Alexandria Engineering Journal (2021) 60, 2347–2353

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Separable solutions of Cattaneo-Hristov heat diffusion equation in a line segment: Cauchy and source problems ☆

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Received 29 June 2020; revised 9 September 2020; accepted 20 December 2020
Available online 6 January 2021

KEYWORDS

Cattaneo-Hristov heat diffusion;
Laplace transform;
Eigenfunction expansion;
Cauchy problem;
Source problem;
Caputo-Fabrizio fractional derivative

Abstract The behavior of Cattaneo-Hristov heat diffusion moving in a line segment under the influence of specified initial and source temperatures has been investigated. The Fourier method has been applied to determine the eigenfunctions thus allowing reducing the problem to a set of time-fractional ordinary differential equations. Analytical solutions by applying the Laplace transform method have been developed.

1. Introduction

Accurate modeling of real phenomena has always been a common interest among scientists. One of the most efficient ways to achieve this is to reveal generalized laws that complement the shortcomings of classical physics laws. In this sense, fractional operators play a quite important role because of their memory and hereditary effects. Besides fractional derivatives naturally emerge in the generalized laws of physics, replacement of the classical derivatives with fractional ones also provide realistic models.

The diffusion phenomenon is one of the main topics that need generalized models. The apparent weakness of the classical diffusion equation is that it causes the unphysical infinite speed of propagation [1,2]. Besides, the classical diffusion equation can be inefficient to describe diffusive transport acting on a heterogeneous or porous medium. These inadequacies have been removed by determining the non-local relations between flux and concentration. Thus, the foreseen generalized law and the relevant fractional diffusion equations are obtained. In this sense, Fujita [3], Mainardi [4], and Povstenko [5] revealed that the time-fractional diffusion equation with Caputo fractional derivative bases on a temporal and/or spatial non-local relation between heat flux and temperature gradient with a "long-tail memory" denoting by power kernel. This is the physical aspect of the requirement of fractional operators in the diffusion phenomenon. However, the statistical aspect of fractional operators corresponds to the non-Brownian motion of particles in the diffusion [6,7]. This behavior of particles is also called as anomalous diffusion. Types of the anomalous diffusion are specified with respect to order of Caputo fractional derivative α . The cases of $0<\alpha<1, \alpha=1, 1<\alpha<2$ and $\alpha=2$ correspond to sub, standard, super and ballistic diffusion equations, respectively [8–11].

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☆ Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2020.12.018
1110-0168

Moreover, fundamental solutions for the initial and boundary value problems related to the fractional diffusion equation with Caputo derivative in various coordinate systems were extensively investigated by Povstenko [12]. To support mathematical solutions, thermoelastic properties resulting from fractional heat diffusion equation were also introduced [13].

Recently, a new perspective has been added to fractional calculus by introducing Caputo-Fabrizio [14] and Atangana-Baleanu [15] derivatives. Non-singular exponential and Mittag–Leffler kernels of these operators provide computation facilities in solution methods. Besides, they are shown to be more effective in modeling processes obeying to the exponential decay law such as heat and mass transfer [16–18], groundwater flow [19–22], epidemiology of infectious diseases, and dynamics of tumor growth [23–30], financial problems [31–35], etc.

The present study addresses the Cattaneo-Hristov heat diffusion problem developed with the fading memory concept demonstrating that the Caputo-Fabrizio derivative [36] naturally appears when the Cattaneo's approach to the Fourier's law [37], suggesting a non-local relation between heat flux and temperature gradient by a time-dependent Jeffrey's kernel [36] has been applied. The time-evolution of Cattaneo-Hristov model was asymptotically studied in [36] by the approximate integral-balance method. Alkahtani and Atangana [38] solved the Cattaneo-Hristov model numerically using three different methods, while Koca and Atangana [39] proposed analytical and numerical solutions only for the elastic part of the model. Further, Hristov modified the Cattaneo constitutive equation by a Jeffrey's-type kernel with a spatial exponential memory [40] and performed solutions of steady-state problems by applying the Laplace transform. In addition, it was demonstrated that when the non-local constitutive relation changes, then the fading memory approach shows a direct link to diffusion equations in terms of the Atangana-Baleanu derivative (Riemann–Liouville sense) [41]. This approach resulted in the Hristov's constructive diffusion equations. Note that, unlike the diffusion equations with Caputo fractional derivative, with the Caputo-Fabrizio ones cannot be classified as subdiffusive or superdiffusive, as they have a different basis in terms of memory kernel [42].

