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A new solution to the Rhoades' open problem with an application

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Abstract. We give a new solution to the Rhoades' open problem on the discontinuity at fixed point via the notion of an S-metric. To do this, we develop a new technique by means of the notion of a Zamfirescu mapping. Also, we consider a recent problem called the "fixed-circle problem" and propose a new solution to this problem as an application of our technique.

1 Introduction and preliminaries

Fixed-point theory has been extensively studied by various aspects. One of these is the discontinuity problem at fixed points (see [1, 2, 3, 4, 5, 6, 24, 25, 26, 27] for some examples). Discontinuous functions have been widely appeared in many areas of science such as neural networks (for example, see [7, 12, 13, 14]). In this paper, we give a new solution to the Rhoades' open problem (see [28] for more details) on the discontinuity at fixed point in the setting of an S-metric space which is a recently introduced generalization of a metric space. S-metric spaces were introduced in [29] by Sedgi et al., as follows:

Definition 1 [29] *Let* X *be a nonempty set and* $\mathcal{S} : X \times X \times X \to [0, \infty)$ *a function satisfying the following conditions for all* $x, y, z, a \in X$ *:*

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S1) $\mathcal{S}(x,y,z)=0$ *if and only if* $x=y=z$,
S2) $\mathcal{S}(x,y,z)\leq\mathcal{S}(x,x,a)+\mathcal{S}(y,y,a)+\mathcal{S}(z,z,a)$.
Then $\mathcal{S}$ *is called an* S*-metric on* X *and the pair* $(X,\mathcal{S})$ *is called an* S*-metric space.*

Relationships between a metric and an S-metric were given as follows:

Lemma 1 [9] *Let* (X,d) *be a metric space. Then the following properties are satisfied:*

1. $\mathcal{S}_d(x,y,z)=d(x,z)+d(y,z)$ *for all* $x,y,z\in X$ *is an* S*-metric on* X.
2. $x_n\to x$ *in* (X,d) *if and only if* $x_n\to x$ *in* $(X,\mathcal{S}_d)$.
3. $\{x_n\}$ *is Cauchy in* (X,d) *if and only if* $\{x_n\}$ *is Cauchy in* $(X,\mathcal{S}_d)$.
4. (X,d) *is complete if and only if* $(X,\mathcal{S}_d)$ *is complete.*

The metric $\mathcal{S}_d$ was called as the S-metric generated by d [17]. Some examples of an S-metric which is not generated by any metric are known (see [9, 17] for more details).

Furthermore, Gupta claimed that every S-metric on X defines a metric d_S on X as follows:

$$d_S(x,y)=\mathcal{S}(x,x,y)+\mathcal{S}(y,y,x), \tag{1}$$

for all $x,y\in X$ [8]. However, since the triangle inequality does not satisfied for all elements of X everywhen, the function $d_S(x,y)$ defined in (1) does not always define a metric (see [17]).

In the following, we see an example of an S-metric which is not generated by any metric.

Example 1 [17] *Let* $X=\mathbb{R}$ *and the function* $\mathcal{S}:X\times X\times X\to[0,\infty)$ *be defined as*

$$\mathcal{S}(x,y,z)=|x-z|+|x+z-2y|,$$

for all $x,y,z\in\mathbb{R}$*. Then* $\mathcal{S}$ *is an* S*-metric which is not generated by any metric and the pair* $(X,\mathcal{S})$ *is an* S*-metric space.*

The following lemma will be used in the next sections.

Lemma 2 [29] *Let* $(X,\mathcal{S})$ *be an* S*-metric space. Then we have*

$$\mathcal{S}(x,x,y)=\mathcal{S}(y,y,x).$$

In this paper, our aim is to obtain a new solution to the Rhoades' open problem on the existence of a contractive condition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point. To do this, we inspire of a result of Zamfirescu given in [33].

On the other hand, a recent aspect to the fixed point theory is to consider geometric properties of the set $\mathrm{Fix}(T)$, the fixed point set of the self-mapping T. Fixed-circle problem (resp. fixed-disc problem) have been studied in this context (see [6, 18, 19, 20, 21, 22, 23, 26, 27, 30, 31]). As an application, we present a new solution to these problems. We give necessary examples to support our theoretical results.

