On Interpolating Sesqui-Harmonic Legendre Curves in Sasakian Space Forms

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Abstract. We consider interpolating sesqui-harmonic Legendre curves in Sasakian space forms. We find the necessary and sufficient conditions for Legendre curves in Sasakian space forms to be interpolating sesqui-harmonic. Finally, we obtain an example for an interpolating sesqui-harmonic Legendre curve in a Sasakian space form.

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1. **Introduction**

A map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds is called a *harmonic map* and a *biharmonic map,* respectively if it is a critical point of the $E(\varphi)$ and $E_2(\varphi)$

$$E(\varphi) = \int_\Omega \|d\varphi\|^2 \, d\nu_g,$$

$$E_2(\varphi) = \int_\Omega \|\tau(\varphi)\|^2 \, d\nu_g,$$

where Ω is a compact domain of M. The harmonic map equation is

$$\tau(\varphi) = tr(\nabla d\varphi) = 0, \tag{1.1}$$

and it is called the *tension field* of φ [5]. The Euler-Lagrange equation of $E_2(\varphi)$ is

$$\tau_2(\varphi) = tr(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_\nabla)\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) = 0, \tag{1.2}$$

and it is called the *bitension field* of φ [11].

In [3], Branding defined and considered interpolating sesqui-harmonic maps between Riemannian manifolds. The author introduced an action functional for maps between Riemannian manifolds that interpolated between the actions for harmonic and biharmonic maps. The map φ is said to be *interpolating sesqui-harmonic* if it is a critical point of $E_{\delta_1,\delta_2}(\varphi)$

$$E_{\delta_1,\delta_2}(\varphi) = \delta_1 \int_\Omega \|d\varphi\|^2 \, d\nu_g + \delta_2 \int_\Omega \|\tau(\varphi)\|^2 \, d\nu_g, \tag{1.3}$$

where Ω is a compact domain of M and $\delta_1, \delta_2 \in \mathbb{R}$ [3]. The interpolating sesqui-harmonic map equation is

$$\tau_{\delta_1,\delta_2}(\varphi) = \delta_2 \tau_2(\varphi) - \delta_1 \tau(\varphi) = 0 \tag{1.4}$$

for $\delta_1, \delta_2 \in \mathbb{R}$ [3]. An interpolating sesqui-harmonic map is biminimal if variations of (1.3) that are normal to the image $\varphi(M) \subset N$ and $\delta_2 = 1$, $\delta_1 > 0$ [13]. For some recent study of biminimal immersions see [8], [13], [14] and [15].

Interpolating sesqui-harmonic curves in a 3-dimensional sphere were studied in [3]. In [6] and [7], Fetcu and Oniciuc considered biharmonic Legendre curves in Sasakian space forms. In [4], Cho, Inoguchi and Lee studied affine biharmonic curves in 3-dimensional pseudo-Hermitian geometry. In [10], Inoguchi and Lee studied affine biharmonic curves in 3-dimensional homogeneous geometries. In [16], the second author and Güvenç studied biharmonic Legendre curves in generalized Sasakian space forms. In [9], Güvenç and the second author studied f-biharmonic Legendre curves in Sasakian space forms. Motivated by the above studies, in the present paper, we consider interpolating sesqui-harmonic Legendre curves in Sasakian space forms. We obtain the necessary and sufficient conditions for Legendre curves in Sasakian space forms to be interpolating sesqui-harmonic. We also give an example for an interpolating sesqui-harmonic Legendre curve in a Sasakian space form.

2. Preliminaries

Let $M = (M^{2n+1}, \phi, \xi, \eta, g)$ be an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g). A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called a *Sasakian manifold* if it is normal, that is,

$$N_\phi = -2d\eta \otimes \xi$$

where N_ϕ is the Nijenhuis tensor field of ϕ [1]. It is well-known that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y) X$$

and

$$\nabla_X \xi = -\phi X$$

[2]. The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature*. When the ϕ-sectional curvature is a constant, then the Sasakian manifold is called a *Sasakian space form* and it is denoted by $M(c)$ [2]. The curvature tensor R of a Sasakian space form $M(c)$ is given by

$$R(X, Y) Z = \frac{c+3}{4} \{g(Y, Z)X - g(X, Z)Y\}$$

$$+\frac{c-1}{4} \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z$$

$$+\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \tag{2.1}$$

for all $X, Y, Z \in TM$ [2].

