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APPROXIMATION BY MATRIX TRANSFORMS IN MORREY SPACES

Abstract. In this work, approximation properties of matrix transforms constructed via the Fourier and Faber series in the subspaces of Morrey spaces are investigated.

Key words: *Morrey space, Morrey-Smirnov class, matrix transform, Faber series, approximation, Lipschitz class*

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1. Introduction and main results. Let $\Gamma \subset \mathbb{C}$ be a rectifiable Jordan curve. For a given $0 < \lambda \leqslant 2$ and $1 \leqslant p < \infty$, the Morrey space $L^{p,\lambda}(\Gamma)$ is defined as the set of all functions $f \in L^p_{loc}(\Gamma)$, such that

$$\|f\|_{L^{p,\lambda}(\Gamma)} := \left\{ \sup_B \frac{1}{|B \cap \Gamma|^{1-\lambda/2}} \int\limits_{B\cap\Gamma} |f(z)|^p \, |dz| \right\}^{1/p} < \infty,$$

where $|B \cap \Gamma|$ denotes the Lebesgue measure of $B \cap \Gamma$ and the supremum is taken over all disks $B \subset \mathbb{C}$ centered on Γ. Let $\mathbb{T} := \{w : |w| = 1\}$ or $\mathbb{T} := [0, 2\pi]$. In the case of $\Gamma = \mathbb{T}$, the Morrey space $L^{p,\lambda}(\mathbb{T})$ can be defined as the set of all functions $f \in L^p_{loc}(0, 2\pi)$ for which

$$\|f\|_{L^{p,\lambda}(\mathbb{T})} = \|f\|_{L^{p,\lambda}(0,2\pi)} := \left\{ \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int\limits_I \left|f\left(e^{i\theta}\right)\right|^p |d\theta| \right\}^{1/p} < \infty,$$

where the supremum is taken over all subintervals $I \subset (0, 2\pi)$. $L^{p,\lambda}(\Gamma)$, $0 < \lambda \leqslant 2$ and $1 \leqslant p < \infty$, becomes a Banach space equipped with the norm $\|\cdot\|_{L^{p,\lambda}(\Gamma)}$. If we choose $\lambda = 2$, then $L^{p,2}(\Gamma)$ coincides with the Lebesgue space $L^p(\Gamma)$; also, if we choose $\lambda = 0$, then $L^{p,0}(\Gamma)$ coincides with $L^\infty(\Gamma)$. Moreover, $L^{p,\lambda_1}(\Gamma) \subset L^{p,\lambda_2}(\Gamma)$ as soon as $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 2$.

Let $G \subset \mathbb{C}$ be a bounded Jordan domain with rectifiable boundary Γ and let $G^{-} := Ext\Gamma$. Denoting by $E^{p}(G)$ the classical Smirnov class of analytic functions in G, we define the Morrey-Smirnov class $E^{p,\lambda}(G)$ as

$$E^{p,\lambda}(G) := \left\{ f \in E^{1}(G) : f \in L^{p,\lambda}(\Gamma) \right\}.$$

Then $E^{p,\lambda}(G)$, $0 < \lambda \leqslant 2$ and $1 \leqslant p < \infty$, becomes a Banach space equipped with the norm $\|f\|_{E^{p,\lambda}(G)} := \|f\|_{L^{p,\lambda}(\Gamma)}$. If we choose $\lambda = 2$, then $E^{p,2}(G)$ coincides with the classical Smirnov class $E^{p}(G)$. Moreover, it can be easily seen that $E^{p,\lambda_1}(G) \subset E^{p,\lambda_2}(G)$ iff $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 2$.

If $G := \mathbb{D} := \{w : |w| < 1\}$, then we obtain the Morrey-Hardy space $H^{p,\lambda}(\mathbb{D}) : = E^{p,\lambda}(\mathbb{D})$, defined on $\mathbb{D}$.

Morrey spaces were introduced by Morrey in [24] and have important applications in differential equations. They are commonly used for study of local behavior of solutions of the elliptic differential equations, especially. Many authors have considered the fundamental problems of potential theory, maximal and singular operator theory in these spaces (see for instance: [1–3], [25], [8], [22]). Also, problems of approximation theory in Morrey spaces have been studied; in particular, in the papers [13], [14], [18–20], [5] the direct and inverse theorems of approximation theory in the Morrey spaces $L^{p,\lambda}(\mathbb{T})$ and also in the Morrey-Smirnov classes $E^{p,\lambda}(G)$ were obtained.

In this work, we study approximation properties of matrix transforms constructed via the Fourier and Faber series, in the subclasses of Morrey spaces and Morrey-Smirnov classes of analytic functions, respectively. Let us give some definitions needed to formulate the main results obtained in this work.

Let $f \in L^{1}(\mathbb{T})$ and let $f(x) \sim a_0/2 + \sum\limits_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ be its Fourier series representation with the Fourier coefficients

$$a_k := \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(t) \cos(kt)\, dt \text{ and } b_k := \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(t) \sin(kt)\, dt.$$

Let also $S_n(f)(x) = \sum\limits_{k=0}^{n} u_k(f)(x)$, $\quad n = 0, 1, 2, \ldots$ be the n-th partial sums of the Fourier series of f, where

$$u_0(f)(x) := \frac{a_0}{2} \text{ and } u_k(f)(x) := (a_k \cos kx + b_k \sin kx), \; k = 1, 2, \ldots$$

Let $A = (a_{n,k})$ be an infinite lower-triangular regular matrix with non-negative elements and let $s_n^{(A)} = \sum_{k=0}^{n} a_{n,k}$ be its n-th row sum for $n = 0, 1, 2, \ldots$ We say that the matrix $A = (a_{n,k})$ has almost monotone increasing (decreasing) rows, if there is a constant K_1 (K_2), depending only on A, such that $a_{n,k} \leqslant K_1 a_{n,m}$ $(a_{n,m} \leqslant K_2 a_{n,k})$, where $0 \leqslant k \leqslant m \leqslant n$. The matrix transform of Fourier series of $f \in L^{p,\lambda}(\mathbb{T})$ with respect to $A = (a_{n,k})$ is defined as

$$T_n^{(A)}(f)(x) = \sum_{k=0}^{n} a_{n,k} S_k(f)(x).$$

If $a_{n,k} := p_{n-k}/P_n$, for a given sequence (p_n) of positive numbers, where $P_n = \sum_{k=0}^{n} p_k$, then the matrix transform $T_n^{(A)}(f)$ coincides with the Nörlund mean

$$N_n(f)(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(f)(x),$$

which reduces to the Cesàro means

$$\sigma_n(f)(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(f)(x)$$

in the case $p_n = 1$ for all $n = 0, 1, 2, \ldots$ Let us define the modulus of smoothness $\Omega(f, \cdot)_{p,\lambda} : [0, \infty) \to [0, \infty)$ defined as

$$\Omega(f, \delta)_{p,\lambda} := \sup_{|t| \leqslant \delta} \|f(\cdot + t) - f(\cdot)\|_{L^{p,\lambda}(\mathbb{T})}, \quad \delta > 0.$$

We use the relation $f = \mathcal{O}(g)$, which means that $f \leqslant cg$ for a positive constant c, independent of f and g.

