


Bilateral-type solutions to the fixed-circle problem with rectified linear units application

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Abstract: In this paper, we prove new fixed-circle (resp. fixed-disc) results using the bilateral type contractions on a metric space. To do this, we modify some known contractive conditions called the Jaggi-type bilateral contraction and the Dass-Gupta type bilateral contraction. We give some examples to show the validity of our obtained results. Also, we construct an application to rectified linear units activation functions used in the neural networks. This application shows the importance of studying “fixed-circle problem”.

Key words: Fixed circle, metric space, bilateral contraction

1. Introduction and definition of the problem

There are some examples of self-mappings which have a unique fixed point or more than one fixed point. For example, let (\mathbb{R}, d) be the usual metric space with the function $d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ defined as

$$d(x, y) = |x - y|,$$

for all $x, y \in \mathbb{R}$. If we consider the self-mappings $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$fx = 1 - x$$

and

$$gx = x^2 - 4x + 6,$$

for all $x \in \mathbb{R}$, then f has a unique fixed point $x_0 = \frac{1}{2}$ and g has two fixed points $x_1 = 2$, $x_2 = 3$. If the number of fixed points of a self-mapping is more than one, the following question occurs:

QUESTION: What are the geometric properties of fixed points in which case a self-mapping has more than one fixed point?

As a new light to the fixed-point theory, by geometric thinking, the above question has been defined as “fixed-circle problem”. This problem was first discussed in [15]. The studying of this problem gains importance both in terms of theoretical mathematical studies and some applied areas.

Now, what is the notion of a fixed circle?

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Let (X, d) be a metric space, $C_{x_0,r} = \{x \in X : d(x, x_0) = r\}$ a circle and $T : X \rightarrow X$ a self-mapping. If $Tx = x$ for every $x \in C_{x_0,r}$ then $C_{x_0,r}$ is called as the fixed circle of T [15].

If we consider the definition of a fixed circle, we see that there are a lot of examples of an activation function used in neural networks. For example, let $(\mathbb{C}, d_{\mathbb{C}})$ be the usual metric space with the function $d_{\mathbb{C}} : \mathbb{C} \times \mathbb{C} \rightarrow [0, \infty)$ defined as

$$d_{\mathbb{C}}(z, w) = |z - w|,$$

for all $z, w \in \mathbb{C}$, where \mathbb{C} is the set of all complex numbers. If we take the self-mapping $T : \mathbb{C} \rightarrow \mathbb{C}$ as

$$Tz = \begin{cases} \frac{1}{\bar{z}} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases},$$

for all $z \in \mathbb{C}$, where \bar{z} is a complex conjugate of the complex number z , then $C_{0,1}$ is the fixed circle of T . In [22], the activation function defined as

$$fz = \frac{1}{\bar{z}},$$

for all $z \in \mathbb{C} - \{0\}$, which has a fixed circle, was used in the complex-valued neural network (CVNN). The purpose of these activation functions is to ensure the existence of fixed points of the complex-valued Hopfield neural network (CVHNN).

For the above reasons, the first solution of the fixed-circle problem was given on metric spaces in [15]. After this study, new solutions have been investigated on both a metric space and some generalized metric spaces (for example, see [1–3, 6, 11–14, 16–21, 23, 24, 26–29]).

By the above motivation, in this paper, we present new solutions to the fixed-circle problem using the bilateral type contractions on a metric space. To do this, we inspire of the given definitions and the obtained results in [5] because in [5], fixed-point theorems were obtained for the cases where the number of fixed points is at least one. In Section 2, we give a brief survey related to the fixed-circle problem. In Section 3, we modify some known contractive conditions called the Jaggi-type bilateral contraction and the Dass-Gupta type bilateral contraction to obtain new fixed-circle (resp. fixed-disc, common fixed-circle, common fixed-disc) results. In Section 4, we construct an application of our theoretical results to rectified linear units activation functions.

2. A survey of the recent solutions

The first solution of the fixed-circle problem was given using Caristi's inequality [4] on metric spaces as follows:

Theorem 2.1 [15] *Let (X, d) be a metric space and $C_{x_0,r}$ a circle on X . Let us define the mapping $\varphi : X \rightarrow [0, \infty)$ as*

$$\varphi(x) = d(x, x_0),$$

for all $x \in X$. If there exists a self-mapping $T : X \rightarrow X$ satisfying

$$(C1) \quad d(x, Tx) \leq \varphi(x) - \varphi(Tx),$$

$$(C2) \quad d(Tx, x_0) \geq r,$$

for each $x \in C_{x_0,r}$, then the circle $C_{x_0,r}$ is a fixed circle of T .

The above theorem can be considered an existence theorem of a fixed circle, that is, this theorem guarantees the existence of a fixed circle of a self-mapping T . The conditions (C1) and (C2) have a geometric

meaning. The condition (C1) guarantees that Tx is not in the exterior of the circle $C_{x_0,r}$ and the condition (C2) guarantees that Tx is not in the interior of the circle $C_{x_0,r}$ for each $x \in C_{x_0,r}$, that is, $T(C_{x_0,r}) \subset C_{x_0,r}$. Using similar geometric approaches, other existence theorems of a fixed circle were given with necessary examples and uniqueness conditions (see [15] for more details). Also, a condition which excludes the identity map, $I : X \rightarrow X$ defined as $I(x) = x$ for all $x \in X$, was investigated in [15].

Using the different auxiliary function, the following fixed-circle theorem was given for the existence of a fixed circle in [18].