A submanifold of a Sasakian manifold M is called an *integral submanifold* if $\eta(X) = 0$, for every tangent vector X. An integral curve of a Sasakian manifold M is called a *Legendre curve* [2].

3. Interpolating sesqui-harmonic Legendre curves in Sasakian space forms

Let $\gamma : I \subset \mathbb{R} \longrightarrow (M^n, g)$ be a curve parametrized by arc length in a Riemannian manifold (M^n, g) . Then γ is called a Frenet curve of osculating order r, $1 \leq r \leq n$, if there exists orthonormal vector fields $\{E_i\}_{i=1,2,...n}$ along γ such that

$$\begin{aligned} E_1 &= T = \gamma', \\ \nabla_T E_1 &= k_1 E_2, \\ \nabla_T E_i &= -k_{i-1} E_{i-1} + k_i E_{i+1}, \quad 2 \leq i \leq n-1, \qquad (3.1) \\ \\ \nabla_T E_n &= -k_{n-1} E_{n-1}, \end{aligned}$$

where the function $\{k_1 = k, k_2 = \tau, k_3, ..., k_{n-1}\}$ are called the curvatures of γ [12].

Firstly, we have the following theorem for an interpolating sesqui-harmonic Legendre curve in a Sasakian space form:

Theorem 3.1. *Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with constant ϕ-sectional curvature c and $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ be a Legendre curve of osculating order r and $m = min\{r, 4\}$. Then γ is interpolating sesqui-harmonic if and only if there exists real numbers δ_1, δ_2 such that*
(1) $c = 1$ or $\phi T \perp E_2$ or $\phi T \in \{E_2, ..., E_m\}$; and
(2) the first m of the following equations are satisfied:

$$-3\delta_2 k_1 k_1' = 0, \tag{3.2}$$

$$\begin{aligned} &\delta_2 \left[k_1'' - k_1^3 - k_1 k_2^2 - \left(\frac{c+3}{4}\right) k_1 \right. \\ &\left. +3\left(\frac{c-1}{4}\right) k_1 \left[g(\phi T, E_2)\right]^2 - \left(\frac{c-1}{4}\right) k_1 \left[\eta\left(E_2\right)\right]^2 \right] - \delta_1 k_1 = 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\delta_2 \left[2k_1' k_2 + k_1 k_2' + 3\left(\frac{c-1}{4}\right) k_1 g(\phi T, E_2) g(\phi T, E_3) \right. \\ &\left. - \left(\frac{c-1}{4}\right) k_1 \eta\left(E_2\right) \eta\left(E_3\right) \right] = 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\delta_2 \left[k_1 k_2 k_3 + 3\left(\frac{c-1}{4}\right) k_1 g(\phi T, E_2) g(\phi T, E_4) \right. \\ &\left. - \left(\frac{c-1}{4}\right) k_1 \eta\left(E_2\right) \eta\left(E_4\right) \right] = 0. \end{aligned} \tag{3.5}$$

Proof. Let $\gamma : I \longrightarrow M$ be a Legendre curve of osculating order r in $M(c)$. By the use of (1.1) and (3.1), we have

$$\tau(\gamma) = \nabla_T T = k_1 E_2. \tag{3.6}$$

From (3.1), we get

$$\nabla_T \nabla_T T = -k_1^2 E_1 + k_1' E_2 + k_1 k_2 E_3,$$

$$\nabla_T\nabla_T\nabla_T T = -3k_1k_1^{'}E_1 + \left(k_1'' - k_1^3 - k_1k_2^2\right)E_2 + \left(2k_1'k_2 + k_1k_2'\right)E_3 + \left(k_1k_2k_3\right)E_4, \tag{3.7}$$

$$R(T,\nabla_T T)T = -\left(\frac{c+3}{4}\right)k_1E_2 - 3\left(\frac{c-1}{4}\right)k_1g(\phi T, E_2)\phi T + \left(\frac{c-1}{4}\right)k_1\eta\left(E_2\right)\xi. \tag{3.8}$$