Definition 1. *Let $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leqslant 2$, $1 < p < \infty$ and $0 < \alpha \leqslant 1$. We say that $f \in Lip^{p,\lambda}(\mathbb{T}, \alpha)$ if $\Omega(f, \delta)_{p,\lambda} = \mathcal{O}(\delta^\alpha)$ for $0 \leqslant \delta$.*

Firstly, we study the approximation properties of the matrix transforms $T_n^{(A)}(f)$ in the subspaces $Lip^{p,\lambda}(\mathbb{T}, \alpha)$, $0 < \alpha \leqslant 1$, and then extend the obtained results to the subclasses of $E^{p,\lambda}(G)$. Our main results are following:

Theorem 1. *Let $0 < \lambda \leqslant 2$ and $1 < p < \infty$. Let $f \in Lip^{p,\lambda}(\mathbb{T}, \alpha)$, $0 < \alpha < 1$, and let $A = (a_{n,k})$ be a lower-triangular matrix with $\left|s_n^{(A)} - 1\right| = \mathcal{O}(n^{-\alpha})$. If one of the conditions:*

(i) A has almost monotone decreasing rows and $(n+1)a_{n,0} = \mathcal{O}(1)$,

(ii) A has almost monotone increasing rows and $(n+1)a_{n,k} = \mathcal{O}(1)$, *where k is the integer part of* $n/2$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}\left(n^{-\alpha}\right).$$

Let (p_n) be a sequence of positive numbers. If (p_n) is almost monotone decreasing, then the matrix $A = (a_{n,k})$ with $a_{n,k} := p_{n-k}/P_n$ has almost monotone increasing rows and

$$(n+1)\,a_{n,k} \leqslant K\frac{(n+1)\,p_k}{P_n} = K_1\frac{K\,(k+1)\,p_k}{P_k} = \mathcal{O}\,(1)\,,$$

where $k = [n/2]$. Thus A satisfies the condition (ii) of Theorem 1. If (p_n) is almost monotone increasing and $(n+1)p_n = \mathcal{O}\,(P_n)$, then A has almost monotone decreasing rows and

$$(n+1)a_{n,0} \leqslant (n+1)\frac{p_n}{P_n} = \frac{1}{P_n}\mathcal{O}\,(P_n) = \mathcal{O}(1).$$

Therefore, A satisfies the condition (i) of Theorem 1 and, hence, we have

Corollary 1. *Let* $0 < \lambda \leqslant 2$ *and* $1 < p < \infty$. *Let also* $f \in Lip^{p,\lambda}(\mathbb{T},\alpha)$ *for* $0 < \alpha < 1$ *and let* (p_n) *be a sequence of positive numbers. If one of the conditions:*

(i) (p_n) *is almost monotone increasing and* $(n+1)p_n = \mathcal{O}\,(P_n)$,

(ii) (p_n) *is almost monotone decreasing*

holds, then $\|f - N_n(f)\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}\,(n^{-\alpha})$.

Theorem 2. *Let* $0 < \lambda \leqslant 2$ *and* $1 < p < \infty$. *Let* $f \in Lip^{p,\lambda}(\mathbb{T},1)$ *and let* $A = (a_{n,k})$ *be a lower-triangular matrix satisfying the relation* $\left|s_n^{(A)} - 1\right| = \mathcal{O}\,(n^{-1})$.

If $\sum_{k=1}^{n-1}(n-k)|a_{n,k-1}-a_{n,k}| = \mathcal{O}(1)$, *then* $\left\|f - T_n^{(A)}(f)\right\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}(n^{-1})$.

Since $\sum_{k=1}^{n-1}(n-k)\,|a_{n,k-1}-a_{n,k}| \leqslant n\sum_{k=1}^{n-1}|a_{n,k-1}-a_{n,k}|$, by Theorem 1 we immediately have Corollary 2.

Corollary 1. *Let* $0 < \lambda \leqslant 2$ *and* $1 < p < \infty$. *Let* $f \in Lip^{p,\lambda}(\mathbb{T},1)$ *and let* $A = (a_{n,k})$ *be a lower-triangular matrix satisfying the relation* $|s_n^{(A)} - 1| = \mathcal{O}(n^{-1})$.

If $\sum\limits_{k=1}^{n-1} |a_{n,k-1} - a_{n,k}| = \mathcal{O}(n^{-1})$, *then* $\|f - T_n^{(A)}(f)\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}(n^{-1})$.

If $\sum_{k=1}^{n-1} |p_k - p_{k+1}| = \mathcal{O}(P_n/n)$, we have $\sum_{k=1}^{n-1} |a_{n,k-1} - a_{n,k}| = \mathcal{O}(n^{-1})$, where $a_{n,k} := p_{n-k}/P_n$ and $P_n = \sum_{k=0}^{n} p_k$ (see, [16]). Hence, Corollary 1 implies

Corollary 2. *Let* $0 < \lambda \leqslant 2$ *and* $1 < p < \infty$. *Let also* $f \in Lip^{p,\lambda}(\mathbb{T}, 1)$ *and let* (p_n) *be a sequence of positive numbers.*

If $\sum\limits_{k=1}^{n-1} |p_k - p_{k+1}| = \mathcal{O}(P_n/n)$, *then* $\|f - N_n(f)\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}(n^{-1})$.

Note that similar results in classical and variable Lebesgue spaces were proved in [4], [23], [21], [11], [15]. We extend the results obtained above to the subclasses of $E^{p,\lambda}(G)$. Moreover, similar results in weighted Orlicz space were proved in [17]. Therefore, we need to give some definitions and auxiliary results.

Definition 2. *Let* Γ *be a smooth Jordan curve and let* $\theta(s)$ *be the angle between the tangent and the positive real axis expressed as a function of arclength* s. *If* $\theta(s)$ *has a modulus of continuity* $\omega(\theta, s)$ *satisfying the Dini smoothness condition* $\int_0^\delta [\omega(\theta, s)/s]\, ds < \infty$, $\delta > 0$, *then we say that* Γ *is a Dini smooth curve.*

We denote the set of Dini smooth curves by $\mathfrak{D}$.