Theorem 2.2 [18] *Let (X, d) be a metric space, \mathbb{R} the set of all real numbers and $C_{x_0,r}$ any circle on X . Let us define the mapping $\varphi_r : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ as*

$$\varphi_r(u) = \begin{cases} u - r & ; u > 0 \\ 0 & ; u = 0 \end{cases} ,$$

for all $u \in \mathbb{R}^+ \cup \{0\}$. If there exists a self-mapping $T : X \rightarrow X$ satisfying

- (1) $d(Tx, x_0) = r$ for each $x \in C_{x_0,r}$,
- (2) $d(Tx, Ty) > r$ for each $x, y \in C_{x_0,r}$ and $x \neq y$,
- (3) $d(Tx, Ty) \leq d(x, y) - \varphi_r(d(x, Tx))$ for each $x, y \in C_{x_0,r}$,

then the circle $C_{x_0,r}$ is a fixed circle of T .

Using the Wardowski's techniques [31] and a Khan-type inequality [8], some fixed-circle results were obtained with different aspects as seen in the following theorems.

Definition 2.3 [31] *Let \mathbb{F} be the family of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ such that*

- (F₁) F is strictly increasing,
- (F₂) For each sequence $\{\alpha_n\}$ in $(0, \infty)$ the following holds

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \text{ if and only if } \lim_{n \rightarrow \infty} F(\alpha_n) = -\infty,$$

- (F₃) There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

Definition 2.4 [28] *If there exist $t > 0$, $F \in \mathbb{F}$ and $x_0 \in X$ such that for all $x \in X$ the following holds:*

$$d(x, Tx) > 0 \implies t + F(d(x, Tx)) \leq F(d(x_0, x)),$$

then T is said to be an F_C -contraction on X .

Theorem 2.5 [28] *Let T be an F_C -contractive self-mapping with $x_0 \in X$ and*

$$r = \inf \{d(x, Tx) : x \neq Tx, x \in X\}. \tag{2.1}$$

Then $C_{x_0,r}$ is a fixed circle of T . Especially, T fixes every circle $C_{x_0,\rho}$ where $\rho < r$.

Definition 2.6 [25] *Let \mathbb{F}_k be the family of all increasing functions $F : (0, \infty) \rightarrow \mathbb{R}$, that is, for all $x, y \in (0, \infty)$, if $x < y$ then $F(x) \leq F(y)$.*

Definition 2.7 [12] Let (X, d) be a metric space and $T : X \rightarrow X$ a self-mapping. T is said to be an F_C -Khan type I contraction if there exist $F \in \mathbb{F}_k$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ if the following condition holds

$$\max \{d(Tx_0, x_0), d(Tx, x)\} \neq 0,$$

then

$$t + F(d(Tx, x)) \leq F \left(h \frac{d(Tx, x)d(Tx_0, x) + d(Tx_0, x)d(Tx, x_0)}{\max \{d(Tx_0, x_0), d(Tx, x)\}} \right),$$

where $h \in [0, \frac{1}{2})$ and if $\max \{d(Tx_0, x_0), d(Tx, x)\} = 0$ then $Tx = x$.

Proposition 2.8 [12] If a self-mapping T on X is an F_C -Khan-type I contraction with $x_0 \in X$ then we get $Tx_0 = x_0$.

Theorem 2.9 [12] Let (X, d) be a metric space, $T : X \rightarrow X$ a self-mapping and r defined as in (2.1). If T is an F_C -Khan-type I contraction with $x_0 \in X$ then $C_{x_0, r}$ is a fixed circle of T .

Similar type contractive conditions and related fixed-circle results were given with some illustrative examples on a metric space (see [12] and [28] for more details).

Now we recall the notions of a disc and a fixed disc on metric spaces. Let (X, d) be a metric space and $T : X \rightarrow X$ a self-mapping. Then the disc is defined by

$$D_{x_0, r} = \{x \in X : d(x, x_0) \leq r\}.$$

If $Tx = x$ for every $x \in D_{x_0, r}$ then $D_{x_0, r}$ is called as the fixed disc of T (see [21] and the references therein).

“Fixed-Circle Problem” has been studied in the sense of fixed-disc results. In this context, there are some studies in the literature. For example, in [21], some fixed-disc theorems were presented by a new approach using the set of simulation functions and some known fixed-point techniques. Also, some possible applications of the obtained theoretical results were discussed in the neural networks. In [23], Pant et al. obtained new fixed-circle (resp. fixed-disc) results using the number

$$m(x, y) = \max \left\{ \begin{array}{l} d(x, y), ad(x, Tx) + (1 - a)d(y, Ty), \\ (1 - a)d(x, Tx) + ad(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \end{array} \right\},$$

where $0 \leq a < 1$ and the Wardowski’s techniques. An application was constructed to discontinuous activation functions using an appropriate fixed circle. In [24], using the similar approach and the following number

$$N(x, y) = \max \left\{ \begin{array}{l} d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Ty)[1+d(x, Tx)]}{1+d(x, y)}, \\ \frac{d(x, Tx)[1+d(y, Ty)]}{1+d(Tx, Ty)} \end{array} \right\},$$

new fixed-disc results were obtained and some applications of these results to real or complex-valued discontinuous activation functions were investigated. In [3], Bisht and Özgür studied some geometric properties of the set of fixed points of a self-mapping with the number

$$L(x, y) = \max \left\{ \begin{array}{l} d(x, y), a \frac{d(x, Tx)(1+d(y, Ty))}{1+d(x, y)} + (1 - a) \frac{d(y, Ty)(1+d(x, Tx))}{1+d(x, y)}, \\ (1 - a) \frac{d(x, Tx)(1+d(y, Ty))}{1+d(x, y)} + a \frac{d(y, Ty)(1+d(x, Tx))}{1+d(x, y)} \end{array} \right\},$$

where $0 < a < 1$. Therefore, they proved some fixed-circle (resp. fixed-disc) theorems with necessary examples and some applications to neural networks.