Using the equations (3.6), (3.7) and (3.8) into the equation (4.1) in [3], we find

$$\tau_{\delta_1,\delta_2}(\gamma) = \left(-3\delta_2k_1k_1'\right)E_1 + \left[\delta_2\left(k_1'' - k_1^3 - k_1k_2^2 + \left(\frac{c+3}{4}\right)k_1\right) - \delta_1k_1\right]E_2 + \delta_2\left(2k_1'k_2 + k_1k_2'\right)E_3 + \delta_2\left(k_1k_2k_3\right)E_4 + 3\left(\frac{c-1}{4}\right)\delta_2k_1g(\phi T, E_2)\phi T - \left(\frac{c-1}{4}\right)\delta_2k_1\eta\left(E_2\right)\xi. \tag{3.9}$$

Taking the scalar product of equation (3.9) with E_2, E_3 and E_4 respectively, then we obtain the desired result. □

Now we shall discuss some special cases of Theorem 3.1:

Case I. $c = 1$.

From Theorem 3.1, we have:

Proposition 3.1. *Let $M(1) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c = 1$ and $\gamma : I \subset \mathbb{R} \longrightarrow M(1)$ be a Legendre curve of osculating order r such that $\frac{\delta_1}{\delta_2} \neq 0$. Then γ is interpolating sesqui-harmonic if and only if*

$$k_1 = \textit{constant} > 0,\ k_2 = \textit{constant},$$
$$k_1^2 + k_2^2 = 1 - \frac{\delta_1}{\delta_2},$$
$$k_2k_3 = 0$$

where $1 - \frac{\delta_1}{\delta_2} > 0$, δ_1, δ_2 is a constant.

Proof. Assume that γ is an interpolating sesqui-harmonic Legendre curve of osculating order r in $M(1)$ such that $\frac{\delta_1}{\delta_2} \neq 0$ and $c = 1$. From Theorem 3.1, we obtain the result. □

Using Proposition 3.1, we have:

Theorem 3.2. *Let $M(1) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c = 1$ and $\gamma : I \subset \mathbb{R} \longrightarrow M(1)$ be a non geodesic Legendre curve of osculating order r. Then*

(1) *It is a Legendre geodesic or*

(2) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre circle with $k_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}}$ where $1 - \frac{\delta_1}{\delta_2} > 0$ is a constant or*

(3) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre helix with $k_1^2 + k_2^2 = 1 - \frac{\delta_1}{\delta_2}$ where $1 - \frac{\delta_1}{\delta_2} > 0$, δ_1, δ_2 is a constant.*

In both cases, if $1-\frac{\delta_1}{\delta_2}<0$, *then such an interpolating sesqui-harmonic Legendre curve does not exist.*

Proof. Let $\gamma : I \longrightarrow M(1)$ be an interpolating sesqui-harmonic curve with $\frac{\delta_1}{\delta_2} \neq 0$. From Theorem 3.1, if we consider the osculating order $r=2$, then γ is a Legendre circle with $k_1=\sqrt{1-\frac{\delta_1}{\delta_2}}$ where $1-\frac{\delta_1}{\delta_2}>0$ is a constant. Similarly, if we consider the osculating order $r=3$, then we obtain that k_2 is a non-zero constant. Thus, γ is a Legendre helix with $k_1^2+k_2^2=1-\frac{\delta_1}{\delta_2}$ where $1-\frac{\delta_1}{\delta_2}>0$ is a constant. On the other hand, assume that γ is a Legendre circle with $k_1=\sqrt{1-\frac{\delta_1}{\delta_2}}$ or a Legendre helix with $k_1^2+k_2^2=1-\frac{\delta_1}{\delta_2}$ where $1-\frac{\delta_1}{\delta_2}>0$ is a constant. Obviously, γ satisfies Theorem 3.1, respectively. It is trivial that $1-\frac{\delta_1}{\delta_2}<0$ cannot be possible. If $1-\frac{\delta_1}{\delta_2}=0$, we obtain a geodesic. This proves the theorem. □