Analytical solutions for the complete Cattaneo-Hristov heat diffusion problem were presented using the Fourier sine transform combining with Laplace transform by Sene [43] and with Elzaki transform by Singh et. al. [44]. Similarly, Sene [45] addressed the analytical solution of the Hristov diffusion equation by integral transforms. It is worth noting that all mentioned studies on the Cattaneo-Hristov model were considered on a half-line domain under the homogeneous initial temperature.

Taking into account the variety of initial conditions, domains, and heat sources applicable to heat conduction problems, commonly solved separately, there is challenge different types of Cauchy and source problems for the Cattaneo-Hristov heat diffusion equation in a line segment to be addressed and solutions developed. The solution procedure applied here is based on the Fourier method of separation of variables, and the Laplace transform.

The paper is organized as follows. In Section 2, we remind the physical background of addressed problem and give necessary mathematical tools. We obtain the analytical solutions related to the Cauchy and source problems for Cattaneo-Hristov heat diffusion in Section 3 and 4, respectively. Additionally, in both sections the results are graphically illustrated and interpreted. Finally, we give concluding remarks in Section 5.

2. Mathematical tools

Cattaneo's viewpoint for accurately modeling the finite speed of heat diffusion in rigid conductors suggests a following non-local relation [37]

$$q(x,t) = -\int_{-\infty}^{t} R(x,t)\bigtriangledown T(x,t-s)ds, \tag{1}$$

where q denotes heat flux, $R(x,t)$ is the kernel function which changes with respect to the prescribed physical problem, and T represents the temperature function. When, R is chosen as time-dependent Jeffrey's kernel as $R(t) = \exp[-(t-s)/\tau]$ where constant τ is the relaxation time, Cattaneo's heat diffusion equation can be obtained as [36]

$$\frac{\partial T(x,t)}{\partial t} = -\frac{a_2}{\tau}\int_0^t \exp\left(-\frac{t-s}{\tau}\right)\frac{\partial T(x,t)}{\partial x}ds,\ a_2 = \frac{k_2}{\rho C_p}, \tag{2}$$

in which k_2 denotes the elastic conductivity, C_p is the specific heat of particles, and ρ is the density of particles. By considering the conservation of internal energy into the Cattaneo's model (2), the generalized heat diffusion equation in Jeffrey's type is obtained as follows [36]

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{a_2}{\tau}\int_0^t \exp\left(-\frac{t-s}{\tau}\right)\frac{\partial^2 T(x,t)}{\partial x^2}ds,$$
$$a_1 = \frac{k_1}{\rho C_p}, \tag{3}$$

where k_1 is the thermal conductivity. Hristov [36] modified the model on the assumption that the relaxation time $\frac{1}{\tau} = \frac{(1-\alpha)}{\alpha} \in [0,\infty), \alpha \in (0,1)$ as

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)\, {}^{CF}D_t^{\alpha}\left(\frac{\partial^2 T(x,t)}{\partial x^2}\right), \tag{4}$$

in which ${}^{CF}D_t^{\alpha}$ denotes the Caputo-Fabrizio fractional derivative of order α .

Definition 1. [14] Let $f \in H^1(0,t)$ and $0 < \alpha < 1$, then the Caputo-Fabrizio fractional derivative is defined by

$${}^{CF}D_t^{\alpha} f(t) = \frac{\mathscr{M}(\alpha)}{1-\alpha}\int_0^t f'(s)\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)ds, \tag{5}$$

where $\mathscr{M}(\alpha)$ is the normalization function such that $\mathscr{M}(0) = \mathscr{M}(1) = 1$.

Definition 2. [46] The Caputo-Fabrizio fractional integral of order α is defined as

$${}^{CF}I^{\alpha} f(t) = \frac{1-\alpha}{\mathscr{B}(\alpha)}f(t) + \frac{\alpha}{\mathscr{B}(\alpha)}\int_0^t f(s)ds,\ \ t \geqslant 0, \tag{6}$$

where $\mathscr{B}(\alpha) = \frac{2}{(2-\alpha)\mathscr{M}(\alpha)}$.