Fixed-circle problem has been also studied on some generalized metric spaces. For example, in [16], the first solutions of this problem were obtained on an S -metric space with some illustrative examples. After this study, new fixed-circle (or fixed-disc) results were given with various aspects (see [11, 14, 17, 26, 27]). Also, some of these results were generalized on an S_b -metric space with geometric approaches [20]. New fixed-disc results were investigated in rectangular and quasi metric spaces (see [1, 2]). Using known fixed-point techniques, some applications were presented to “Fixed-Circle Problem” on 2-cone Banach spaces, M_b -metric spaces, rectangular M -metric spaces and parametric N_b -metric spaces (see [6, 13, 19, 29]).

In some of the above studies, some open problems were left to improve fixed-circle problem. For example, in [23], the following question was left:

Let (X, d) be a metric space and x_1, \dots, x_n any elements of X . Is there a circle on X with the elements x_1, \dots, x_n ? If so, what is the maximum value of such n ?

After examining all these studies, in the next section, we investigate new solutions to fixed-circle problem with different techniques on metric spaces.

3. Main results

In this section, we investigate new solutions to the fixed-circle problem. To do this, we use the following numbers given in [5].

Let (X, d) be a metric space and $T : X \rightarrow X$ a self-mapping.

$$R_T(x, y) = \max \left\{ d(x, y), \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \right\}$$

and

$$Q_T(x, y) = \max \left\{ d(x, y), \frac{(1 + d(x, Tx))d(y, Ty)}{1 + d(x, y)} \right\}.$$

3.1. Some bilateral type fixed-circle and fixed-disc results

We introduce new contractive conditions to obtain new fixed-circle theorems.

Definition 3.1 *If there exist a function $\varphi : X \rightarrow (0, \infty)$ and $x_0 \in X$ such that*

$$d(x, Tx) > 0 \implies d(x, Tx) \leq [\varphi(x) - \varphi(Tx)] R_T(x_0, x),$$

for all $x \in X - \{x_0\}$, then T is called a Jaggi-type bilateral x_0 -contractive mapping.

Theorem 3.2 *Let $T : X \rightarrow X$ be a Jaggi-type bilateral x_0 -contractive mapping with $x_0 \in X$ and r defined as*

$$r = \inf \left\{ \frac{d(x, Tx)}{\varphi(x)} : x \neq Tx, x \in X \right\}.$$

If $Tx_0 = x_0$, then T fixes the circle $C_{x_0, r}$.

Proof Let $r = 0$. Then we get $C_{x_0,r} = \{x_0\}$. Using the hypothesis, we have $Tx_0 = x_0$. Let $r > 0$ and $x \in C_{x_0,r}$ be any point such that $x \neq Tx$. By the Jaggi-type bilateral x_0 -contraction hypothesis, we get

$$\begin{aligned} d(x, Tx) &\leq [\varphi(x) - \varphi(Tx)] R_T(x_0, x) \\ &= [\varphi(x) - \varphi(Tx)] \max \left\{ d(x_0, x), \frac{d(x_0, Tx_0)d(x, Tx)}{d(x_0, x)} \right\} \\ &= [\varphi(x) - \varphi(Tx)] d(x_0, x) \\ &= [\varphi(x) - \varphi(Tx)] r \end{aligned}$$

and using the definition of r , we obtain

$$d(x, Tx) \leq [\varphi(x) - \varphi(Tx)] \frac{d(x, Tx)}{\varphi(x)} < d(x, Tx),$$

a contradiction. It should be $Tx = x$, that is, T fixes the circle $C_{x_0,r}$. □

Theorem 3.2 can be also considered a fixed-disc result.

Corollary 3.3 *Let $T : X \rightarrow X$ be a Jaggi type bilateral x_0 -contractive mapping with $x_0 \in X$ and r defined as in Theorem 3.2. If $Tx_0 = x_0$, then T fixes the disc $D_{x_0,r}$.*

Proof By the similar arguments used in the proof of Theorem 3.2, the proof can be easily seen. □

We give the following example.

Example 3.4 *Let $X = \mathbb{R}$ be the usual metric space. Let us define the self-mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ as*

$$Tx = \begin{cases} x & ; \quad x \in [-1, 1] \\ 0 & ; \quad x \in (-\infty, -1) \cup (1, \infty) \end{cases} ,$$

for all $x \in \mathbb{R}$. Then T is a Jaggi-type bilateral x_0 -contractive mapping with $x_0 = 0$ and the function $\varphi : \mathbb{R} \rightarrow (0, \infty)$ as

$$\varphi(x) = \begin{cases} 1 & ; \quad x \in [-1, 1] \\ 2|x| & ; \quad x \in (-\infty, -1) \cup (1, \infty) \end{cases} ,$$

for all $x \in \mathbb{R}$. Indeed, we obtain

$$d(x, Tx) = |x - 0| = |x| > 0$$

and

$$d(x, Tx) = |x| \leq [2|x| - 1] |x| = [\varphi(x) - \varphi(Tx)] R_T(0, x),$$

for all $x \in (-\infty, -1) \cup (1, \infty)$. We get

$$\begin{aligned} r &= \inf \left\{ \frac{d(x, Tx)}{\varphi(x)} : x \neq Tx, x \in \mathbb{R} \right\} \\ &= \inf \left\{ \frac{|x|}{2|x|} : x \in (-\infty, -1) \cup (1, \infty) \right\} = \frac{1}{2}. \end{aligned}$$

Hence, T satisfies the conditions of Theorem 3.2 and Corollary 3.3. Consequently, T fixes the circle $C_{0,\frac{1}{2}} = \{-\frac{1}{2}, \frac{1}{2}\}$ and the disc $D_{0,\frac{1}{2}} = [-\frac{1}{2}, \frac{1}{2}]$.