Let φ be the conformal mapping of G^- onto $\mathbb{D}^-$, normalized by the conditions $\varphi(\infty) = \infty$ and $\lim\limits_{z\to\infty} \varphi(z)/z > 0$, and let $\psi := \varphi^{-1}$ be its inverse. Since Γ is a rectifiable Jordan curve, the derivatives φ' and ψ' have definite nontangential boundary values a.e. on Γ and $\mathbb{T}$, and the boundary functions are integrable with respect to the Lebesgue measure on Γ and $\mathbb{T}$, respectively [10, p. 419–438]. On the other hand, if $\Gamma \in \mathfrak{D}$, then, by [27], there are positive constants $c_i > 0$, $i = 1, 2, 3, 4$, such that

$$0 < c_1 \leqslant |\psi'(w)| \leqslant c_2 < \infty, \ \ 0 < c_3 \leqslant |\varphi'(z)| \leqslant c_4 < \infty, \tag{1}$$

a.e. on $\mathbb{T}$ and Γ, respectively. Using (1), it is easily to see that if $\Gamma \in \mathfrak{D}$, then there exist some positive constants c_2' and c_3'', such that for any arc $\gamma \subset \Gamma$ the relation $c_2'|\gamma| \leqslant |\varphi(\gamma)| \leqslant c_3''|\gamma|$, where $|\gamma|$ and $|\varphi(\gamma)|$ are the linear Lebesgue measures of γ and its image under the conformal mapping φ, holds. Then, denoting $f_0(w) := (f \circ \psi)(w)$, $w \in \mathbb{T}$, we have the implication

$$f \in L^{p,\lambda}(\Gamma) \Leftrightarrow f_0 \in L^{p,\lambda}(\mathbb{T}). \tag{2}$$

For a given $f \in L^{p,\lambda}(\Gamma)$, the Cauchy-type integral

$$f_0^+(w) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w} d\tau, \quad w \in \mathbb{D}$$

is analytic in $\mathbb{D}$. If $f \in L^{p,\lambda}(\Gamma)$, then, by (2) $f_0 \in L^{p,\lambda}(\mathbb{T})$ and by Corollary 1 proved in [13], we have $f_0^+ \in H^{p,\lambda}(\mathbb{D})$.

Let $f \in E^{p,\lambda}(G)$, $0 < \lambda \leqslant 2$, $1 < p < \infty$ and $0 < \alpha \leqslant 1$. Denoting $\Omega(f,\delta)_{G,p,\lambda} := \Omega(f_0^+,\delta)_{p,\lambda}$, we say that

$$f \in Lip^{p,\lambda}(G,\alpha) \text{ if } \Omega(f,\delta)_{G,p,\lambda} = \mathcal{O}(\delta^\alpha) \text{ for } 0 \leqslant \delta.$$

Now let F_k, $k = 0, 1, 2, \ldots$, be the Faber polynomials for $\overline{G}$, defined by the series representation (see, [26]):

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}, \quad w \in \mathbb{D}^- \text{ and } z \in G. \tag{3}$$

On the other hand, by the Cauchy integral formula:

$$f(z) = \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(w)\psi'(w)}{\psi(w) - z} dw, \quad z \in G,$$

for every $f \in E^{p,\lambda}(G)$. Comparing this formula with (3), we have

$$f(z) \sim \sum_{k=0}^{\infty} a_k F_k(z), z \in G, \tag{4}$$

where

$$a_k = a_k(f) : = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw, \qquad k = 0, 1, 2, \ldots \tag{5}$$

The series (4) is called the Faber series of $f \in E^{p,\lambda}(G)$ and the coefficients a_k, $k = 0, 1, 2, \ldots$, are the Faber coefficients of $f \in E^{p,\lambda}(G)$. For $f \in E^{p,\lambda}(G)$, we define the n-th partial sums of series (4) as

$$S_n^G(f)(z) := \sum_{k=0}^{n} a_k(f) F_k(z), \quad n = 1, 2, 3, \ldots,$$

and n-th matrix transform

$$T_{G,n}^{(A)}(f)(z) := \sum_{k=0}^{n} a_{n,k} S_k^G(f)(z), \quad n = 1, 2, 3, \ldots,$$

of the Faber series with respect to the infinite lower-triangular regular matrix $A = (a_{n,k})$ with non-negative elements $a_{n,k}$. If $a_{n,k} := p_{n-k}/P_n$ for a given sequence (p_n) of positive numbers, where $P_n = \sum_{k=0}^{n} p_k$, then the matrix transforms $T_{G,n}^{(A)}(f)$, $n = 0, 1, \ldots$, coincide with the Nörlund means

$$N_n^G(f)(z) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k^G(f)(z)$$

of Faber series. Now, let $s_n^{(A)} = \sum_{k=0}^{n} a_{n,k}$, $n = 0, 1, \ldots$ be the n-th row sum of the matrix A.

Theorem 3. *Let $\Gamma \in \mathfrak{D}$, $0 < \lambda \leqslant 2$, $1 < p < \infty$. Let $f \in Lip^{p,\lambda}(G, \alpha)$, $0 < \alpha < 1$, and let A be a lower-triangular matrix with $|s_n^{(A)} - 1| = \mathcal{O}(n^{-\alpha})$, $A = (a_{n,k})$.*

If one of the following conditions:

(i) A has almost monotone decreasing rows and $(n+1)\, a_{n,0} = \mathcal{O}(1)$,

(ii) A has almost monotone increasing rows and $(n+1)\, a_{n,k} = \mathcal{O}(1)$, where k is the integer part of $n/2$,

holds, then

$$\left\| f - T_{G,n}^{(A)}(f) \right\|_{L^{p,\lambda}(\Gamma)} = \mathcal{O}\left(n^{-\alpha}\right).$$

Corollary 1. *Let $\Gamma \in \mathfrak{D}$, $0 < \lambda \leqslant 2$, $1 < p < \infty$. Let also $f \in Lip^{p,\lambda}(G, \alpha)$, $0 < \alpha < 1$, and let (p_n) be a sequence of positive numbers. If one of the following conditions:*

(i) (p_n) is almost monotone increasing and $(n+1)\, p_n = \mathcal{O}(P_n)$,

(ii) (p_n) is almost monotone decreasing,

holds, then $\| f - N_n^G(f) \|_{L^{p,\lambda}(\Gamma)} = \mathcal{O}(n^{-\alpha})$.