The relation between Caputo-Fabrizio fractional derivative and integral is given by the relation

$$ {}^{CF}I^{\alpha}\left({}^{CF}D_t^{\alpha}f(t)\right) = f(t) - f(0). \quad (7) $$

The Laplace transform of the Caputo-Fabrizio fractional derivative is

$$ \mathscr{L}\left[{}^{CF}D_t^{(\alpha+n)}f\right](s) = \frac{s^{n+1}\mathscr{L}[f](s) - s^n f(0) - s^{n-1}f'(0) - \cdots - f^{(n)}(0)}{s + \alpha(1-s)}, \quad (8) $$

where $0 \leqslant \alpha \leqslant 1$ and $n = 0, 1, 2, \ldots$

3. The Cauchy Problem

Consider the following Cattaneo-Hristov heat diffusion in a line segment $0 < x < L$

$$ \frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)\,{}^{CF}D_t^{\alpha}\left(\frac{\partial^2 T(x,t)}{\partial x^2}\right), \ 0 < \alpha < 1 \quad (9) $$

under the non-homogeneous initial condition

$$ T(x,0) = T_0(x), \quad (10) $$

and the zero temperature changes at the end of segment

$$ \frac{\partial T(0,t)}{\partial x} = \frac{\partial T(L,t)}{\partial x} = 0. \quad (11) $$

The analytical solution assumes the temperature function constructed in the separable form as:

$$ T(x,t) = X(x)\mathscr{T}(t). \quad (12) $$

Then, substituting Eq. (12) into Eq. (9) the following ordinary differential equations yield

$$ X''(x) + \lambda^2 X = 0, \quad (13) $$

$$ \frac{d\mathscr{T}}{dt} + a_1\lambda^2\mathscr{T}(t) + a_2(1-\alpha)\lambda^2\,{}^{CF}D_t^{\alpha}\mathscr{T}(t) = 0, \quad (14) $$

where λ represents the separation constant.

Similarly, we can obtain the boundary conditions for Eq. (13) by substituting Eq. (11) into Eq. (12) as follows

$$ X'(0) = X'(L) = 0. \quad (15) $$

The solution of Eq. (13) under the conditions (15) allows finding the relevant eigenvalues and eigenfunctions, namely

$$ \lambda_n = \frac{n\pi}{L}, n = 1, 2, \ldots, \quad (16) $$

$$ X_n(x) = \cos\left(\frac{n\pi}{L}x\right). \quad (17) $$

This allows the temperature to be expressed as

$$ T(x,t) = \sum_{n=1}^{\infty}\mathscr{T}_n(t)\cos\left(\frac{n\pi}{L}x\right). \quad (18) $$

Now, substituting Eq. (18) into Eq. (9) and using the orthogonality property of the set $\left\{\cos\left(\frac{n\pi}{L}x\right) : n = 1, 2, \ldots\right\}$, we are able to rewritethe fractional ordinary differential equation of Caputo-Fabrizio type given by Eq. (14) in sense of eigenvalues as

$$ \frac{d\mathscr{T}_n}{dt} + A_n\mathscr{T}_n(t) + B_n\,{}^{CF}D_t^{\alpha}\mathscr{T}_n(t) = 0, n = 1, 2, \ldots, \quad (19) $$

where the abbreviations mean

$$ A_n = a_1\frac{n^2\pi^2}{L^2},\ B_n = a_2(1-\alpha)\frac{n^2\pi^2}{L^2}. $$

The solution of Eq. (19), adopting Eq. (10) into Eq. (18) and using the orthogonality property leads to the initial condition

$$ \mathscr{T}_n(0) = \frac{2}{L}\int_0^L T_0(x)\cos\left(\frac{n\pi}{L}x\right)dx. \quad (20) $$

Now, applying the Laplace transform to the fractional order initial value problem given by Eqs. (19), (20) we get the following algebraic equation in s-domain, namely

$$ \mathscr{L}[\mathscr{T}_n](s) = \frac{as + d}{as^2 + bs + c}\mathscr{T}_n(0), \quad (21) $$

where the coefficients are

$$ a = 1-\alpha,\ b = \alpha + (1-\alpha)A_n + B_n,\ c = \alpha A_n,\ d = \alpha + B_n. \quad (22) $$

The inverse transform results in a solution of the problem (19), (20) expressed as

$$ \mathscr{T}_n(t) = \exp\left(-\frac{b}{2a}t\right)\left[\cosh\left(\frac{\sqrt{b^2-4ac}}{2a}t\right) + \frac{2d-b}{\sqrt{b^2-4ac}}\sinh\left(\frac{\sqrt{b^2-4ac}}{2a}t\right)\right]\mathscr{T}_n(0). \quad (23) $$

The complete analytical solution can be obtained by substituting Eq. (23) into Eq. (18).