Definition 3.5 If there exists a function $\varphi : X \rightarrow (0, 1)$ and $x_0 \in X$ such that

$$d(x, Tx) > 0 \implies d(x, Tx) \leq [\varphi(x) - \varphi(Tx)] Q_T(x_0, x),$$

for all $x \in X - \{x_0\}$, then T is called a Dass-Gupta type I bilateral x_0 -contractive mapping.

Theorem 3.6 $T : X \rightarrow X$ be a Dass-Gupta type I bilateral x_0 -contractive mapping with $x_0 \in X$ and r defined as in Theorem 3.2. If $Tx_0 = x_0$, then T fixes the circle $C_{x_0, r}$.

Proof Let $r = 0$. Then we get $C_{x_0, r} = \{x_0\}$. Using the hypothesis, we have $Tx_0 = x_0$. Let $r > 0$ and $x \in C_{x_0, r}$ be any point such that $x \neq Tx$. By the Dass-Gupta-type I bilateral x_0 -contraction hypothesis and the definition of r , we obtain

$$\begin{aligned} d(x, Tx) &\leq [\varphi(x) - \varphi(Tx)] Q_T(x_0, x) \\ &= [\varphi(x) - \varphi(Tx)] \max \left\{ d(x_0, x), \frac{(1 + d(x_0, Tx_0)) d(x, Tx)}{1 + d(x_0, x)} \right\} \\ &= [\varphi(x) - \varphi(Tx)] \max \left\{ r, \frac{d(x, Tx)}{1 + r} \right\} \\ &< \varphi(x) \max \left\{ r, \frac{d(x, Tx)}{1 + r} \right\} \leq \varphi(x) \max \left\{ \frac{d(x, Tx)}{\varphi(x)}, \frac{d(x, Tx)}{1 + r} \right\} \\ &= \varphi(x) \frac{d(x, Tx)}{\varphi(x)} = d(x, Tx), \end{aligned}$$

a contradiction. It should be $Tx = x$, that is, T fixes the circle $C_{x_0, r}$. □

Theorem 3.6 can be considered another fixed-disc theorem.

Corollary 3.7 Let $T : X \rightarrow X$ be a Dass-Gupta-type I bilateral x_0 -contractive mapping with $x_0 \in X$ and r defined as in Theorem 3.2. If $Tx_0 = x_0$, then T fixes the disc $D_{x_0, r}$.

Proof By the similar arguments used in the proof of Theorem 3.6, the proof can be easily obtained. □

We give the following illustrative example.

Example 3.8 Let $X = \{-2, -\frac{4}{3}, -1, 0, 1, \frac{4}{3}, 2, 3\}$ be the metric space with the usual metric d . Let us define the self-mapping $T : X \rightarrow X$ as

$$Tx = \begin{cases} x & ; x \in \{-\frac{4}{3}, -1, 0, 1, \frac{4}{3}, 3\} \\ x + 1 & ; x \in \{-2, 2\} \end{cases},$$

for all $x \in X$. Then T is a Dass-Gupta-type I bilateral x_0 -contractive mapping with $x_0 = 0$ and the function $\varphi : \mathbb{R} \rightarrow (0, 1)$ as

$$\varphi(x) = \begin{cases} \frac{1}{4} & ; x \in \{-\frac{4}{3}, -1, 0, 1, \frac{4}{3}, 3\} \\ \frac{3}{4} & ; x \in \{-2, 2\} \end{cases},$$

for all $x \in X$. We get

$$r = \inf \left\{ \frac{1}{3} : x \in \{-2, 2\} \right\} = \frac{4}{3}.$$

Hence, T satisfies the conditions of Theorem 3.6 and Corollary 3.7. Consequently, T fixes the circle $C_{0, \frac{4}{3}} = \{-\frac{4}{3}, \frac{4}{3}\}$ and the disc $D_{0, \frac{4}{3}} = \{-\frac{4}{3}, -1, 0, 1, \frac{4}{3}\}$.

Definition 3.9 If there exist a function $\varphi : X \rightarrow (0, \infty)$ and $x_0 \in X$ such that

$$d(x, Tx) > 0 \implies d(x, Tx) \leq [\varphi(x) - \varphi(Tx)] Q_T(x, x_0),$$

for all $x \in X - \{x_0\}$, then T is called a Dass-Gupta type II bilateral x_0 -contractive mapping.

Theorem 3.10 Let $T : X \rightarrow X$ be a Dass-Gupta-type II bilateral x_0 -contractive mapping with $x_0 \in X$ and r defined as in Theorem 3.2. If $Tx_0 = x_0$, then T fixes the circle $C_{x_0, r}$.

Proof From the similar approaches used in the proof of Theorem 3.6, the proof can be easily seen. □

As a consequence of Theorem 3.10, we obtain the following corollary.

Corollary 3.11 Let $T : X \rightarrow X$ be a Dass-Gupta-type II bilateral x_0 -contractive mapping with $x_0 \in X$ and r defined as in Theorem 3.2. If $Tx_0 = x_0$, then T fixes the disc $D_{x_0, r}$.

Proof By the similar arguments used in the proof of Theorem 3.6, the proof can be easily obtained. □

Example 3.12 Let $X = \mathbb{R}$ be the usual metric space. Let us define the self-mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ as

$$Tx = \begin{cases} x & ; \quad x \in [-2, \infty) \\ 0 & ; \quad x \in (-\infty, -2) \end{cases} ,$$

for all $x \in \mathbb{R}$. Then T is a Dass-Gupta type II bilateral x_0 -contractive mapping with $x_0 = 0$ and the function $\varphi : \mathbb{R} \rightarrow (0, \infty)$ as

$$\varphi(x) = \begin{cases} \frac{1}{2} & ; \quad x \in [-2, \infty) \\ |x| & ; \quad x \in (-\infty, -2) \end{cases} ,$$

for all $x \in \mathbb{R}$. We obtain

$$r = \inf \left\{ \frac{|x|}{|x|} : x \in (-\infty, -2) \right\} = 1.$$

Thus, T satisfies the conditions of Theorem 3.10 and Corollary 3.11. Consequently, T fixes the circle $C_{0,1} = \{-1, 1\}$ and the disc $D_{0,1} = [-1, 1]$.