Theorem 4. *Let $\Gamma \in \mathfrak{D}$, $0 < \lambda \leqslant 2$ and $1 < p < \infty$. Let also $f \in Lip^{p,\lambda}(G, 1)$ and let $A = (a_{n,k})$ be a lower-triangular matrix satisfying the relation $|s_n^{(A)} - 1| = \mathcal{O}(n^{-1})$.*

If $\sum_{k=1}^{n-1} (n-k)\, |a_{n,k-1} - a_{n,k}| = \mathcal{O}(1)$, then

$$\left\| f - T_{G,n}^{(A)}(f) \right\|_{L^{p,\lambda}(\Gamma)} = \mathcal{O}(n^{-1}).$$

Corollary 1. *Let $\Gamma \in \mathfrak{D}$, $0 < \lambda \leqslant 2$ and $1 < p < \infty$. Let $f \in Lip^{p,\lambda}(G,1)$ and let $A = (a_{n,k})$ be a lower-triangular matrix satisfying the condition $|s_n^{(A)} - 1| = \mathcal{O}\left(n^{-1}\right)$.*
If $\sum_{k=1}^{n-1} |a_{n,k-1} - a_{n,k}| = \mathcal{O}(n^{-1})$, then $\left\| f - T_{G,n}^{(A)}(f) \right\|_{L^{p,\lambda}(\Gamma)} = \mathcal{O}(n^{-1})$.

Corollary 2. *Let $\Gamma \in \mathfrak{D}$, $0 < \lambda \leqslant 2$ and $1 < p < \infty$. Let $f \in Lip^{p,\lambda}(G,1)$ and let (p_n) be a sequence of positive numbers.*
If $\sum_{k=1}^{n-1} |p_k - p_{k+1}| = \mathcal{O}\left(P_n/n\right)$, then $\left\| f - N_n^G(f) \right\|_{L^{p,\lambda}(\Gamma)} = \mathcal{O}\left(n^{-1}\right)$.

2. Auxiliary results.

Lemma 1. *Let $0 < \lambda \leqslant 2$ and $1 < p < \infty$. Then there exists a constant c such that for every $f \in L^{p,\lambda}(\mathbb{T})$ the inequality $\|S_n(f)\|_{L^{p,\lambda}(\mathbb{T})} \leqslant c\|f\|_{L^{p,\lambda}(\mathbb{T})}$ holds.*

Proof. Let I be any subinterval of $\mathbb{T}$ with the characteristic function χ_I. By [7], the maximal function $M\chi_I$ belongs to $A_1(\mathbb{T})$, i.e., almost everywhere on $\mathbb{T}$ the inequality $M(M\chi_I) \leqslant cM\chi_I$ holds. Considering the boundedness [12] of $S_n(f)$ in the weighted Lebesgue space, we have

$$\int_I |S_n(f)(x)|^p\, dx = \int_{\mathbb{T}} |S_n(f)(x)|^p \chi_I(x)\, dx \leqslant$$
$$\leqslant \int_{\mathbb{T}} |S_n(f)(x)|^p M\chi_I(x)\, dx \leqslant c \int_{\mathbb{T}} |f(x)|^p M\chi_I(x)\, dx.$$

Then, by the equivalence (see, also: [8])

$$M\chi_I(x) \approx \chi_I(x) + \sum_{k=0}^{\infty} 2^{-2k} \chi_{(2^{k+1}I \backslash 2^k I)}(x)$$

we obtain

$$\sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_I |S_n(f)(x)|^p\, dx \leqslant$$
$$\leqslant c_6 \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_{\mathbb{T}} |f(x)|^p \left(\chi_I(x) + \sum_{k=0}^{\infty} 2^{-2k} \chi_{(2^{k+1}I \backslash 2^k I)}(x)\right) dx \leqslant$$
$$\leqslant c_6 \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_I |f(x)|^p\, dx +$$

$$+c_6 \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_{\mathbb{T}} |f(x)|^p \sum_{k=0}^{\infty} 2^{-2k} \chi_{(2^{k+1}I \backslash 2^k I)}(x)\, dx =$$

$$= c_6 \Bigg(\|f\|^p_{L^{p,\lambda}(\mathbb{T})} + \sum_{k=0}^{\infty} 2^{-2k} \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_{2^{k+1}I 2^k I} |f(x)|^p (x)\, dx \Bigg) \leqslant$$

$$\leqslant c_6 \Bigg(\|f\|^p_{L^{p,\lambda}(\mathbb{T})} + \sum_{k=0}^{\infty} 2^{-2k} \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_{2^{k+1}I} |f(x)|^p (x)\, dx \Bigg) \leqslant$$

$$\leqslant c_6 \Bigg(\|f\|^{p+}_{L^{p,\lambda}(\mathbb{T})} +$$

$$+ \sum_{k=0}^{\infty} 2^{-2k+(k+1)(1-\frac{\lambda}{2})} \sup_I \frac{1}{|2^{k+1}I|^{1-\frac{\lambda}{2}}} \int_{2^{k+1}I} |f(x)|^p (x)\, dx \Bigg) \leqslant$$

$$\leqslant c_6 \Bigg(\|f\|^p_{L^{p,\lambda}(\mathbb{T})} + \sum_{k=0}^{\infty} 2^{-2k+(k+1)(1-\frac{\lambda}{2})} \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_I |f(x)|^p (x) dx \Bigg) \leqslant$$

$$\leqslant c_7 \|f\|^p_{L^{p,\lambda}(\mathbb{T})},$$

because of $\sum_{k=0}^{\infty} 2^{-2k+(k+1)(1-\frac{\alpha}{2})} < \infty$. □

Let $f \in L^1(\mathbb{T})$ and let $\widetilde{f}$ be its conjugate function defined as

$$\widetilde{f}(x) := \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(t)}{2\tan\left(\frac{t-x}{2}\right)} dt.$$

The conjugate operator $\widetilde{f}$ is bounded in the weighted Lebesgue space [12]. Applying the same method used in the proof of Lemma 1, we obtain

Lemma 2. *Let $0 < \lambda \leqslant 2$ and $1 < p < \infty$. Then there exists a constant c such that for every $f \in L^{p,\lambda}(\mathbb{T})$ the inequality $\|\widetilde{f}\|_{L^{p,\lambda}(\mathbb{T})} \leqslant c \|f\|_{L^{p,\lambda}(\mathbb{T})}$ holds.*

Lemma 3. *Let $0 < \alpha \leqslant 1$, $0 < \lambda \leqslant 2$, $1 < p < \infty$. If $f \in Lip^{p,\lambda}(\mathbb{T}, \alpha)$, $r = 1, 2, \dots,$ then $\|f - S_n(f)\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}(n^{-\alpha})$.*

