=0}^{n}a_k\left[\varphi(\zeta)\right]^k}{\zeta-z'}d\zeta\\&\quad-\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}f_0^+\left(\varphi(\zeta)\right)}{\zeta-z'}d\zeta\\&\quad+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}f_0^-\left(\varphi(\zeta)\right)}{\zeta-z'}d\zeta+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\zeta)}{\zeta-z'}d\zeta.\end{aligned}$$

Since

$$\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}f_0^-\left(\varphi(\zeta)\right)\in E^1(G_1^-),$$

we get

$$-\left[\varphi'(z')\right]^{1/(p-\varepsilon)}f_0^-\left(\varphi(z')\right)=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}f_0^-\left(\varphi(\zeta)\right)}{\zeta-z'}d\zeta.$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n}a_kF_{k,p,\varepsilon}(z')&=\left[\varphi'(z')\right]^{1/(p-\varepsilon)}\sum_{k=0}^{n}a_k\left[\varphi(z')\right]^k\\&\quad-\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}\left[f_0^+\left(\varphi(\zeta)\right)-\sum_{k=0}^{n}a_k\left[\varphi(\zeta)\right]^k\right]}{\zeta-z'}d\zeta\\&\quad-\left[\ \varphi'(z')\right]^{1/(p-\varepsilon)}f_0^-\left(\varphi(z')\right)+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\zeta)}{\zeta-z'}d\zeta.\end{aligned}\tag{15}$$

If $z' \in G_2^-$, then by using relations (2) and (12) we obtain

$$\sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z') = -\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{1/(p-\varepsilon)}\left[\varphi_1(\xi)\right]^{-2/(p-\varepsilon)}\sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k}{\xi - z'}d\xi$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{1/(p-\varepsilon)}\left[\varphi_1(\xi)\right]^{-2/(p-\varepsilon)}\left(f_1^+\big(\varphi_1(\xi)\big) - \sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi - z'}d\xi$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{1/(p-\varepsilon)}\left[\varphi_1(\xi)\right]^{-2/(p-\varepsilon)} f_1^-\big(\varphi_1(\xi)\big)}{\xi - z'}d\xi - \frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi - z'}d\xi$$

and, by virtue of the Cauchy integral formula for

$$\big(\varphi_1(\xi)\big)^{-2/(p-\varepsilon)}\big(\varphi_1'(\xi)\big)^{1/(p-\varepsilon)} f_1^-\big(\varphi_1(\xi)\big) \in E^1(G_2),$$

we find

$$\sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z') = -\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi - z'}d\xi$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(f_1^+\big(\varphi_1(\xi)\big) - \sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi - z'}d\xi. \tag{16}$$

Thus, for $z' \in G_1^-$, by (15), (16), and (9), we get

$$\sum_{k=0}^{n} a_k F_{k,p,\varepsilon}(z') + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z')$$

$$= \left[\varphi'(z')\right]^{1/(p-\varepsilon)} \sum_{k=0}^{n} a_k \left[\varphi(z')\right]^k$$

$$- \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\left[\varphi'(\zeta)\right]^{1/(p-\varepsilon)}\left[f_0^+\big(\varphi(\zeta)\big) - \sum_{k=0}^{n} a_k\left[\varphi(\zeta)\right]^k\right]}{\zeta - z'} - d\zeta$$

$$- \left[\varphi'(z')\right]^{1/(p-\varepsilon)} f_0^-\big(\varphi(z')\big)$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(f_1^+\big(\varphi_1(\xi)\big) - \sum_{k=1}^{n}\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi - z'}d\xi.$$

Further, taking the limit as $z' \to z \in \Gamma_1$ along all nontangential paths outside Γ_1, we obtain

$$\begin{aligned} f(z) &- \sum_{k=0}^{n} a_k F_{k,p,\varepsilon}(z) - \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z) \\ &= \left[\varphi'(z)\right]^{1/(p-\varepsilon)} \left[f_0^+ \left(\varphi(z)\right) - \sum_{k=0}^{n} a_k \left[\varphi(z)\right]^k \right] \\ &\quad - \frac{1}{2} \left[\varphi'(z)\right]^{1/(p-\varepsilon)} \left[f_0^+ \left(\varphi(z)\right) - \sum_{k=0}^{n} a_k \left[\varphi(z)\right]^k \right] \\ &\quad + S_{\Gamma_1} \left[\left[\varphi'\right]^{1/(p-\varepsilon)} \left(\left(f_0^+ \circ \varphi\right) - \sum_{k=0}^{n} a_k \left[\varphi\right]^k \right) \right](z) \\ &\quad - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\left[\varphi_1(\xi)\right]^{\frac{1}{p-\varepsilon}} \left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}} \left(\sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(\xi)\right]^k - f_1^+\left(\varphi_1(\xi)\right) \right)}{\xi - z} d\xi \end{aligned} \tag{17}$$

a.e. on Γ_1. Since $\omega \in A_p(\Gamma)$, applying Theorem A for Γ_1 and using (17) and the Minkowski inequality, we get

$$\left\|f - R_n(f)\right\|_{L^{p),\theta}(\Gamma_1,\omega)} \le c \left\{ \left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p),\theta}(\mathbb{T},\omega_0)} + \left\| f_1^+(w) - \sum_{k=1}^{n} \widetilde{a}_k w^k \right\|_{L^{p),\theta}(\mathbb{T},\omega_1)} \right\}.$$