Now we give the following corollaries.

Corollary 3.13 Let r be defined as in Theorem 3.2. If there exist a function $\varphi : X \rightarrow (0, \infty)$ and $x_0 \in X$ such that

$$d(x, Tx) > 0 \implies d(x, Tx) \leq [\varphi(x) - \varphi(Tx)] \left(\alpha_1 d(x, x_0) + \alpha_2 \frac{d(x, Tx)d(x_0, Tx_0)}{d(x, x_0)} \right),$$

for all $x \in X - \{x_0\}$ where α_1, α_2 are two nonnegative real numbers with a sum 1 and $Tx_0 = x_0$, then T fixes the circle $C_{x_0, r}$. Especially, T fixes the disc $D_{x_0, r}$.

Proof Using the following inequality,

$$\alpha_1 d(x, x_0) + \alpha_2 \frac{d(x, Tx)d(x_0, Tx_0)}{d(x, x_0)} \leq R_T(x_0, x),$$

we derive the desired corollary. □

Corollary 3.14 *Let r be defined as in Theorem 3.2. If there exists a function $\varphi : X \rightarrow (0, \infty)$ and $x_0 \in X$ such that*

$$d(x, Tx) > 0 \implies d(x, Tx) \leq [\varphi(x) - \varphi(Tx)] \left(\alpha_1 d(x, x_0) + \alpha_2 \frac{(1 + d(x, Tx)) d(x_0, Tx_0)}{1 + d(x, x_0)} \right),$$

for all $x \in X - \{x_0\}$ where α_1, α_2 are two nonnegative real numbers with a sum 1 and $Tx_0 = x_0$, then T fixes the circle $C_{x_0, r}$. Especially, T fixes the disc $D_{x_0, r}$.

Proof Using the following inequality,

$$\alpha_1 d(x, x_0) + \alpha_2 \frac{(1 + d(x, Tx)) d(x_0, Tx_0)}{1 + d(x, x_0)} \leq Q_T(x, x_0),$$

we see this corollary. □

Corollary 3.15 *Let r be defined as in Theorem 3.2. If there exists a function $\varphi : X \rightarrow (0, \infty)$ and $x_0 \in X$ such that*

$$d(x, Tx) > 0 \implies d(x, Tx) \leq [\varphi(x) - \varphi(Tx)] d(x, x_0),$$

for all $x \in X - \{x_0\}$ with $Tx_0 = x_0$, then T fixes the circle $C_{x_0, r}$. Especially, T fixes the disc $D_{x_0, r}$.

Proof Using the following inequality,

$$d(x, x_0) \leq Q_T(x, x_0),$$

we prove the desired result. □

Corollary 3.16 *Let r be defined as in Theorem 3.2. If there exists a function $\varphi : X \rightarrow (0, 1)$ and $x_0 \in X$ such that*

$$d(x, Tx) > 0 \implies d(x, Tx) \leq [\varphi(x) - \varphi(Tx)] \left(\frac{(1 + d(x_0, Tx_0)) d(x, Tx)}{1 + d(x_0, x)} \right),$$

for all $x \in X - \{x_0\}$ with $Tx_0 = x_0$, then T fixes the circle $C_{x_0, r}$. Especially, T fixes the disc $D_{x_0, r}$.

Proof Using the following inequality,

$$\frac{(1 + d(x_0, Tx_0)) d(x, Tx)}{1 + d(x_0, x)} \leq Q_T(x_0, x),$$

we derive the desired corollary. □

3.2. Two bilateral type common fixed-circle theorems

In this section, we suppose that (X, d) is a metric space and $T, S : X \rightarrow X$ are two self-mappings on X . If $Tx = Sx = x$ for all $x \in C_{x_0, r}$ (resp. $x \in D_{x_0, r}$), then $C_{x_0, r}$ (resp. $D_{x_0, r}$) is called the common fixed circle [12] (resp. the common fixed disc [21]) of the pair (T, S) .

Now we modify the number $R_T(x, y)$ for the pair (T, S) as follows:

$$R_{T,S}(x, y) = \max \left\{ d(x, y), \frac{d(x, Tx)d(y, Sy)}{d(x, y)} \right\}.$$

Let us define the following numbers:

$$\begin{aligned} r_T &= \inf \left\{ \frac{d(x, Tx)}{\varphi(x)} : Tx \neq x, \varphi(x) > 0, x \in X \right\}, \\ r_S &= \inf \left\{ \frac{d(x, Sx)}{\varphi(x)} : Sx \neq x, \varphi(x) > 0, x \in X \right\}, \\ r_{T,S} &= \inf \left\{ \frac{d(Tx, Sx)}{\varphi(x)} : Tx \neq Sx, \varphi(x) > 0, x \in X \right\} \end{aligned}$$

and

$$r^* = \min \{r_T, r_S, r_{T,S}\}.$$

We prove the following theorem.