Proof. Let T_n^* , $n = 1, 2, \dots,$ be the best-approximation trigonometric polynomial to $f \in Lip^{p,\lambda}(\mathbb{T},\alpha)$ in Π_n, where Π_n is the set of trigonometric polynomials of degree not exceeding n. Then, from the direct theorem proved in [13], we have $\|f - T_n^*\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}(n^{-\alpha})$. Hence, applying

Lemma 1, we have

$$\begin{aligned}\|f-S_n(f)\|_{L^{p,\lambda}(\mathbb{T})} &\leqslant \|f-T_n^*\|_{L^{p,\lambda}(\mathbb{T})}+\|T_n^*-S_n(f)\|_{L^{p,\lambda}(\mathbb{T})}\leqslant\\ &\leqslant \|f-T_n^*\|_{L^{p,\lambda}(\mathbb{T})}+\|S_n(T_n^*-f)\|_{L^{p,\lambda}(\mathbb{T})}=\\ &= \mathcal{O}\left(\|f-T_n^*\|_{L^{p,\lambda}(\mathbb{T})}\right)=\mathcal{O}\left(n^{-\alpha}\right).\end{aligned}$$

Thus, the lemma is proved. □

Let $W_1^{p,\lambda}(\mathbb{T}):=\left\{f : f \text{ be absolutely continuous and } f'\in L^{p,\lambda}(\mathbb{T})\right\}$ be the Sobolev-Morrey space defined on $\mathbb{T}$.

Lemma 4. *Let $0<\lambda\leqslant 2$ and $1<p<\infty$. Then $f\in Lip^{p,\lambda}(\mathbb{T},1)\Leftrightarrow f\in W_1^{p,\lambda}(\mathbb{T})$.*

Proof. Let $f\in Lip^{p,\lambda}(\mathbb{T},1)$. Since $L^{p,\lambda}(\mathbb{T},1)\subset L^p(\mathbb{T})$, we have $\|f\|_{L^p(\mathbb{T})}\leqslant c\|f\|_{L^{p,\lambda}(\mathbb{T})}$ and then $\Omega(f,\delta)_{p,2}\leqslant c\Omega(f,\delta)_{p,\lambda}=\mathcal{O}(\delta)$. This relation shows that f is an absolute continuous function on $\mathbb{T}$ and, moreover, $f'\in L^p(\mathbb{T})$. Since $[f(x+t)-f(x)]/t\to f'(x)$, $t\to 0$, a. e. on $\mathbb{T}$, we have

$$\frac{2}{\delta}\int\limits_{\delta/2}^{\delta}\frac{|f(x+t)-f(x)|}{t}dt\to|f'(x)|,\ \delta\to 0^+,$$

and, then, by the Fatou Lemma in $L^{p,\lambda}(\mathbb{T},1)$

$$\begin{aligned}\|f'\|_{L^{p,\lambda}(\mathbb{T})} &= \left\|\lim_{\delta\to 0^+}\frac{2}{\delta}\int\limits_{\delta/2}^{\delta}\frac{|f(x+t)-f(x)|}{t}dt\right\|_{L^{p,\lambda}(\mathbb{T})}\leqslant\\ &\leqslant \lim_{\delta\to 0^+}\inf\left\|\frac{2}{\delta}\int\limits_{\delta/2}^{\delta}\frac{|f(x+t)-f(x)|}{t}dt\right\|_{L^{p,\lambda}(\mathbb{T})}\leqslant\\ &\leqslant \lim_{\delta\to 0^+}\inf\frac{4}{\delta}\left\|\frac{1}{\delta}\int\limits_{0}^{\delta}|f(x+t)-f(x)|\,dt\right\|_{L^{p,\lambda}(\mathbb{T})}\leqslant\\ &\leqslant c\lim_{\delta\to 0^+}\inf\frac{4}{\delta}\Omega(f,\delta)_{p,\lambda}=\mathcal{O}(1);\end{aligned}$$

hence, $f'\in W_1^{p,\lambda}(\mathbb{T})$. Conversely, if $f\in W_1^{p,\lambda}(\mathbb{T})$, then, by absolute continuity of f, we have $f(x+t)-f(x)=\int\limits_0^t f'(x+u)\,du$. Considering the boundedness of the maximal operator in the Morrey spaces in [6], we have

$$\Omega(f,\delta)_{p,\lambda} = \sup_{|h|\leqslant\delta} \|f(x+t) - f(x)\|_{L^{p,\lambda}(\mathbb{T})} \leqslant$$
$$\leqslant \sup_{|h|\leqslant\delta} \left\| \int_0^t |f'(x+u)|\,du \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant c\delta \left\| \frac{1}{\delta}\int_0^\delta |f'(x+u)|\,du \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant$$
$$\leqslant c\delta \left\| f' \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant c\delta.$$

Thus $f \in Lip^{p,\lambda}(\mathbb{T},1)$. □

Lemma 5. *Let* $0 < \lambda \leqslant 2$ *and* $1 < p < \infty$. *If* $f \in Lip^{p,\lambda}(\mathbb{T},1)$, *then* $\|S_n(f) - \sigma_n(f)\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}(n^{-1})$.

Proof. Let $f \in Lip^{p,\lambda}(\mathbb{T},1)$. By Lemma 4, we have $f \in W_1^{p,\lambda}(\mathbb{T})$. If f has the Fourier series $\sum\limits_{k=0}^{\infty} u_k(f)$, then the Fourier series of the conjugate function $\widetilde{f'}$ is $\sum\limits_{k=1}^{\infty} k u_k(f)$. After simple computations, we have

$$S_k(f) - \sigma_n(f) =$$
$$= \sum_{k=0}^{n} u_k(f) - \frac{1}{n+1}\sum_{k=0}^{n}\sum_{\nu=0}^{k} u_\nu(f) = \sum_{k=0}^{n}\left(u_k(f) - \frac{1}{n+1}\sum_{\nu=0}^{k} u_\nu(f)\right) =$$
$$= \sum_{k=0}^{n}\left(1 - \frac{n+1-k}{n+1}\right) u_k(f) = \sum_{k=0}^{n} \frac{k}{n+1} u_k(f),$$

and, hence,

$$S_n(f) - \sigma_n(f) = \sum_{k=1}^{n} \frac{k}{n+1} u_k(f). \tag{6}$$

Using here Lemmas 1 and 2, we obtain

$$\left\| S_n(f) - \sigma_n(f) \right\|_{L^{p,\lambda}(\mathbb{T})} = \left\| \sum_{k=1}^{n} \frac{k}{n+1} u_k(f) \right\|_{L^{p,\lambda}(\mathbb{T})} =$$
$$= \frac{1}{n+1} \left\| S_n\left(\widetilde{f'}\right) \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant \frac{c}{n} \|f'\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}\left(n^{-1}\right).$$

Thus, the lemma is proved. □

Lemma 6. [11] *Let $A = (a_{n,k})$ be infinite lower-triangular matrix and $0 < \alpha < 1$. If one of the conditions:*

(i) A has almost monotone decreasing rows and $(n+1)\, a_{n,0} = \mathcal{O}(1)$,

(ii) A has almost monotone increasing rows and $(n+1)a_{n,r} = \mathcal{O}(1)$, where r is the integer part of $n/2$ and $\left| s_n^{(A)} - 1 \right| = \mathcal{O}(n^{-\alpha})$,

holds, then $\sum_{k=1}^{n} k^{-\alpha} a_{n,k} = \mathcal{O}(n^{-\alpha})$.