The Faber coefficients a_k and $\widetilde{a}_k$ are the Taylor coefficients of f_0^+ and f_1^+, respectively, at the origin. Since $\omega_0 \in A_p(\mathbb{T})$ and $\omega_1 \in A_p(\mathbb{T})$, by using Lemmas 3 and 2, we get

$$\left\|f - R_n(f)\right\|_{L^{p),\theta}(\Gamma_1,\omega)} \le c \left\{ \Omega_r\left(f_0, \frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1, \frac{1}{n}\right)_{p),\theta,\omega_1} \right\}. \tag{18}$$

Let $z'' \in G_2$. Then by virtue of (8) and (12), we obtain

$$\begin{aligned} \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z'') &= \left[\varphi_1'(z'')\right]^{\frac{1}{p-\varepsilon}} \left[\varphi_1(z'')\right]^{-\frac{2}{p-\varepsilon}} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(z'')\right]^k \\ &\quad - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\left[\varphi_1'(\xi)\right]^{\frac{1}{p-\varepsilon}} \left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(\xi)\right]^k}{\xi - z''} d\xi \\ &= \left[\varphi_1'(z'')\right]^{\frac{1}{p-\varepsilon}} \left[\varphi_1(z)\right]^{-\frac{2}{p-\varepsilon}} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(z'')\right]^k \end{aligned}$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{\left[\varphi_1'(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(f_1^+\left(\varphi_1(\xi)\right)-\sum_{k=1}^n\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi-z''}d\xi$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{\left[\varphi_1'(\xi)\right]^{\frac{1}{p-\varepsilon}}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}f_1^-\left(\varphi_1(\xi)\right)}{\xi-z''}d\xi-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi-z''}d\xi.$$

Since

$$\left(\varphi_1(\xi)\right)^{-2/(p-\varepsilon)}\left(\varphi_1'(\xi)\right)^{1/(p-\varepsilon)}f_1^-\left(\varphi_1(\xi)\right)\in E^1(G_2),$$

we find

$$\left[\varphi_1'(z'')\right]^{1/(p-\varepsilon)}\left[\varphi_1(z'')\right]^{-2/(p-\varepsilon)}f_1^-\left(\varphi_1(z'')\right)$$

$$=\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{\left[\varphi_1'(\xi)\right]^{1/(p-\varepsilon)}\left[\varphi_1(\xi)\right]^{-2/(p-\varepsilon)}f_1^-\left(\varphi_1(\xi)\right)}{\xi-z''}d\xi.$$

This equality implies that

$$\sum_{k=1}^n\widetilde{a}_k\widetilde{F}_{k,p,\varepsilon}(1/z'')=\left[\varphi_1'(z'')\right]^{1/(p-\varepsilon)}\left[\varphi_1(z'')\right]^{-\frac{2}{p-\varepsilon}}\sum_{k=1}^n\widetilde{a}_k\left[\varphi_1(z'')\right]^k$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{\left[\varphi_1'(\xi)\right]^{1/(p-\varepsilon)}\left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}}\left(f_1^+\left(\varphi_1(\xi)\right)-\sum_{k=1}^n\widetilde{a}_k\left[\varphi_1(\xi)\right]^k\right)}{\xi-z''}d\xi$$

$$-\left[\varphi_1'(z'')\right]^{1/(p-\varepsilon)}\left[\varphi_1(z'')\right]^{-\frac{2}{p-\varepsilon}}f_1^-\left(\varphi_1(z'')\right)-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi-z''}d\xi. \tag{19}$$