Theorem 3.17 *Suppose that there exist a function $\varphi : X \rightarrow (0, \infty)$ and $x_0 \in X$ such that*

$$d(Tx, Sx) > 0 \implies d(Tx, Sx) \leq [\varphi(x) - \varphi(Tx)] R_{T,S}(x_0, x),$$

for all $x \in X - \{x_0\}$. If $Tx_0 = Sx_0 = x_0$ and T is a Jaggi-type bilateral x_0 -contractive mapping (or S is a Jaggi-type bilateral x_0 -contractive mapping) with $x_0 \in X$, then C_{x_0, r^} is a common fixed circle of the pair (T, S) . Especially, D_{x_0, r^*} is a common fixed disc of the pair (T, S) .*

Proof Let $r^* = 0$. Then we have $C_{x_0, r^*} = \{x_0\}$, so using the hypothesis, C_{x_0, r^*} is a common fixed circle of the pair (T, S) . Let $r^* > 0$ and $x \in C_{x_0, r^*}$ be any point such that $Tx \neq Sx$. Thus, we get $d(Tx, Sx) > 0$. Using the hypothesis, we have

$$\begin{aligned} d(Tx, Sx) &\leq [\varphi(x) - \varphi(Tx)] R_{T,S}(x_0, x) \\ &= [\varphi(x) - \varphi(Tx)] \max \left\{ d(x_0, x), \frac{d(x_0, Tx_0)d(x, Sx)}{d(x_0, x)} \right\} \\ &= [\varphi(x) - \varphi(Tx)] d(x_0, x) \\ &= [\varphi(x) - \varphi(Tx)] r^* \\ &< \varphi(x) r_{T,S} \leq \varphi(x) \frac{d(Tx, Sx)}{\varphi(x)} = d(Tx, Sx), \end{aligned}$$

which is a contradiction. Hence, x is a coincidence point of the pair (T, S) , that is,

$$Tx = Sx. \tag{3.1}$$

Finally, if T is a Jaggi type bilateral x_0 -contractive mapping (or S is a Jaggi type bilateral x_0 -contractive mapping) then by Theorem 3.2 we get

$$Tx = x \text{ (or } Sx = x). \tag{3.2}$$

Consequently, using the equalities (3.1) and (3.2), we obtain

$$Tx = Sx = x,$$

for all $x \in C_{x_0, r^*}$, that is, C_{x_0, r^*} is a common fixed circle of the pair (T, S) . The last part of this proof, it can be easily seen. □

We give the following illustrative example of the above theorem.

Example 3.18 Let $X = \mathbb{R}$ be the usual metric space. Let us consider the self-mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ defined as in Example 3.4. Let us define another self-mapping $S : \mathbb{R} \rightarrow \mathbb{R}$ as

$$Sx = \begin{cases} x & ; \quad x \in [-1, 1] \\ \frac{x}{2} & ; \quad x \in (-\infty, -1) \cup (1, \infty) \end{cases} ,$$

for all $x \in \mathbb{R}$. Then the pair (T, S) satisfies the conditions of Theorem 3.17 with $x_0 = 0$ and the function $\varphi : \mathbb{R} \rightarrow (0, \infty)$ as

$$\varphi(x) = \begin{cases} 1 & ; \quad x \in [-1, 1] \\ 2|x| & ; \quad x \in (-\infty, -1) \cup (1, \infty) \end{cases} ,$$

for all $x \in \mathbb{R}$. Also, we obtain

$$\begin{aligned} r_T &= \inf \left\{ \frac{d(x, Tx)}{\varphi(x)} : Tx \neq x, \varphi(x) > 0, x \in X \right\} \\ &= \inf \left\{ \frac{|x|}{2|x|} : x \in (-\infty, -1) \cup (1, \infty) \right\} = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} r_S &= \inf \left\{ \frac{d(x, Sx)}{\varphi(x)} : Sx \neq x, \varphi(x) > 0, x \in X \right\} \\ &= \inf \left\{ \frac{|\frac{x}{2}|}{2|x|} : x \in (-\infty, -1) \cup (1, \infty) \right\} = \frac{1}{4}, \end{aligned}$$

$$\begin{aligned} r_{T,S} &= \inf \left\{ \frac{d(Tx, Sx)}{\varphi(x)} : Tx \neq Sx, \varphi(x) > 0, x \in X \right\} \\ &= \inf \left\{ \frac{|\frac{x}{2}|}{2|x|} : x \in (-\infty, -1) \cup (1, \infty) \right\} = \frac{1}{4} \end{aligned}$$

and

$$r^* = \min \{r_T, r_S, r_{T,S}\} = \frac{1}{4}.$$

Consequently, $C_{0, \frac{1}{4}} = \{-\frac{1}{4}, \frac{1}{4}\}$ (resp. $D_{0, \frac{1}{4}} = [-\frac{1}{4}, \frac{1}{4}]$) is a common fixed circle (resp. is a common fixed disc) of the pair (T, S) .

Let us modify the number $Q_T(x, y)$ for the pair (T, S) as follows:

$$Q_{T,S}(x, y) = \max \left\{ d(x, y), \frac{(1 + d(x, Tx)) d(y, Sy)}{1 + d(x, y)} \right\}.$$

Now we give the following theorem using the numbers $Q_{T,S}(x, y)$ and r^* .

Theorem 3.19 *Assume that there exist a function $\varphi : X \rightarrow (0, \infty)$ and $x_0 \in X$ such that*

$$d(Tx, Sx) > 0 \implies d(Tx, Sx) \leq [\varphi(x) - \varphi(Tx)] Q_{T,S}(x, x_0),$$

for all $x \in X - \{x_0\}$. If $Tx_0 = Sx_0 = x_0$ and T is a Dass-Gupta type II bilateral x_0 -contractive mapping (or S is a Dass-Gupta type II bilateral x_0 -contractive mapping) with $x_0 \in X$, then C_{x_0, r^*} is a common fixed circle of the pair (T, S) . Especially, D_{x_0, r^*} is a common fixed disc of the pair (T, S) .