Lemma 7. [16] *If $A = (a_{n,k})$ is an infinite lower-triangular matrix with non-negative elements $a_{n,k}$, then for every positive integer r and n such that $1 \leqslant r \leqslant n-1$, the equality*

$$\sum_{k=r}^{n-1} \Big(\sum_{m=r}^{k} |a_{n,m-1} - a_{n,m}| \Big) = \sum_{k=r}^{n-1} (n-k)\, |a_{n,k-1} - a_{n,k}|$$

holds.

Let $\mathcal{P}$ be the set of all polynomials with no restrictions on the degree and $\mathcal{P}(\mathbb{D})$ be the trace of all members of $\mathcal{P}$ on $\mathbb{D}$.

We define the operator $\Upsilon : \mathcal{P}(\mathbb{D}) \to E^{p,\lambda}(G)$ as

$$\Upsilon(P)(z) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{P(w)\, \psi'(w)}{\psi(w) - z} dw = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{P(\varphi(\varsigma))}{\varsigma - z} d\varsigma, \quad z \in G.$$

If $P(w) := \sum\limits_{k=0}^{n} b_k w^k$, then, by (3), we have

$$\Upsilon\Big(\sum_{k=0}^{n} b_k w^k \Big) = \frac{1}{2\pi i} \sum_{k=0}^{n} b_k \int\limits_{\mathbb{T}} \frac{w^k \psi'(w)}{\psi(w) - z} dw = \sum_{k=0}^{n} b_k F_k(z).$$

If $\Gamma \in \mathfrak{D}$, $0 < \lambda \leqslant 2$, and $1 < p < \infty$, then the linear operator $\Upsilon : \mathcal{P}(\mathbb{D}) \to E^{p,\lambda}(G)$ is bounded (see [14]). Hence, extending the operator Υ from $\mathcal{P}(\mathbb{D})$ to $H^{p,\lambda}(\mathbb{D})$ as a linear and bounded operator, we obtain the extended operator $\Upsilon : H^{p,\lambda}(\mathbb{D}) \to E^{p,\lambda}(G)$ with the representation

$$\Upsilon(f)(z) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f(w)\, \psi'(w)}{\psi(w) - z} dw, \quad z \in G, \quad f \in H^{p,\lambda}(\mathbb{D}).$$

Theorem 5. [14] *Let $\Gamma \in \mathfrak{D}$, $0 < \lambda \leqslant 2$ and $1 < p < \infty$. Then the operator $\Upsilon(P) : H^{p,\lambda}(\mathbb{D}) \to E^{p,\lambda}(G)$ is linear, bounded, one to one, and onto. Moreover, $\Upsilon\left(f_0^{+}\right) = f$ for every $f \in E^{p,\lambda}(G)$.*

3. Proofs of the Main Results.

Proof of Theorem 1. Let $f \in Lip^{p,\lambda}(\mathbb{T}, \alpha)$, $0 < \alpha < 1$, $0 < \lambda \leqslant 2$, $1 < p < \infty$ and let $A = (a_{n,k})$ be a lower-triangular matrix with $|s_n^{(A)} - 1| = \mathcal{O}(n^{-\alpha})$. Suppose that one of the conditions (i) and (ii) holds. By definition of $T_n^{(A)}(f)$ and $s_n^{(A)}$, we have

$$\begin{aligned}
T_n^{(A)}(f)(x) - f(x) &= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) - f(x) = \\
&= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) - f(x) + s_n^{(A)} f(x) - s_n^{(A)} f(x) = \\
&= \sum_{k=0}^{n} a_{n,k} \left[S_k(f)(x) - f(x)\right] + \left(s_n^{(A)} - 1\right) f(x).
\end{aligned}$$

Since $|s_n^{(A)} - 1| = \mathcal{O}(n^{-\alpha})$, by Lemmas 3 and 6 we obtain

$$\begin{aligned}
&\left\| f - T_n^{(A)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant a_{n,0} \left\| S_0(f) - f \right\|_{L^{p,\lambda}(\mathbb{T})} + \\
&\quad + \sum_{k=1}^{n} a_{n,k} \left\| S_k(f) - f \right\|_{L^{p,\lambda}(\mathbb{T})} + \left| s_n^{(A)} - 1 \right| \left\| f \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant \\
&\quad \leqslant \mathcal{O}\left(\frac{1}{n+1}\right) + c \sum_{k=1}^{n} a_{n,k} k^{-\alpha} + \mathcal{O}\left(n^{-\alpha}\right) = \mathcal{O}\left(n^{-\alpha}\right). \ \square
\end{aligned}$$

Proof of Theorem 2. Let $f \in Lip^{p,\lambda}(\mathbb{T}, 1)$, $0 < \lambda \leqslant 2$, $p \in (1, \infty)$, $A = (a_{n,k})$ be a lower-triangular matrix, $|s_n^{(A)} - 1| = \mathcal{O}(n^{-1})$. By Lemma 3, we have:

$$\begin{aligned}
\left\| f - T_n^{(A)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} &\leqslant \left\| S_n(f) - T_n^{(A)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} + \left\| f - S_n(f) \right\|_{L^{p,\lambda}(\mathbb{T})} = \\
&= \left\| S_n(f) - T_n^{(A)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} + \mathcal{O}\left(n^{-1}\right). \quad (7)
\end{aligned}$$

If $A_{n,k} := \sum_{m=k}^{n} a_{n,m}$, then

$$\begin{aligned}
T_n^{(A)}(f)(x) &= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) = \sum_{k=0}^{n} a_{n,k} \left(\sum_{m=0}^{k} u_m(f)(x) \right) = \\
&= \sum_{k=0}^{n} \left(\sum_{m=k}^{n} a_{n,k} \right) u_k(f)(x) = \sum_{k=0}^{n} A_{n,k} u_k(f)(x).
\end{aligned}$$