2 Main results

From now on, we assume that $(X, \mathcal{S})$ is an S-metric space and $T : X \to X$ is a self-mapping. In this section, we use the numbers defined as

$$M_z(x, y) = \max\left\{ad(x, y), \frac{b}{2}[d(x, Tx) + d(y, Ty)], \frac{c}{2}[d(x, Ty) + d(y, Tx)]\right\}$$

and

$$M_z^S(x, y) = \max\left\{\begin{array}{c} a\mathcal{S}(x, x, y), \frac{b}{2}[\mathcal{S}(x, x, Tx) + \mathcal{S}(y, y, Ty)], \\ \frac{c}{2}[\mathcal{S}(x, x, Ty) + \mathcal{S}(y, y, Tx)] \end{array}\right\},$$

where $a, b \in [0, 1)$ and $c \in \left[0, \frac{1}{2}\right]$.

We give the following theorem as a new solution to the Rhoades' open problem.

Theorem 1 *Let $(X, \mathcal{S})$ be a complete S-metric space and T a self-mapping on X satisfying the conditions*

i) *There exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and*

$$\mathcal{S}(Tx, Tx, Ty) \leq \phi\left(M_z^S(x, y)\right),$$

for all $x, y \in X$,

ii) *There exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M_z^S(x, y) < \varepsilon + \delta$ implies $\mathcal{S}(Tx, Tx, Ty) \leq \varepsilon$ for a given $\varepsilon > 0$.*

Then T has a unique fixed point $u \in X$. Also, T is discontinuous at u if and only if $\lim\limits_{x \to u} M_z^S(x, u) \neq 0$.

Proof. At first, we define the number

$$\xi = \max\left\{a, \frac{2}{2-b}, \frac{c}{2-2c}\right\}.$$

Clearly, we have $\xi < 1$.

By the condition (i), there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and

$$\mathcal{S}(Tx, Tx, Ty) \leq \phi\left(M_z^S(x, y)\right),$$

for all $x, y \in X$. Using the properties of ϕ, we obtain

$$\mathcal{S}(Tx, Tx, Ty) < M_z^S(x, y), \tag{2}$$

whenever $M_z^S(x, y) > 0$.

Let us consider any $x_0 \in X$ with $x_0 \neq Tx_0$ and define a sequence $\{x_n\}$ as $x_{n+1} = Tx_n = T^n x_0$ for all $n = 0, 1, 2, 3, ...$ Using the condition (i) and the inequality (2), we get

$$\begin{aligned}
\mathcal{S}(x_n, x_n, x_{n+1}) &= \mathcal{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq \phi\left(M_z^S(x_{n-1}, x_n)\right) \\
&< M_z^S(x_{n-1}, x_n) \\
&= \max\left\{\begin{array}{c} a\mathcal{S}(x_{n-1}, x_{n-1}, x_n), \\ \frac{b}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, Tx_{n-1}) + \mathcal{S}(x_n, x_n, Tx_n)\right], \\ \frac{c}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, Tx_n) + \mathcal{S}(x_n, x_n, Tx_{n-1})\right] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} a\mathcal{S}(x_{n-1}, x_{n-1}, x_n), \\ \frac{b}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})\right], \\ \frac{c}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1}) + \mathcal{S}(x_n, x_n, x_n)\right] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} a\mathcal{S}(x_{n-1}, x_{n-1}, x_n), \\ \frac{b}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})\right], \\ \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1}) \end{array}\right\}.
\end{aligned} \tag{3}$$

Assume that $M_z^S(x_{n-1}, x_n) = a\mathcal{S}(x_{n-1}, x_{n-1}, x_n)$. Then using the inequality (3), we have

$$\mathcal{S}(x_n, x_n, x_{n+1}) < a\mathcal{S}(x_{n-1}, x_{n-1}, x_n) \leq \xi\mathcal{S}(x_{n-1}, x_{n-1}, x_n) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n)$$

and so

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n). \tag{4}$$

Let $M_z^S(x_{n-1}, x_n) = \frac{b}{2}[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})]$. Again using the inequality (3), we get

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{b}{2}[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})],$$

which implies

$$\left(1 - \frac{b}{2}\right)\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{b}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_n)$$

and hence

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{b}{2-b}\mathcal{S}(x_{n-1}, x_{n-1}, x_n) \le \xi\mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