Proof By the similar arguments used in the proof of Theorem 3.17, we can easily prove it. □

Example 3.20 *Let $X = \mathbb{R}$ be the usual metric space. Let us consider the self-mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ defined as in Example 3.12. Let us define another self-mapping $S : \mathbb{R} \rightarrow \mathbb{R}$ as*

$$Sx = \begin{cases} x & ; \quad x \in [-2, \infty) \\ \frac{x}{3} & ; \quad x \in (-\infty, -2) \end{cases},$$

for all $x \in \mathbb{R}$. Then the pair (T, S) satisfies the conditions of Theorem 3.19 with $x_0 = 0$ and the function $\varphi : \mathbb{R} \rightarrow (0, \infty)$ as

$$\varphi(x) = \begin{cases} \frac{1}{2} & ; \quad x \in [-2, \infty) \\ |x| & ; \quad x \in (-\infty, -2) \end{cases},$$

for all $x \in \mathbb{R}$. Also, we obtain

$$r_T = \inf \left\{ \frac{|x|}{|x|} : x \in (-\infty, -2) \right\} = 1,$$

$$r_S = \inf \left\{ \frac{|\frac{2x}{3}|}{|x|} : x \in (-\infty, -2) \right\} = \frac{2}{3},$$

$$r_{T,S} = \inf \left\{ \frac{|\frac{x}{3}|}{|x|} : x \in (-\infty, -2) \right\} = \frac{1}{3}$$

and

$$r^* = \min \{r_T, r_S, r_{T,S}\} = \frac{1}{3}.$$

Consequently, $C_{0, \frac{1}{3}} = \{-\frac{1}{3}, \frac{1}{3}\}$ (resp. $D_{0, \frac{1}{3}} = [-\frac{1}{3}, \frac{1}{3}]$) is a common fixed circle (resp. is a common fixed disc) of the pair (T, S) .

4. An application to rectified linear units activation functions

Activation functions are of great importance in the neural networks to learn and make sense of something. The main aim of these functions is to convert an input signal of a node in the neural networks to an output signal. There are a lot of examples of activation functions used in the neural networks. One of the most popular activation functions is “rectified linear units (ReLU)” (see [7, 9, 10, 30] and the references therein). In this section, we focus on “leakly rectified linear unit (LReLU)” and “parametric rectified linear unit (PReLU)”. Using these activation functions, we give an application of the main result obtained in Section 3. To do this, at first, we recall the notions of LReLU and PReLU, respectively:

$$LReLU(x) = f(x) = \begin{cases} x & ; x > 0 \\ 0.01x & ; x \leq 0 \end{cases}$$

and

$$PReLU(x) = g(x) = \begin{cases} x & ; x > 0 \\ \alpha x & ; x \leq 0 \end{cases} .$$

If $\alpha \leq 1$ then $PReLU(x)$ is equivalent to

$$h(x) = \max(x, \alpha x)$$

and if $\alpha = 0$ then $PReLU(x)$ is a $ReLU$.

Now we consider the activation function $PReLU(x)$ with $\alpha \in (0, 1)$. Then the function satisfies the conditions of Theorem 3.2 on the usual metric space with $x_0 = 1$ and the function $\varphi : X \rightarrow (0, \infty)$ as

$$\varphi(x) = \begin{cases} 1 & ; x = 0 \\ |x| & ; x \neq 0 \end{cases} ,$$

for all $x \in \mathbb{R}$. Indeed, for $x \in (-\infty, 0)$, we get

$$d(x, g(x)) = |x - \alpha x| = |(1 - \alpha)x| > 0,$$

$$R_T(1, x) = \max \left\{ d(1, x), \frac{d(1, g(1))d(x, g(x))}{d(1, x)} \right\} = |1 - x|$$

and

$$\begin{aligned} d(x, g(x)) &= d(x, \alpha x) = |x - \alpha x| = |x| |1 - \alpha| \\ &< |x| |1 - \alpha| |1 - x| = |x| (1 - \alpha) |1 - x| \\ &= (|x| - \alpha |x|) |1 - x| = (|x| - |\alpha x|) |1 - x| \\ &= [\varphi(x) - \varphi(\alpha x)] |1 - x| = [\varphi(x) - \varphi(g(x))] R_T(1, x). \end{aligned}$$

Hence, g is a Jaggi-type bilateral x_0 -contractive mapping. Also, we obtain

$$\begin{aligned} r &= \inf \left\{ \frac{d(x, g(x))}{\varphi(x)} : x \neq g(x), x \in \mathbb{R} \right\} \\ &= \left\{ \frac{|x - \alpha x|}{|x|} : x \in (-\infty, 0) \right\} \\ &= \left\{ \frac{|x| |1 - \alpha|}{|x|} : x \in (-\infty, 0) \right\} = |1 - \alpha| = 1 - \alpha. \end{aligned}$$

Consequently, the activation function $PReLU(x)$ fixes the circle $C_{1,1-\alpha} = \{\alpha, 2 - \alpha\}$ and the disc $D_{1,1-\alpha} = [\alpha, 2 - \alpha]$. On the other hand, if we take $\alpha = 0.01$, then the activation function $LReLU(x)$ fixes the circle $C_{1,0.99} = \{0.01, 1.99\}$ and the disc $D_{1,0.99} = [0.01, 1.99]$.