On the other hand, since $s_n^{(A)}=\sum_{k=0}^{n}a_{n,k}$, we get:

$$\begin{aligned} S_n(f)(x) &= \sum_{m=0}^{n} u_m(f)(x) = \\ &= A_{n,0}\sum_{k=0}^{n} u_k(f)(x) + (1-A_{n,0})\sum_{k=0}^{n} u_k(f)(x) = \\ &= \sum_{k=0}^{n} A_{n,0}u_k(f)(x) + \left(1-s_n^{(A)}\right) S_n(f)(x). \end{aligned}$$

Thus,

$$\begin{aligned} T_n^{(A)}(f)(x) - S_n(f)(x) = \sum_{k=1}^{n}(A_{n,k}-A_{n,0})\,u_k(f)(x) + \\ + \left(s_n^{(A)}-1\right) S_n(f)(x). \end{aligned}$$

By Lemma 1 and by the condition $|s_n^{(A)}-1| = \mathcal{O}(n^{-1})$:

$$\left\| S_n(f) - T_n^{(A)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant \left\| \sum_{k=1}^{n}(A_{n,k}-A_{n,0})\,u_k(f) \right\|_{L^{p,\lambda}(\mathbb{T})} + \mathcal{O}\left(n^{-1}\right). \tag{8}$$

Setting $b_{n,k} := \frac{A_{n,k}-A_{n,0}}{k}$, $k=1,2,\ldots,n$ and applying the Abel transform, we have

$$\begin{aligned} \sum_{k=1}^{n}(A_{n,k}-A_{n,0})\,u_k(f) = \sum_{k=1}^{n} b_{n,k}k u_k(f) = b_{n,n}\sum_{m=1}^{n} m u_m(f) + \\ + \sum_{k=1}^{n-1}(b_{n,k}-b_{n,k+1})\left(\sum_{m=1}^{k} m u_m(f)\right), \end{aligned}$$

and, then:

$$\begin{aligned} \left\| \sum_{k=1}^{n}(A_{n,k}-A_{n,0})\,u_k(f) \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant |b_{n,n}| \left\| \sum_{m=1}^{n} m u_m(f) \right\|_{L^{p,\lambda}(\mathbb{T})} + \\ + \sum_{k=1}^{n-1}|b_{n,k}-b_{n,k+1}| \left(\left\| \sum_{m=1}^{k} m u_m(f) \right\|_{L^{p,\lambda}(\mathbb{T})} \right). \end{aligned} \tag{9}$$

Now, using (6) and applying Lemma 5, we have

$$\begin{aligned}\left\| \sum_{m=1}^{n} m u_m (f) \right\|_{L^{p,\lambda}(\mathbb{T})} &= (n+1) \left\| S_n (f) - \sigma_n (f) \right\|_{L^{p,\lambda}(\mathbb{T})} = \\ &= (n+1) \mathcal{O}\left(n^{-1}\right) = \mathcal{O}(1); \end{aligned} \tag{10}$$

later, by the condition $\left| s_n^{(A)} - 1 \right| = \mathcal{O}\left(n^{-1}\right)$:

$$\begin{aligned}|b_{n,n}| = \frac{|A_{n,n} - A_{n,0}|}{n} = \frac{\left|a_{n,n} - s_n^{(A)}\right|}{n} = \\ = \frac{1}{n}\left(s_n^{(A)} - a_{n,n}\right) \leqslant \frac{1}{n} s_n^{(A)} = \mathcal{O}\left(n^{-1}\right). \end{aligned} \tag{11}$$

Since, by the relations (8)–(11),

$$\left\| S_n (f) - T_n^{(A)} (f) \right\|_{L^{p,\lambda}(\mathbb{T})} \leqslant \mathcal{O}\Big(n^{-1} + \sum_{k=1}^{n-1} |b_{n,k} - b_{n,k+1}|\Big), \tag{12}$$

to complete the proof it remains to prove that $\sum\limits_{k=1}^{n-1} |b_{n,k} - b_{n,k+1}| = \mathcal{O}\left(n^{-1}\right)$. After simple calculations, we have

$$b_{n,k} - b_{n,k+1} = \frac{1}{k(k+1)} \Big[(k+1)\, a_{n,k} - \sum_{m=0}^{k} a_{n,m} \Big] \tag{13}$$

and, later, iteration easily shows that for $k = 1, 2, \ldots, n$,

$$\left| \sum_{m=0}^{k} a_{n,m} - (k+1)\, a_{n,k} \right| \leqslant \sum_{m=1}^{k} m \left| a_{n,m-1} - a_{n,m} \right|. \tag{14}$$

If the condition $\sum_{k=1}^{n-1} (n-k) \left| a_{n,k-1} - a_{n,k} \right| = \mathcal{O}(1)$ of Theorem 2 holds, then by (13) and (14) we get

$$\begin{aligned}\sum_{k=1}^{n-1} |b_{n,k} - b_{n,k+1}| &\leqslant \sum_{k=1}^{r} \frac{1}{k(k+1)} \sum_{m=1}^{k} m \left| a_{n,m-1} - a_{n,m} \right| + \\ &+ \sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=1}^{k} m \left| a_{n,m-1} - a_{n,m} \right|, \end{aligned} \tag{15}$$

for $r := [n/2]$. For the first term of the right-hand side, applying the Abel transform we have

$$\sum_{k=1}^{r} \frac{1}{k(k+1)} \sum_{m=1}^{k} m \left| a_{n,m-1} - a_{n,m} \right| \leqslant$$
$$\leqslant \sum_{k=1}^{r} \left| a_{n,k-1} - a_{n,k} \right| = \sum_{k=1}^{r} \frac{1}{(n-k)} (n-k) \left| a_{n,k-1} - a_{n,k} \right| \leqslant$$
$$\leqslant \frac{1}{(n-r)} \sum_{k=1}^{r} (n-k) \left| a_{n,k-1} - a_{n,k} \right| \leqslant$$
$$\leqslant \frac{1}{(n-r)} \mathcal{O}(1) = \mathcal{O}\left(n^{-1}\right). \quad (16)$$

For the second term, we can write

$$\sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=1}^{k} m \left| a_{n,m-1} - a_{n,m} \right| \leqslant$$
$$\leqslant \sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=1}^{r} m \left| a_{n,m-1} - a_{n,m} \right| +$$
$$+ \sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=r}^{k} m \left| a_{n,m-1} - a_{n,m} \right| := I_{n_1} + I_{n_2}.$$

Since $\sum_{k=1}^{r} \left| a_{n,k-1} - a_{n,k} \right| = \mathcal{O}\left(n^{-1}\right)$, based on (16) we have

$$I_{n_1} \leqslant \sum_{k=r}^{n-1} \frac{1}{k+1} \sum_{m=1}^{r} \left| a_{n,m-1} - a_{n,m} \right| =$$
$$= \mathcal{O}\left(n^{-1}\right) \sum_{k=r}^{n-1} \frac{1}{k+1} = \mathcal{O}\left(n^{-1}\right)(n-r) \frac{1}{r+1} = \mathcal{O}\left(n^{-1}\right).$$