As expected, the initial condition $T_0(x)$ is significant in the solution of Eq. (19) and hence in the desired temperature $T(x,t)$ in Eq. (18). To investigate the effect of the initial conditions onto the whole solution, we consider the following cases.

Case 1. Linear initial temperature $T_0(x) = x : \mathscr{T}_n(0)$ initial conditions are calculated from Eq. (20) as

$$ \mathscr{T}_n(0) = \begin{cases} -\frac{4L}{n^2\pi^2}, & for\,odd\,n \\ 0, & else. \end{cases} \quad (24) $$

Case 2. Sinusoidal initial temperature $T_0(x) = \cos\left(\frac{\pi x}{L}\right)$: $\mathscr{T}_n(0)$ conditions are calculated from Eq. (20) as

$$ \mathscr{T}_n(0) = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1. \end{cases} \quad (25) $$

In both cases for the Cauchy problem, the series solution for the temperature distribution can be obtained by substitution of Eqs. (24), (25) into Eq. (23).

4. The source problem

In order to investigate the heat source effect on the Cattaneo-Hristov heat diffusion problem (9)–(11), we are motivated to

$$ \frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)\,{}^{CF}D_t^{\alpha}\left(\frac{\partial^2 T(x,t)}{\partial x^2}\right) + f(x,t), \quad (26) $$

under the homogeneous initial and boundary conditions, respectively, given as

$$T(x,0)=0, \tag{27}$$

$$\frac{\partial T(0,t)}{\partial x}=\frac{\partial T(L,t)}{\partial x}=0. \tag{28}$$

For convenience, we assume the heat source function allows aFourier series expansion similar to the temperature function given by Eq. (18)

$$f(x,t)=\sum_{n=1}^{\infty} f_n(t)\cos\left(\frac{n\pi}{L}x\right). \tag{29}$$

Hence, the substitution of the series (18) and (29) into the Eq. (26) and the use of the orthogonality property of the set $\left\{\cos\left(\frac{n\pi}{L}x\right): n=1,2,\ldots\right\}$ yield the following fractional ordinary differential equation involving the Caputo-Fabrizio operator:

$$\frac{d\mathcal{T}_n}{dt}+A_n\mathcal{T}_n(t)+B_n\,{}^{CF}D_t^{\alpha}\mathcal{T}_n(t)+f_n(t)=0,\ n=1,2,\ldots \tag{30}$$

By choosing the heat source $f(x,t)$ arbitrarily and adopting this into the Eq. (29)with the orthogonality property, we have

$$f_n(t)=\frac{2}{L}\int_0^L f(x,t)\cos\left(\frac{n\pi}{L}x\right)dx. \tag{31}$$

To arrive at the solution of Eq. (30) by the Laplace transform, the functions $f_n(t)$ have to be calculated by considering two types of heat sources considered next.

Case 3. Let us take a linear time varying heat source which has an increasing penetration

$$f(x,t)=tx+t, \tag{32}$$

and so $f_n(t)$ can be easily calculated from Eq. (31)

$$f_n(t)=\begin{cases}-\frac{4L}{n^2\pi^2}t, & for\, odd\, n,\\ 0, & else.\end{cases} \tag{33}$$

Therefore, the substitution of $f_n(t)$ into Eq. (30) and the application of Laplace transform yield the algebraic equation in s-domain

$$\mathcal{L}[\mathcal{T}_n](s)=\frac{4L}{n^2\pi^2}\frac{as+\alpha}{as^4+bs^3+cs^2}, \tag{34}$$

where a,b and c are given in Eq. (22). Inverting the Laplace transform, we get the solution of Eq. (30)as

$$\begin{aligned}\mathcal{T}_n(t)= \frac{4L}{n^2\pi^2}\Bigg\{&\exp\left(-\frac{b}{2a}t\right)\left[\frac{-ac(2\alpha+b)+\alpha b^2}{c^2\sqrt{b^2-4ac}}\sinh\left(\frac{\sqrt{b^2-4ac}}{2a}t\right)\right.\\ &\left.+\frac{\alpha b-ac}{c^2}\cosh\left(\frac{\sqrt{b^2-4ac}}{2a}t\right)\right]+\frac{ac+\alpha ct-\alpha b}{c^2}\Bigg\}.\end{aligned} \tag{35}$$

Thus, we obtain the complete series solution for the temperature function $T(x,t)$ by substituting the result (35) into Eq. (18).