This yields

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n). \tag{5}$$

Suppose that $M_z^S(x_{n-1}, x_n) = \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1})$. Then using the inequality (3), Lemma 2 and the condition (S2), we obtain

$$\begin{aligned}
\mathcal{S}(x_n, x_n, x_{n+1}) &< \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1}) = \frac{c}{2}\mathcal{S}(x_{n+1}, x_{n+1}, x_{n-1}) \\
&\le \frac{c}{2}[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + 2\mathcal{S}(x_{n+1}, x_{n+1}, x_n)] \\
&= \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + c\mathcal{S}(x_{n+1}, x_{n+1}, x_n) \\
&= \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + c\mathcal{S}(x_n, x_n, x_{n+1}),
\end{aligned}$$

which implies

$$(1 - c)\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

Considering this, we find

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{c}{2(1-c)}\mathcal{S}(x_{n-1}, x_{n-1}, x_n) \le \xi\mathcal{S}(x_{n-1}, x_{n-1}, x_n)$$

and so

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n). \tag{6}$$

If we set $\alpha_n = \mathcal{S}(x_n, x_n, x_{n+1})$, then by the inequalities (4), (5) and (6), we find

$$\alpha_n < \alpha_{n-1}, \tag{7}$$

that is, α_n is a strictly decreasing sequence of positive real numbers whence the sequence α_n tends to a limit $\alpha \geq 0$.

Assume that $\alpha > 0$. There exists a positive integer $k \in \mathbb{N}$ such that $n \geq k$ implies

$$\alpha < \alpha_n < \alpha + \delta(\alpha). \tag{8}$$

Using the condition (ii) and the inequality (7), we get

$$\mathcal{S}\left(Tx_{n-1}, Tx_{n-1}, Tx_n\right) = \mathcal{S}\left(x_n, x_n, x_{n+1}\right) = \alpha_n < \alpha, \tag{9}$$

for $n \geq k$. Then the inequality (9) contradicts to the inequality (8). Therefore, it should be $\alpha = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. Let us fix an $\varepsilon > 0$. Without loss of generality, we suppose that $\delta\left(\varepsilon\right) < \varepsilon$. There exists $k \in \mathbb{N}$ such that

$$\mathcal{S}\left(x_n, x_n, x_{n+1}\right) = \alpha_n < \frac{\delta}{4},$$

for $n \geq k$ since $\alpha_n \to 0$. Using the mathematical induction and the Jachymski's technique (see [10, 11] for more details), we show

$$\mathcal{S}\left(x_k, x_k, x_{k+n}\right) < \varepsilon + \frac{\delta}{2}, \tag{10}$$

for any $n \in \mathbb{N}$. At first, the inequality (10) holds for $n = 1$ since

$$\mathcal{S}\left(x_k, x_k, x_{k+1}\right) = \alpha_k < \frac{\delta}{4} < \varepsilon + \frac{\delta}{2}.$$

Assume that the inequality (10) holds for some n. We show that the inequality (10) holds for $n+1$. By the condition (S2), we get

$$\mathcal{S}\left(x_k, x_k, x_{k+n+1}\right) \leq 2\mathcal{S}\left(x_k, x_k, x_{k+1}\right) + \mathcal{S}\left(x_{k+n+1}, x_{k+n+1}, x_{k+1}\right).$$

From Lemma 2, we have

$$\mathcal{S}\left(x_{k+n+1}, x_{k+n+1}, x_{k+1}\right) = \mathcal{S}\left(x_{k+1}, x_{k+1}, x_{k+n+1}\right)$$

and so it suffices to prove

$$\mathcal{S}\left(x_{k+1}, x_{k+1}, x_{k+n+1}\right) \leq \varepsilon.$$

To do this, we show

$$M_z^S(x_k, x_{k+n}) \leq \varepsilon + \delta.$$

Then we find

$$a\mathcal{S}(x_k, x_k, x_{k+n}) < \mathcal{S}(x_k, x_k, x_{k+n}) < \varepsilon + \frac{\delta}{2},$$