Case II. $c \neq 1$ and $\phi T \perp E_2$.
From Theorem 3.1, we can state:

Proposition 3.2. *Let* $M(c)=(M^{2n+1},\phi,\xi,\eta,g)$ *be a Sasakian space form with* $c\neq 1$, $\phi T \perp E_2$ *and* $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ *be a Legendre curve of osculating order* r *such that* $\frac{\delta_1}{\delta_2}\neq 0$. *Then* γ *is interpolating sesqui-harmonic if and only if*

$$k_1 = \textit{constant} > 0,\ k_2 = \textit{constant},$$

$$k_1^2+k_2^2=\frac{c+3}{4}-\frac{\delta_1}{\delta_2},$$

$$k_2k_3=0$$

where δ_1, δ_2 *is a constant.*

Proof. Let γ be an interpolating sesqui-harmonic Legendre curve of osculating order r in $M(c)$ such that $c\neq 1$, $\phi T \perp E_2$ and $\frac{\delta_1}{\delta_2}\neq 0$. From Theorem 3.1, we get the result. □

From [7], we have the following lemma:

Lemma 3.1. [7] *Let* γ *be a Legendre Frenet curve of osculating order* 3 *in a Sasakian space form* $M(c)$ *and* $\phi T \perp E_2$. *Then* $\{T=E_1,E_2,E_3,\phi T,\nabla_T \phi T,\xi\}$ *is linearly independent at any point of* γ *and therefore* $n\geq 3$.

Hence we can state:

Theorem 3.3. *Let* $M(c)=(M^{2n+1},\phi,\xi,\eta,g)$ *be a Sasakian space form with* $c\neq 1$, $\phi T \perp E_2$ *and* $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ *a Legendre curve of osculating order* r.

(1) *If* $c\leq -3$ *and* $\frac{\delta_1}{\delta_2}\geq 0$, *then* γ *is interpolating sesqui-harmonic if and only if it is a geodesic.*

(2) *If* $c>-3$ *and* $\frac{\delta_1}{\delta_2}<0$, *then* γ *is interpolating sesqui-harmonic if and only if either*

(*a*) *γ is of osculating order $r = 2$, $n \geq 2$ and γ is a circle with $k_1^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$, in which case $\{T, E_2, \phi T, \nabla_T \phi T, \xi\}$ are linearly independent, or*

(*b*) *γ is of osculating order $r = 3$, $n \geq 3$ and γ is a helix with $k_1^2 + k_2^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$, in which case $\{T, E_2, E_3, \phi T, \nabla_T \phi T, \xi\}$ are linearly independent, where δ_1, $\delta_2 \in \mathbb{R}$.*

Proof. (1) From Proposition 3.2, if we take $c \leq -3$ and $\frac{\delta_1}{\delta_2} \geq 0$, it is easy to see that γ is interpolating sesqui-harmonic if and only if it is a geodesic.

(2) Assume that $c > -3$, $\frac{\delta_1}{\delta_2} < 0$ and $\gamma : I \longrightarrow M(c)$ be an interpolating sesqui-harmonic curve. From Proposition 3.2, if we take $n \geq 2$ and γ is of osculating order $r = 2$, then γ is a circle with $k_1^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$. Using Lemma 3.1, we have that $\{T, E_2, \phi T, \nabla_T \phi T, \xi\}$ are linearly independent. Similarly, if we take $n \geq 3$ and γ is of osculating order $r = 3$, then we obtain that k_2 is a non-zero constant. Thus, γ is a helix with $k_1^2 + k_2^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$. Using Lemma 3.1, we have that $\{T, E_2, E_3, \phi T, \nabla_T \phi T, \xi\}$ are linearly independent. Conversely, assume that γ is a Legendre circle with $k_1^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$ or a Legendre helix with $k_1^2 + k_2^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$. Obviously, γ satisfies Theorem 3.1, respectively. Hence, we obtain the desired result. □

Case III. $c \neq 1$ and $\phi T \parallel E_2$.