Case 4. Consider a time-dependent exponentially decaying heat source

$$f(x,t)=x\exp(-t),$$

Further, $f_n(t)$ can be obtained as

$$f_n(t)=\begin{cases}-\frac{4L}{n^2\pi^2}\exp(-t), & for\, odd\, n,\\ 0, & else.\end{cases} \tag{37}$$

Now, substituting the functions $f_n(t)$ into Eq. (30) and applying the Laplace transform the result is

$$\mathcal{L}[\mathcal{T}_n](s)=\frac{4L}{n^2\pi^2}\frac{as+\alpha}{(as^2+bs+c)(s+1)}. \tag{38}$$

The coefficients a,b,c are given by Eq. (22). The inverse Laplace transform gives $\mathcal{T}_n(t)$

$$\begin{aligned}\mathcal{T}_n(t)= \frac{4L}{n^2\pi^2}\Bigg\{&\exp\left(-\frac{b}{2a}t\right)\left[\frac{2a(\alpha+c)-b(a+\alpha)}{\sqrt{b^2-4ac}}\sinh\left(\frac{\sqrt{b^2-4ac}}{2a}t\right)\right.\\ &\left.+\frac{a-\alpha}{a-b+c}\cosh\left(\frac{\sqrt{b^2-4ac}}{2a}t\right)\right]-\frac{a-\alpha}{a-b+c}\exp(-t)\Bigg\}.\end{aligned} \tag{39}$$

Finally, thesubstitution Eq. (39),taking into account Eq. (18), allows the complete solution to be obtained.

The results arising from the Cauchy and source problems are displayed in Figs. 1–5. To show the convergence of the solution series, we consider exemplary Case 1 and plot $T(x,t)$ function with respect to increasing values of N. Thus, Fig. 1a exhibits that a few numbers of series terms are suffi-

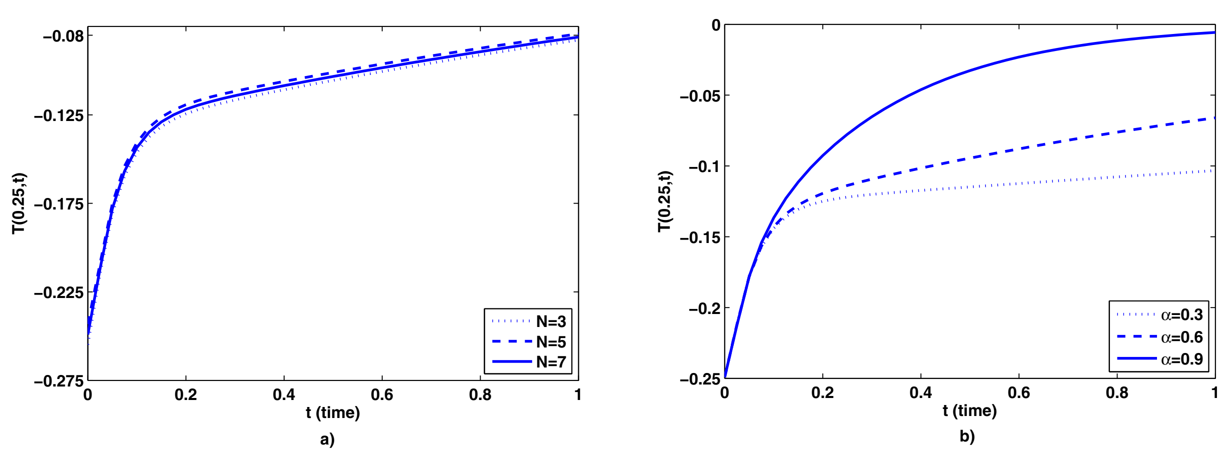

Fig. 1 (a) Contribution of the series terms for $\alpha = 0.5$. (b) Effect of fractional orders α on the temperature.