$$\begin{aligned}
&\frac{b}{2}\left[\mathcal{S}(x_k, x_k, x_{k+1}) + \mathcal{S}(x_{k+n}, x_{k+n}, x_{k+n+1})\right] \\
<&\mathcal{S}(x_k, x_k, x_{k+1}) + \mathcal{S}(x_{k+n}, x_{k+n}, x_{k+n+1}) \\
<&\frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{c}{2}\left[\mathcal{S}(x_k, x_k, x_{k+n+1}) + \mathcal{S}(x_{k+n}, x_{k+n}, x_{k+1})\right] \\
&\leq \frac{c}{2}\left[4\mathcal{S}(x_k, x_k, x_{k+1}) + \mathcal{S}(x_{k+1}, x_{k+1}, x_{k+1+n}) + \mathcal{S}(x_k, x_k, x_{k+n})\right] \\
&= c\left[2\mathcal{S}(x_k, x_k, x_{k+1}) + \frac{\mathcal{S}(x_{k+1}, x_{k+1}, x_{k+1+n})}{2} + \frac{\mathcal{S}(x_k, x_k, x_{k+n})}{2}\right] \qquad (11) \\
&< c\left[\frac{\delta}{2} + \varepsilon + \frac{\delta}{2}\right] < \varepsilon + \delta.
\end{aligned}$$

Using the definition of $M_z^S(x_k, x_{k+n})$, the condition (ii) and the inequalities (10) and (11), we obtain

$$M_z^S(x_k, x_{k+n}) \leq \varepsilon + \delta$$

and so

$$\mathcal{S}\left(x_{k+1}, x_{k+1}, x_{k+n+1}\right) \leq \varepsilon.$$

Hence we get

$$\mathcal{S}(x_k, x_k, x_{k+n+1}) < \varepsilon + \frac{\delta}{2},$$

whence $\{x_n\}$ is Cauchy. From the completeness hypothesis, there exists a point $u \in X$ such that $x_n \to u$ for $n \to \infty$. Also we get

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u.$$

Now we prove that u is a fixed point of T. On the contrary, assume that u is not a fixed point of T. Then using the condition (i) and the property of ϕ, we obtain

$$\mathcal{S}(Tu, Tu, Tx_n) \quad \leq \quad \phi(M_z^S(u, x_n)) < M_z^S(u, x_n)$$

$$= \max\left\{ \begin{array}{c} a\mathcal{S}(u,u,x_n), \frac{b}{2}\left[\mathcal{S}(u,u,Tu)+\mathcal{S}(x_n,x_n,Tx_n)\right], \\ \frac{c}{2}\left[\mathcal{S}(u,u,Tx_n)+\mathcal{S}(x_n,x_n,Tu)\right] \end{array} \right\}.$$

Using Lemma 2 and taking limit for $n \to \infty$, we find

$$\mathcal{S}(Tu,Tu,u) < \max\left\{\frac{b}{2}\mathcal{S}(u,u,Tu), \frac{c}{2}\mathcal{S}(u,u,Tu)\right\} < \mathcal{S}(Tu,Tu,u),$$

a contradiction. It should be $Tu = u$. We show that u is the unique fixed point of T. Let v be another fixed point of T such that $u \neq v$. From the condition (i) and Lemma 2, we have

$$\begin{aligned} \mathcal{S}(Tu,Tu,Tv) &= \mathcal{S}(u,u,v) \leq \phi(M_z^S(u,v)) < M_z^S(u,v) \\ &= \max\left\{ \begin{array}{c} a\mathcal{S}(u,u,v), \frac{b}{2}\left[\mathcal{S}(u,u,Tu)+\mathcal{S}(v,v,Tv)\right], \\ \frac{c}{2}\left[\mathcal{S}(u,u,Tv)+\mathcal{S}(v,v,Tu)\right] \end{array} \right\} \\ &= \max\{a\mathcal{S}(u,u,v), c\mathcal{S}(u,u,v)\} < \mathcal{S}(u,u,v), \end{aligned}$$

a contradiction. So it should be $u = v$. Therefore, T has a unique fixed point $u \in X$.