Now we estimate I_{n_2}. By Lemma 7

$$I_{n_2} \leqslant \sum_{k=r}^{n-1} \frac{1}{k+1} \sum_{m=r}^{k} \left| a_{n,m-1} - a_{n,m} \right| \leqslant$$

$$\leqslant \frac{1}{r+1}\sum_{k=r}^{n-1}\left(\sum_{m=r}^{k}|a_{n,m-1}-a_{n,m}|\right) \leqslant \frac{2}{n}\sum_{k=r}^{n-1}\left(\sum_{m=r}^{k}|a_{n,m-1}-a_{n,m}|\right) \leqslant$$

$$\leqslant \frac{2}{n}\sum_{k=r}^{n-1}(n-k)\,|a_{n,k-1}-a_{n,k}| \leqslant \frac{2}{n}\sum_{k=1}^{n-1}(n-k)\,|a_{n,k-1}-a_{n,k}| = \mathcal{O}\left(n^{-1}\right).$$

Thus

$$\sum_{k=r}^{n-1}\frac{1}{k\,(k+1)}\sum_{m=1}^{k}m\,|a_{n,m-1}-a_{n,m}| \leqslant I_{n_1}+I_{n_2} = \mathcal{O}\left(n^{-1}\right). \tag{17}$$

Hence, by the relations (15) – (17) we have

$$\sum_{k=1}^{n-1}|b_{n,k}-b_{n,k+1}| = \mathcal{O}\left(n^{-1}\right). \tag{18}$$

Now, the relations (7), (12), and (18) imply the desired inequality. □

Proof of Theorem 3. Let $\Gamma \in \mathfrak{D}$, $0<\lambda\leqslant 2$, $1<p<\infty$, and $f \in Lip^{p,\lambda}(G,\alpha)$ for $0<\alpha<1$. Let $A=(a_{n,k})$ be a lower-triangular matrix with $|s_n^{(A)}-1| = O(n^{-\alpha})$. Since $f\in E^{p,\lambda}(G)$, we have $f_0^+ \in H^{p,\lambda}(\mathbb{D}) \subset H^1(\mathbb{D})$, which implies that the boundary function of f_0^+ belongs to $L^{p,\lambda}(\mathbb{T})$. Let $\sum\limits_{k=0}^{\infty}\beta_k\left(f_0^+\right)w^k$, $w\in\mathbb{D}$, be the Taylor-series expansion of the function f_0^+ on the unit disk $\mathbb{D}$. By Theorem 3.4 in [9, p. 38], we get:

$$c_k\left(f_0^+\right) = \begin{cases}\beta_k\left(f_0^+\right), & k\geqslant 0,\\ 0, & k<0,\end{cases}$$

where $\sum\limits_{k=-\infty}^{\infty}c_k\left(f_0^+\right)e^{ikt}$ is the Fourier series of the boundary function of $f_0^+ \in L^{p,\lambda}(\mathbb{T}) \subset L^1(\mathbb{T})$. Therefore, we have $f_0^+(w) = \sum\limits_{k=-\infty}^{\infty}c_k\left(f_0^+\right)w^k$. Assuming that $f_0(w) = f_0^+(w) - f_0^-(w)$ a. e. on $\mathbb{T}$, we get:

$$\begin{aligned} a_k(f) &= \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0(w)}{w^{k+1}}dw = \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^+(w)}{w^{k+1}}dw - \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^-(w)}{w^{k+1}}dw = \\ &= \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^+(w)}{w^{k+1}}dw = \beta_k\left(f_0^+\right), \end{aligned}$$

which shows that the Faber coefficients $a_k(f)$, $k = 0, 1, 2, \ldots$, are the Taylor coefficients of f_0^+ at the origin, that is

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f)\, w^k, w \in \mathbb{D}. \tag{19}$$

If $\sum\limits_{k=0}^{\infty} a_k(f) F_k(z)$ is the Faber-series expansion of $f \in E^{p,\lambda}(G)$, then, by (19) and (3), we get

$$\Upsilon\left(\sum_{k=0}^{n} c_k\left(f_0^+\right) w^k\right) = S_n^G(f)(z) \text{ and } \Upsilon\left(T_n^{(A)}\left(f_0^+\right)\right) = T_{G,n}^{(A)}(f). \tag{20}$$

Taking into account the conditions of Theorem 3, we have $f_0^+ \in Lip^{p,\lambda}(\mathbb{T}, \alpha)$. Now, applying Theorem 1 for f_0^+ and Theorem 5, we have:

$$\left\|f - T_{G,n}^{(A)}(f)\right\|_{L^{p,\lambda}(\Gamma)} = \left\|\Upsilon\left(f_0^+\right) - \Upsilon\left(T_n^{(A)}\left(f_0^+\right)\right)\right\|_{L^{p,\lambda}(\Gamma)} =$$

$$= \left\|\Upsilon\left(f_0^+ - T_n^{(A)}\left(f_0^+\right)\right)\right\|_{L^{p,\lambda}(\Gamma)} \leqslant c\left\|f_0^+ - T_n^{(A)}\left(f_0^+\right)\right\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}\left(n^{-\alpha}\right). \square$$

Proof of Theorem 4. Let $\Gamma \in \mathfrak{D}$, $0 < \lambda \leqslant 2$, $1 < p < \infty$. The condition $f \in Lip^{p,\lambda}(G, 1)$ of Theorem 4, by definition of classes $Lip^{p,\lambda}(G, 1)$, means that $f_0^+ \in Lip^{p,\lambda}(\mathbb{T}, 1)$. Then, applying Theorem 2 for f_0^+, by (20) and Theorem 5, we have

$$\left\|f - T_{G,n}^{(A)}(f)\right\|_{L^{p,\lambda}(\Gamma)} = \left\|\Upsilon\left(f_0^+\right) - \Upsilon\left(T_n^{(A)}\left(f_0^+\right)\right)\right\|_{L^{p,\lambda}(\Gamma)} =$$

$$= \left\|\Upsilon\left(f_0^+ - T_n^{(A)}\left(f_0^+\right)\right)\right\|_{L^{p,\lambda}(\Gamma)} \leqslant c\left\|f_0^+ - T_n^{(A)}\left(f_0^+\right)\right\|_{L^{p,\lambda}(\mathbb{T})} = \mathcal{O}\left(n^{-1}\right). \square$$

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