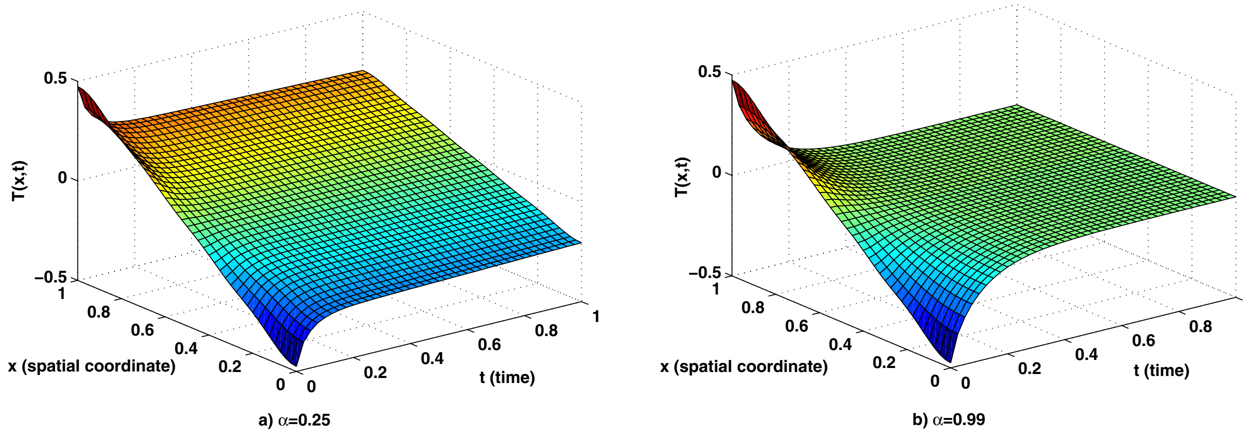

Fig. 2 Solution of the Cauchy problem for Case 1.

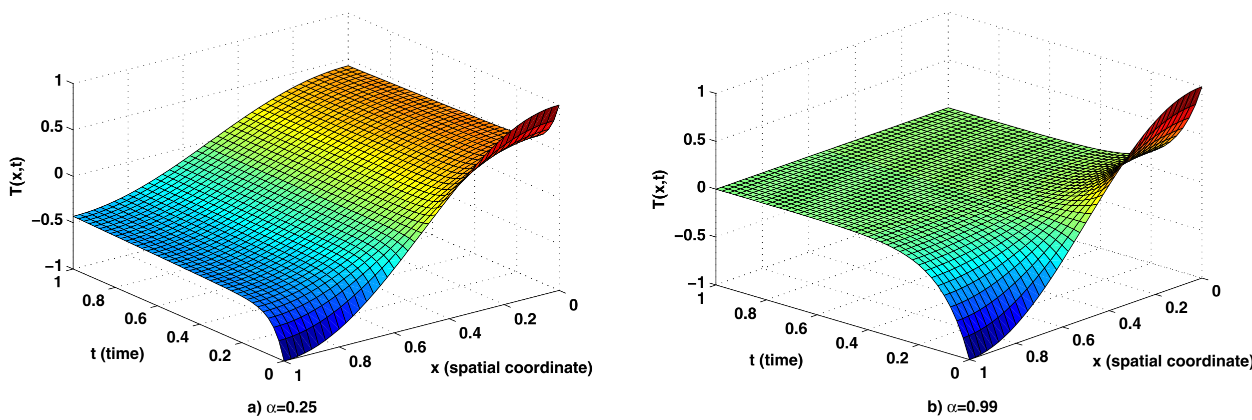

Fig. 3 Solution of the Cauchy problem for Case 2.