Finally, we prove that T is discontinuous at u if and only if $\lim\limits_{x \to u} M_z^S(x,u) \neq 0$. To do this, we can easily show that T is continuous at u if and only if $\lim\limits_{x \to u} M_z^S(x,u) = 0$. Suppose that T is continuous at the fixed point u and $x_n \to u$. Hence we get $Tx_n \to Tu = u$ and using the condition (S2), we find

$$\mathcal{S}(x_n,x_n,Tx_n) \leq 2\mathcal{S}(x_n,x_n,u) + \mathcal{S}(Tx_n,Tx_n,u) \to 0,$$

as $x_n \to u$. So we get $\lim\limits_{x_n \to u} M_z^S(x_n,u) = 0$. On the other hand, assume $\lim\limits_{x_n \to u} M_z^S(x_n,u) = 0$. Then we obtain $\mathcal{S}(x_n,x_n,Tx_n) \to 0$ as $x_n \to u$, which implies $Tx_n \to Tu = u$. Consequently, T is continuous at u. □

We give an example.

Example 2 *Let* $X = \{0,2,4,8\}$ *and* $(X,\mathcal{S})$ *be the* S*-metric space defined as in Example 1. Let us define the self-mapping* $T : X \to X$ *as*

$$Tx = \left\{ \begin{array}{ccc} 4 & ; & x \leq 4 \\ 2 & ; & x > 4 \end{array} \right.,$$

for all $x \in \{0,2,4,8\}$*. Then* T *satisfies the conditions of Theorem 1 with* $a = \frac{3}{4}, b = c = 0$ *and has a unique fixed point* $x = 4$*. Indeed, we get the following table :*

$$\begin{array}{lcl} \mathcal{S}(Tx,Tx,Ty) = 0 & \text{and} & 3 \leq M_z^S(x,y) \leq 6 \text{ when } x,y \leq 4 \\ \mathcal{S}(Tx,Tx,Ty) = 4 & \text{and} & 6 \leq M_z^S(x,y) \leq 12 \text{ when } x \leq 4, y > 4 \\ \mathcal{S}(Tx,Tx,Ty) = 4 & \text{and} & 6 \leq M_z^S(x,y) \leq 12 \text{ when } x > 4, y \leq 4 \end{array}.$$

Hence T *satisfies the conditions of Theorem 1 with*

$$\phi(t) = \begin{cases} 5 & ; \quad t \geq 6 \\ \frac{t}{2} & ; \quad t < 6 \end{cases}$$

and

$$\delta(\varepsilon) = \begin{cases} 6 & ; \quad \varepsilon \geq 3 \\ 6 - \varepsilon & ; \quad \varepsilon < 3 \end{cases} .$$

Now we give the following results as the consequences of Theorem 1.

Corollary 1 *Let* $(X, \mathcal{S})$ *be a complete* S*-metric space and* T *a self-mapping on* X *satisfying the conditions*

i) $\mathcal{S}(Tx, Tx, Ty) < M_z^S(x, y)$ *for any* $x, y \in X$ *with* $M_z^S(x, y) > 0$,

ii) *There exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\varepsilon < M_z^S(x, y) < \varepsilon + \delta$ *implies* $\mathcal{S}(Tx, Tx, Ty) \leq \varepsilon$ *for a given* $\varepsilon > 0$.

Then T *has a unique fixed point* $u \in X$. *Also,* T *is discontinuous at* u *if and only if* $\lim_{x \to u} M_z^S(x, u) \neq 0$.

Corollary 2 *Let* $(X, \mathcal{S})$ *be a complete* S*-metric space and* T *a self-mapping on* X *satisfying the conditions*

i) *There exists a function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\phi(\mathcal{S}(x, x, y)) < \mathcal{S}(x, x, y)$ *and* $\mathcal{S}(Tx, Tx, Ty) \leq \phi(\mathcal{S}(x, x, y))$,

ii) *There exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\varepsilon < t < \varepsilon + \delta$ *implies* $\phi(t) \leq \varepsilon$ *for any* $t > 0$ *and a given* $\varepsilon > 0$.

Then T *has a unique fixed point* $u \in X$.

The following theorem shows that the power contraction of the type $M_z^S(x, y)$ allows also the possibility of discontinuity at the fixed point.