From Theorem 3.1, we have:

Proposition 3.3. *Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ be a Legendre curve of osculating order r with $\phi T \parallel E_2$ and $\frac{\delta_1}{\delta_2} \neq 0$. Then γ is interpolating sesqui-harmonic if and only if*

$$k_1 = constant > 0, \;\; k_2 = constant,$$

$$k_1^2 + k_2^2 = c - \frac{\delta_1}{\delta_2},$$

$$k_2 k_3 = 0$$

where δ_1, δ_2 is a constant.

Proof. Assume γ is an interpolating sesqui-harmonic Legendre curve in $M(c)$ such that $c \neq 1$, $\phi T \parallel E_2$ and $\frac{\delta_1}{\delta_2} \neq 0$. From Theorem 3.1, we get the result. □

Hence we can state:

Theorem 3.4. *Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ a Legendre curve of osculating order r such that $\phi T \parallel E_2$. Then $\{T, \phi T, \xi\}$ is the Frenet frame field of γ.*

(1) *If $c < 1$ and $\frac{\delta_1}{\delta_2} \geq 0$, then γ is interpolating sesqui-harmonic if and only if it is a geodesic.*

(2) *If $c > 1$ and $\frac{\delta_1}{\delta_2} < 0$, then γ is interpolating sesqui-harmonic if and only if it is a helix with $k_1^2 = c - 1 - \frac{\delta_1}{\delta_2}$, ($k_2 = 1$) where δ_1, $\delta_2 \in \mathbb{R}$.*

Proof. If we take $\phi T \parallel E_2$, we get $g(\phi T, E_2) = \pm 1$, $g(\phi T, E_3) = g(\phi T, E_4) = 0$.

(1) From Proposition 3.3 and the above equations and if we take $c \leq 1$ and $\frac{\delta_1}{\delta_2} \geq 0$, it is easy to see that γ is interpolating sesqui-harmonic if and only if it is a geodesic.

(2) If $c > 1$, $\frac{\delta_1}{\delta_2} < 0$ from Proposition 3.3 and the above equations, we have k_1 =constant and $k_1^2 = c - 1 - \frac{\delta_1}{\delta_2}$, and $k_2 = 1$. Conversely, assume that γ is a Legendre helix with $k_1^2 = c - 1 - \frac{\delta_1}{\delta_2}$ and $k_2 = 1$. Then γ satisfies Theorem 3.1 obviously. This completes the proof of the theorem. □

Case IV. $c \neq 1$ and $g(\phi T, E_2) \neq 0, 1, -1$.

Proposition 3.4. *Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c \neq 1$, $g(\phi T, E_2) \neq 0, 1, -1$ and $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ a Legendre curve of osculating order r such that $4 \leq r \leq 2n+1$, $n \geq 2$. Then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$
\begin{aligned}
k_1 &= \textit{constant} > 0, \\
k_1^2 + k_2^2 &= \frac{c+3}{4} + \frac{3(c-1)}{4} f^2 - \frac{\delta_1}{\delta_2}, \\
k_2' &= -\frac{3(c-1)}{4} f g(E_3, \phi T), \\
k_2 k_3 &= -\frac{3(c-1)}{4} f g(E_4, \phi T).
\end{aligned}
$$

Proof. Assume that γ is an interpolating sesqui-harmonic Legendre Frenet curve such that $g(\phi T, E_2)$ is not a constant equal to $0, 1$ or -1. In this case, we get $4 \leq r \leq 2n+1$, $n \geq 2$ and $\phi T \in span\{E_2, E_3, E_4\}$.