Let $z''\in G_1$. Thus, by (1) and (11), we get

$$\sum_{k=0}^n a_kF_{k,p,\varepsilon}(z'')=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}}\sum_{k=0}^n a_k\left[\varphi(\zeta)\right]^k}{\zeta-z''}d\zeta$$

$$=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}}\left(\sum_{k=0}^n a_k\left[\varphi(\zeta)\right]^k-f_0^+\left(\varphi(\zeta)\right)\right)}{\zeta-z''}d\zeta$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}}f_0^-\left(\varphi(\zeta)\right)}{\zeta-z''}d\zeta+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\zeta)}{\zeta-z''}d\zeta,$$

and, by using the Cauchy integral formula for $\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}} f_0^-(\varphi(\zeta)) \in E^1(G_1^-)$, we obtain

$$\sum_{k=0}^{n} a_k F_{k,p,\varepsilon}(z'') = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z''} d\zeta$$

$$+ \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}} \left(\sum_{k=0}^{n} a_k \left[\varphi(\zeta)\right]^k - f_0^+\left(\varphi(\zeta)\right)\right)}{\zeta - z''} d\zeta. \tag{20}$$

For $z'' \in G_2$, relations (19), (20), and (9) imply that

$$\sum_{k=0}^{n} a_k F_{k,p,\varepsilon}(z'') + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z'')$$

$$= \left[\varphi_1'(z'')\right]^{1/(p-\varepsilon)} \left[\varphi_1(z'')\right]^{-\frac{2}{p-\varepsilon}} \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(z'')\right]^k$$

$$+ \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\left[\varphi_1'(\xi)\right]^{1/(p-\varepsilon)} \left[\varphi_1(\xi)\right]^{-\frac{2}{p-\varepsilon}} \left(f_1^+\left(\varphi_1(\xi)\right) - \sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(\xi)\right]^k\right)}{\xi - z''} d\xi$$

$$- \left[\varphi_1'(z'')\right]^{1/(p-\varepsilon)} \left[\varphi_1(z'')\right]^{-\frac{2}{p-\varepsilon}} f_1^-\left(\varphi_1(z'')\right)$$

$$+ \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}} \left(\sum_{k=0}^{n} a_k \left[\varphi(\zeta)\right]^k - f_0^+\left(\varphi(\zeta)\right)\right)}{\zeta - z''} d\zeta.$$

Passing to the limit as $z'' \to z \in \Gamma_2$ along all nontangential paths inside Γ_2, by virtue of (11), we obtain

$$f(z) - \sum_{k=0}^{n} a_k F_{k,p,\varepsilon}(z) - \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p,\varepsilon}(1/z)$$

$$= \left[\varphi_1'(z)\right]^{1/(p-\varepsilon)} \left[\varphi_1(z)\right]^{-\frac{2}{p-\varepsilon}} f_1^+\left(\varphi_1(z)\right)$$

$$- \frac{1}{2} \left[\varphi_1'(z)\right]^{1/(p-\varepsilon)} \left[\varphi_1(z)\right]^{-\frac{2}{p-\varepsilon}} \left[\sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1(z)\right]^k - f_1^+\left(\varphi_1(z)\right)\right]$$

$$- S_{\Gamma_2}\left[\left[\varphi_1'\right]^{1/(p-\varepsilon)} \left[\varphi_1\right]^{-\frac{2}{p-\varepsilon}} \left(\sum_{k=1}^{n} \widetilde{a}_k \left[\varphi_1\right]^k - \left(f_1^+ \circ \varphi_1\right)\right)\right](z)$$

$$- \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\left(\varphi'(\zeta)\right)^{\frac{1}{p-\varepsilon}} \left(\sum_{k=0}^{n} a_k \left[\varphi(\zeta)\right]^k - f_0^+\left(\varphi(\zeta)\right)\right)}{\zeta - z} d\zeta \tag{21}$$

a.e. on Γ_2. Since $\omega \in A_p(\Gamma)$, by applying Theorem A for Γ_2 and using (17) and the Minkowski inequality, we get

$$\|f - R_n(f)\|_{L^{p),\theta}(\Gamma_2,\omega)} \leq c\left\{\left\|f_0^+(w) - \sum_{k=0}^{n} a_k w^k\right\|_{L^{p),\theta}(\mathbb{T},\omega_0)} + \left\|f_1^+(w) - \sum_{k=1}^{n} \widetilde{a}_k w^k\right\|_{L^{p),\theta}(\mathbb{T},\omega_1)}\right\}.$$

The Faber coefficients a_k and $\widetilde{a}_k$ are the Taylor coefficients of f_0^+ and f_1^+, respectively, at the origin. Since $\omega_0 \in A_p(\mathbb{T})$ and $\omega_1 \in A_p(\mathbb{T})$, by using Lemmas 3 and 2, we finally obtain

$$\|f - R_n(f)\|_{L^{p),\theta}(\Gamma_2,\omega)} \leq c\left\{\Omega_r\left(f_0, \frac{1}{n}\right)_{p),\theta,\omega_0} + \Omega_r\left(f_1, \frac{1}{n}\right)_{p),\theta,\omega_1}\right\}. \tag{22}$$

Hence, (18) and (22) complete the proof of Theorem 1.

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