Theorem 2 *Let* $(X, \mathcal{S})$ *be a complete* S*-metric space and* T *a self-mapping on* X *satisfying the conditions*

i) *There exists a function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\phi(t) < t$ *for each* $t > 0$ *and*

$$\mathcal{S}(T^m x, T^m x, T^m y) \leq \phi\left(M_z^{S^*}(x, y)\right),$$

where

$$M_z^{S^*}(x, y) = \max \left\{ \begin{array}{c} a\mathcal{S}(x, x, y), \frac{b}{2}[\mathcal{S}(x, x, T^m x) + \mathcal{S}(y, y, T^m y)], \\ \frac{c}{2}[\mathcal{S}(x, x, T^m y) + \mathcal{S}(y, y, T^m x)] \end{array} \right\}$$

for all $x, y \in X$,

ii) *There exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\varepsilon < M_z^{S^*}(x, y) < \varepsilon + \delta$ *implies* $\mathcal{S}(T^m x, T^m x, T^m y) \leq \varepsilon$ *for a given* $\varepsilon > 0$.

Then T *has a unique fixed point* $u \in X$*. Also,* T *is discontinuous at* u *if and only if* $\lim\limits_{x \to u} M_z^{S^*}(x, u) \neq 0$.

Proof. By Theorem 1, the function T^m has a unique fixed point u. Hence we have

$$Tu = TT^m u = T^m Tu$$

and so Tu is another fixed point of T^m. From the uniqueness of the fixed point, we obtain $Tu = u$, that is, T has a unique fixed point u. □

We note that if the S-metric $\mathcal{S}$ generates a metric d then we consider Theorem 1 on the corresponding metric space as follows:

Theorem 3 *Let* (X, d) *be a complete metric space and* T *a self-mapping on* X *satisfying the conditions*

i) *There exists a function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\phi(t) < t$ *for each* $t > 0$ *and*

$$d(Tx, Ty) \leq \phi\left(M_z(x, y)\right),$$

for all $x, y \in X$,

ii) *There exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\varepsilon < M_z(x, y) < \varepsilon + \delta$ *implies* $d(Tx, Ty) \leq \varepsilon$ *for a given* $\varepsilon > 0$.

Then T *has a unique fixed point* $u \in X$*. Also,* T *is discontinuous at* u *if and only if* $\lim\limits_{x \to u} M_z(x, u) \neq 0$.

Proof. By the similar arguments used in the proof of Theorem 1, the proof can be easily obtained. □

3 An application to the fixed-circle problem

In this section, we investigate new solutions to the fixed-circle problem raised by Özgür and Taş in [19] related to the geometric properties of the set $Fix(T)$ for a self mapping T on an S-metric space $(X, \mathcal{S})$. Some fixed-circle or fixed-disc results, as the direct solutions of this problem, have been studied using various methods on a metric space or some generalized metric spaces (see [15, 16, 20, 21, 22, 23, 26, 27, 30, 31, 32]).

Now we recall the notions of a circle and a disc on an S-metric space as follows:

$$C^S_{x_0, r} = \{x \in X : \mathcal{S}(x, x, x_0) = r\}$$

and

$$D^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) \leq r\},$$

where $r \in [0, \infty)$ [20, 29].

If $Tx = x$ for all $x \in C^S_{x_0,r}$ (resp. $x \in D^S_{x_0,r}$) then the circle $C^S_{x_0,r}$ (resp. the disc $D^S_{x_0,r}$) is called as the fixed circle (resp. fixed disc) of T (for more details see [15, 20]).

We begin with the following definition.

Definition 2 *A self-mapping T is called an $\mathcal{S}$-Zamfirescu type x_0-mapping if there exist $x_0 \in X$ and $a, b \in [0, 1)$ such that*

$$\mathcal{S}(Tx, Tx, x) > 0 \Longrightarrow \mathcal{S}(Tx, Tx, x) \leq \max \left\{ \begin{array}{c} a\mathcal{S}(x, x, x_0), \\ \frac{b}{2} \left[\mathcal{S}(Tx_0, Tx_0, x) + \mathcal{S}(Tx, Tx, x_0)\right] \end{array} \right\},$$

for all $x \in X$.

We define the following number:

$$\rho := \inf \{\mathcal{S}(Tx, Tx, x) : Tx \neq x, x \in X\}. \tag{12}$$

Now we prove that the set $\mathrm{Fix}(T)$ contains a circle (resp. a disc) by means of the number ρ.