Hence, we can take $f(t) = g(\phi T, E_2)$. So by a differentiation, we obtain

$$
\begin{aligned}
f'(t) &= g(\nabla_T \phi T, E_2) + g(\phi T, \nabla_T E_2) \\
&= -k_1 g(T, \phi T) + k_2 g(E_3, \phi T) + g(E_2, \xi) + k_1 g(E_2, \phi E_2).
\end{aligned}
$$

Since γ is a Legendre curve and ϕ is anti-symmetric, we have $\eta(E_2) = 0$, $g(T, \phi T) = 0$ and $g(E_2, \phi E_2) = 0$. Thus we obtain

$$
f'(s) = k_2 g(E_3, \phi T). \tag{3.10}
$$

Additionally, we can write

$$
\phi T = g(\phi T, E_2) E_2 + g(\phi T, E_3) E_3 + g(\phi E_4, E_4) E_4. \tag{3.11}
$$

From Theorem 3.1, the equations (3.10) and (3.11), the curve γ is interpolating sesqui-harmonic if and only if

$$
\begin{aligned}
k_1 &= \text{constant}, \\
k_1^2 + k_2^2 &= \frac{c+3}{4} + \frac{3(c-1)}{4} f^2 - \frac{\delta_1}{\delta_2}, \\
k_2' &= -\frac{3(c-1)}{4} f g(E_3, \phi T),
\end{aligned}
$$

$$k_2 k_3 = -\frac{3\,(c-1)}{4} f g\,(E_4, \phi T)\,.$$

If $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ satisfies the converse statement, it is obvious that the first four of the equations in Theorem 3.1 are satisfied. Thus γ is interpolating sesqui-harmonic. This proves the theorem. □

Using the equation (3.10) and the third equation of Proposition 3.4, we obtain

$$k_2' = -\frac{3\,(c-1)}{4} f g\,(E_3, \phi T) = -\frac{3\,(c-1)}{4} f \frac{f'}{k_2}$$

$$k_2 k_2' = -\frac{3\,(c-1)}{4} f f'$$

$$k_2^2 = -\frac{3\,(c-1)}{4} f^2 + w_0 \tag{3.12}$$

where w_0 =constant. Substituting the equation (3.12) in the second equation of Proposition 3.4, we get

$$k_1^2 = \frac{c+3}{4} + \frac{3\,(c-1)}{2} f^2 - \frac{\delta_1}{\delta_2} - w_0.$$

Then we have f =constant. Thus k_2 =constant > 0, $g\,(E_3, \phi T) = 0$ and then $\phi T = f E_2 + g\,(\phi T, E_4)\, E_4$. We obtain that there exists a unique constant $\alpha_0 \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ such that $f = \cos\alpha_0$ and $g\,(E_4, \phi T) = \sin\alpha_0$.

So we can state:

Theorem 3.5. *Let $M(c) = (M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian space form with $c \neq 1$, $n \geq 2$ and $\gamma : I \subset \mathbb{R} \longrightarrow M(c)$ a Legendre curve of osculating order r such that $g(\phi T, E_2) \neq 0, 1, -1$.*

(1) If $c \leq -3$ and $\frac{\delta_1}{\delta_2} \geq 0$, then γ is interpolating sesqui-harmonic if and only if it is a geodesic.

(2) If $c > -3$ and $\frac{\delta_1}{\delta_2} < 0$, then γ is interpolating sesqui-harmonic if and only if $\phi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$,

$$k_1, k_2, k_3 = \textit{constant} > 0,$$

$$k_1^2 + k_2^2 = \frac{c+3}{4} + \frac{3\,(c-1)}{4} \cos^2\alpha_0 - \frac{\delta_1}{\delta_2},$$

$$k_2 k_3 = -\frac{3\,(c-1)}{8} \sin 2\alpha_0,$$

where $\alpha_0 \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ is constant such that $(c + 3 + 3\,(c-1)\cos^2\alpha_0)\,\delta_2 - 4\delta_1 > 0$ and $3\,(c-1)\sin 2\alpha_0 < 0$.