References

- [1] Aydi H, Taş N, Özgür NY, Mlaiki N. Fixed-discs in rectangular metric spaces. *Symmetry* 2019; 11 (2): 294. doi: 10.3390/sym11020294
- [2] Aydi H, Taş N, Özgür NY, Mlaiki N. Fixed-discs in quasi-metric spaces. *Fixed Point Theory* (to appear).
- [3] Bisht RK, Özgür N. Geometric properties of discontinuous fixed point set of $(\varepsilon - \delta)$ contractions and applications to neural networks. *Aequationes Mathematicae* (accepted). doi: 10.1007/s00010-019-00680-7
- [4] Caristi J. Fixed point theorems for mappings satisfying inwardness conditions. *Transactions of the American Mathematical Society* 1976; 215: 241-251.
- [5] Chen CM, Joonaghany GH, Karapınar E, Khojasteh F. On bilateral contractions. *Mathematics* 2019; 7: 38.
- [6] Dhamodharan D, Taş N, Krishnakumar R. Common fixed point theorems satisfying implicit relations on 2-cone Banach space with an application. *Mathematical Sciences and Applications E-Notes* 2019; 7 (1): 9-18.
- [7] He K, Zhang X, Ren S, Sun J. Delving deep into rectifiers: Surpassing human-level performance on imagenet classification. In: *Proceedings of the IEEE international conference on computer vision 2015*; 1026-1034.
- [8] Khan MS. A fixed point theorem for metric spaces. *Rendiconti dell'Istituto di Matematica dell'Università di Trieste* 1976; 8: 69-72.
- [9] Lee HS, Park HS. A generalization method of partitioned activation function for complex number. arXiv:1802.02987.
- [10] Maas AL, Hannun AY, Ng AY. Rectifier nonlinearities improve neural network acoustic models. *Proceedings of the 30th International Conference on Machine Learning, Atlanta, Georgia, USA, 2013*; 30: 3.
- [11] Mlaiki N, Çelik U, Taş N, Özgür NY, Mukheimer A. Wardowski type contractions and the fixed-circle problem on S -metric spaces. *Journal of Mathematics* 2018; Art. ID 9127486, 9 pp. doi: 10.1155/2018/9127486
- [12] Mlaiki N, Taş N, Özgür NY. On the fixed-circle problem and Khan type contractions. *Axioms* 2018; 7 (4): 80. doi: 10.3390/axioms7040080
- [13] Mlaiki N, Özgür NY, Mukheimer A, Taş N. A new extension of the M_b -metric spaces. *Journal of Mathematical Analysis* 2018; 9 (2): 118-133.
- [14] Mlaiki N, Özgür NY, Taş N. New fixed-point theorems on an S -metric space via simulation functions. *Mathematics* 2019; 7: 583.
- [15] Özgür NY, Taş N. Some fixed-circle theorems on metric spaces. *Bulletin of the Malaysian Mathematical Sciences Society* 2019; 42 (4): 1433-1449. doi: 10.1007/s40840-017-0555-z
- [16] Özgür NY, Taş N. Fixed-circle problem on S -metric spaces with a geometric viewpoint. *Facta Universitatis. Series: Mathematics and Informatics* 2019; 34 (3), 459-472. doi: 10.22190/FUMI1903459O
- [17] Özgür NY, Taş N, Çelik U. New fixed-circle results on S -metric spaces. *Bulletin of Mathematical Analysis and Applications* 2017; 9 (2): 10-23.
- [18] Özgür NY, Taş N. Some fixed-circle theorems and discontinuity at fixed circle. *AIP Conference Proceedings*, 1926, 020048, 2018. doi: 10.1063/1.5020497
- [19] Özgür NY, Mlaiki N, Taş N, Souayah N. A new generalization of metric spaces: rectangular M -metric spaces. *Mathematical Sciences* 2018; 12: 223-233.

- [20] Özgür NY, Taş N. Generalizations of metric spaces: from the fixed-point theory to the fixed-circle theory. In: Rassias T (editor). Applications of Nonlinear Analysis. Springer Optimization and Its Applications, Springer, Cham 2018; 134.
- [21] Özgür NY. Fixed-disc results via simulation functions. Turkish Journal of Mathematics 2019; 43: 2794-2805. doi: 10.3906/mat-1812-44
- [22] Özdemir N, Iskender BB, Özgür NY. Complex valued neural network with Möbius activation function. Communications in Nonlinear Science and Numerical Simulation 2011; 16: 4698-4703.
- [23] Pant RP, Özgür NY, Taş N. On discontinuity problem at fixed point. Bulletin of the Malaysian Mathematical Sciences Society 2020; 43: 499-517. doi: 10.1007/s40840-018-0698-6
- [24] Pant RP, Özgür NY, Taş N. Discontinuity at fixed points with applications. Bulletin of the Belgian Mathematical Society-Simon Stevin 2019; 26: 571-789.
- [25] Piri H, Rahrovi S, Marasi H, Kumam P. A fixed point theorem for F-Khan-contractions on complete metric spaces and application to integral equations. Journal of Nonlinear Science and Applications 2017; 10: 4564-4573.
- [26] Taş N. Suzuki-Berinde type fixed-point and fixed-circle results on S -metric spaces. Journal of Linear and Topological Algebra 2018; 7 (3): 233-244.
- [27] Taş N. Various types of fixed-point theorems on S -metric spaces. Journal of Balıkesir University Institute of Science and Technology 2018; 20 (2): 211-223. doi: 10.25092/baunfbed.426665
- [28] Taş N, Özgür NY, Mlaiki N. New types of F_C -contractions and the fixed-circle problem. Mathematics 2018; 6 (10): 188. doi: 10.3390/math6100188
- [29] Taş N, Özgür NY. Some fixed-point results on parametric N_b -metric spaces. Korean Mathematical Society. Communications 2018; 33 (3): 943-960. doi: 10.4134/CKMS.c170294
- [30] Toth L. Phone recognition with deep sparse rectifier neural networks. In: IEEE International Conference on Acoustics, Speech and Signal Processing 2013; 6985-6989.
- [31] Wardowski D. Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory and Applications 2012; 94: 6 pp.