Theorem 4 *If T is an $\mathcal{S}$-Zamfirescu type x_0-mapping with $x_0 \in X$ and the condition*

$$\mathcal{S}(Tx, Tx, x_0) \leq \rho$$

holds for each $x \in C^S_{x_0,\rho}$ then $C^S_{x_0,\rho}$ is a fixed circle of T, that is, $C^S_{x_0,\rho} \subset \mathrm{Fix}(T)$.

Proof. At first, we show that x_0 is a fixed point of T. On the contrary, let $Tx_0 \neq x_0$. Then we have $\mathcal{S}(Tx_0, Tx_0, x_0) > 0$. By the definition of an $\mathcal{S}$-Zamfirescu type x_0-mapping and the condition $(S1)$, we obtain

$$\begin{aligned} \mathcal{S}(Tx_0, Tx_0, x_0) &\leq \max\left\{a\mathcal{S}(x_0, x_0, x_0), \frac{b}{2}\left[\mathcal{S}(Tx_0, Tx_0, x_0) + \mathcal{S}(Tx_0, Tx_0, x_0)\right]\right\} \\ &= b\mathcal{S}(Tx_0, Tx_0, x_0), \end{aligned}$$

a contradiction because of $b \in [0, 1)$. This shows that $Tx_0 = x_0$.

We have two cases:

Case 1: If $\rho = 0$, then we get $C^S_{x_0,\rho} = \{x_0\}$ and clearly this is a fixed circle of T.

Case 2: Let $\rho > 0$ and $x \in C^S_{x_0,\rho}$ be any point such that $Tx \neq x$. Then we have

$$\mathcal{S}(Tx, Tx, x) > 0$$

and using the hypothesis we obtain,

$$\begin{aligned}\mathcal{S}(Tx, Tx, x) &\leq \max\left\{a\mathcal{S}(x, x, x_0), \frac{b}{2}\left[\mathcal{S}(Tx_0, Tx_0, x) + \mathcal{S}(Tx, Tx, x_0)\right]\right\} \\ &\leq \max\{a\rho, b\rho\} < \rho,\end{aligned}$$

which is a contradiction with the definition of ρ. Hence it should be $Tx = x$ whence $C^S_{x_0,\rho}$ is a fixed circle of T. □

Corollary 3 *If T is an $\mathcal{S}$-Zamfirescu type x_0-mapping with $x_0 \in X$ and the condition*

$$\mathcal{S}(Tx, Tx, x_0) \leq \rho$$

holds for each $x \in D^S_{x_0,\rho}$ then $D^S_{x_0,\rho}$ is a fixed disc of T, that is, $D^S_{x_0,\rho} \subset Fix(T)$.

Now we give an illustrative example to show the effectiveness of our results.

Example 3 *Let $X = \mathbb{R}$ and $(X, \mathcal{S})$ be the S-metric space defined as in Example 1. Let us define the self-mapping $T : X \to X$ as*

$$Tx = \begin{cases} x & ; \ x \in [-3, 3] \\ x + 1 & ; \ x \notin [-3, 3] \end{cases},$$

for all $x \in \mathbb{R}$. Then T is an $\mathcal{S}$-Zamfirescu type x_0-mapping with $x_0 = 0$, $a = \frac{1}{2}$ and $b = 0$. Indeed, we get

$$\mathcal{S}(Tx, Tx, x) = 2\,|Tx - x| = 2 > 0,$$

for all $x \in (-\infty, -3) \cup (3, \infty)$. So we obtain

$$\begin{aligned}\mathcal{S}(Tx, Tx, x) &= 2 \leq \max\left\{aS\left(x, x, 0\right), \frac{b}{2}\left[\mathcal{S}(0, 0, x) + \mathcal{S}(x + 1, x + 1, 0)\right]\right\} \\ &= \frac{1}{2}.2\,|x|\,.\end{aligned}$$

Also we have

$$\rho = \inf\{\mathcal{S}(Tx, Tx, x) : Tx \neq x, x \in X\} = 2$$

and

$$\mathcal{S}(Tx, Tx, 0) = \mathcal{S}(x, x, 0) \leq 2,$$

for all $x \in C^S_{0,2} = \{x : \mathcal{S}(x, x, 0) = 2\} = \{x : 2\,|x| = 2\} = \{x : |x| = 1\}$. Consequently, T fixes the circle $C^S_{0,2}$ and the disc $D^S_{0,2}$.

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