Remark 3.1. *For $c \neq 1$ and $g(\phi T, E_2) \neq 0, 1, -1$, there are also interpolating sesqui-harmonic curves which are not helices.*

Now, we give brief information about the Sasakian space form $\mathbb{R}^{2n+1}(-3)$ [2]:

Let us take $M=\mathbb{R}^{2n+1}$ with the standard coordinate functions $(x_1,...,x_n,y_1,...,y_n,z)$, the contact structure $\eta=\frac{1}{2}(dz-\sum_{i=1}^{n}y_i dx_i)$, the characteristic vector field $\xi=2\frac{\partial}{\partial z}$ and the tensor field ϕ given by

$$\phi=\begin{bmatrix}0 & \delta_{ij} & 0\\ -\delta_{ij} & 0 & 0\\ 0 & y_j & 0\end{bmatrix}.$$

The Riemannian metric is $g=\eta\otimes\eta+\frac{1}{4}\sum_{i=1}^{n}\left((dx_i)^2+(dy_i)^2\right)$. Thus, $\mathbb{R}^{2n+1}(-3)$ is a Sasakian space form with constant $\phi-$sectional curvature $c=-3$. The vector fields

$$X_i=2\frac{\partial}{\partial y_i},\ X_{i+n}=\phi X_i=2(\frac{\partial}{\partial x_i}+y_i\frac{\partial}{\partial z}),\ 1\leq i\leq n,\ \xi=2\frac{\partial}{\partial z}, \tag{3.13}$$

form a g-orthonormal basis and Levi-Civita connection is obtained as

$$\nabla_{X_i}X_j=\nabla_{X_{i+n}}X_{j+n}=0,\ \nabla_{X_i}X_{j+n}=\delta_{ij}\xi,\ \nabla_{X_{i+n}}X_j=-\delta_{ij}\xi, \tag{3.14}$$

$$\nabla_{X_i}\xi=\nabla_{\xi}X_i=-X_{n+i},\ \nabla_{X_{i+m}}\xi=\nabla_{\xi}X_{i+n}=X_i \tag{3.15}$$

(see [1]).

Now, we give an example for interpolating sesqui-harmonic Legendre curves in $\mathbb{R}^5(-3)$:

Example. Let $\gamma=(\gamma_1,...,\gamma_5)$ be a unit speed Legendre curve in $\mathbb{R}^5(-3)$. We can write the tangent vector field T of

$$\gamma T=\frac{1}{2}\left\{\gamma_3'X_1+\gamma_4'X_2+\gamma_1'X_3+\gamma_2'X_4+(\gamma_5'-\gamma_1'\gamma_3-\gamma_2'\gamma_4)\,\xi\right\}.$$

Using the above equation, $\eta(T)=0$ and $g(T,T)=1$, we have

$$\gamma_5'=\gamma_1'\gamma_3+\gamma_2'\gamma_4$$

and

$$(\gamma_1')^2+...+(\gamma_5')^2=4.$$

So for a Legendre curve (3.14), (3.15) and (3.13) gives us

$$\nabla_T T=\frac{1}{2}\left(\gamma_3''X_1+\gamma_4''X_2+\gamma_1''X_3+\gamma_2''X_4\right), \tag{3.16}$$

and

$$\phi T=\frac{1}{2}\left(-\gamma_1'X_1-\gamma_2'X_2+\gamma_3'X_3+\gamma_4'X_4\right). \tag{3.17}$$

From (3.16) and (3.17), $\phi T\perp E_2$ if and only if

$$\gamma_1'\gamma_3''+\gamma_2'\gamma_4''=\gamma_3'\gamma_1''+\gamma_4'\gamma_2''.$$

So we can state the following example:

Let us take $\gamma(t)=(\sin 2t,-\cos 2t,0,0,1)$ in $\mathbb{R}^5(-3)$. By the use of Theorem 3.1 and the above equations, γ is an interpolating sesqui-harmonic Legendre curve with osculating order $r = 2$, $k_1 = 2$, $\delta_1 = -8$, $\delta_2 = 2$ and $\phi T \perp E_2$. We can see that Theorem 3.1 are verified. From the equations (3-1) in [7], the curve γ is not biharmonic. Hence the biharmonicity and interpolating sesqui-harmonic of γ are different.

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