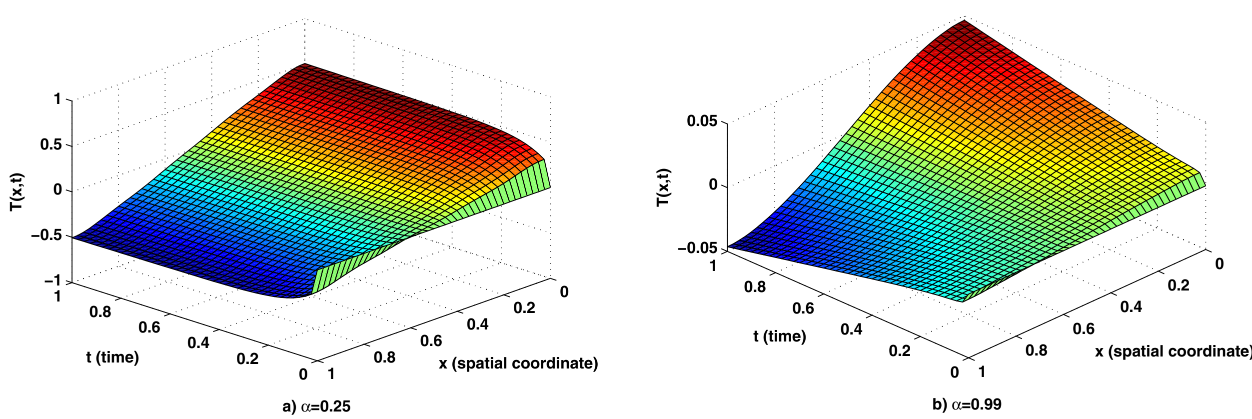

Fig. 4 Solution of the source problem for Case 3.

cient to calculate the solution. For this reason, all the remaining figures devoted to displaying the solutions of Cauchy and source problems for varying α values are drawn for $N = 7$. What makes the Cattaneo-Hristov heat diffusion equation (with a non-singular kernel) different from the Caputo type fractional heat equation (with a singular kernel), not only the type of memory function, but the fact that the fractional order α participate a function as multiplier of the fading memory term in the model (see the analysis in [36]). For this purpose, the effect of fractional order α is illustrated in Fig. 1b which implies that the solution of the Cattaneo-Hristov diffusion equation evolves to the classical diffusion equation when $\alpha \to 1$. Also, it is clear that the relaxation effect in the temperature distribution decreases when the $\alpha \to 0$ since the multiplier function of the Caputo-Fabrizio term is approaching zero.

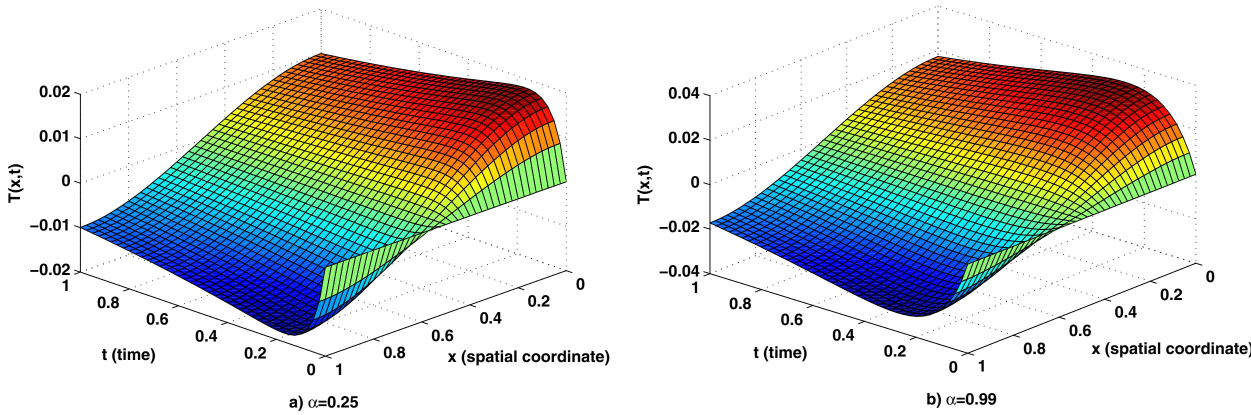

Fig. 5 Solution of the source problem for Case 4.

5. Concluding remarks

In this article, the Cauchy and source problems of the complete Cattaneo-Hristov heat diffusion acting in a line segment have been firstly investigated and solved analytically. For this aim, the eigenfunctions have been determined by applying the Fourier method and thus the main problem has been reduced into the set of time-fractional ordinary differential equations. Then, the Laplace transform method has been used to arrive at the analytical solution. In both of the Cauchy and source problems, the effect of fractional order has been demonstrated by giving illustrative figures. Utilizing the convergence of the series shown graphically, all the figures have been plotted by truncated solution series. Remarkably, the fractional parameter has a critical effect on the relaxation term related to the fading memory concept applied to the model build-up. It has been observed that the relaxation of temperature increases while the values of fractional order approaches to one. Since keeping the temperature distribution caused by the Cattaneo-Hristov model at the desired value can be a remarkable problem, the heat source can be considered as a control function but development of source control problems draws future work.

Declaration of Competing Interest

None.

Acknowledgement

The authors would like to sincerely thank Prof. Dr. Jordan Hristov for his valuable criticism and proofreading of the manuscript.

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