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Commutator subgroups of generalized Hecke and extended generalized Hecke groups,II

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Abstract: Let $p_1, \cdots, p_n$ be integers where $n \geq 2$ and each $p_i \geq 2$. Let also $H(p_1, \cdots, p_n)$ be the generalized Hecke group associated to all $p_i \geq 2$. In this paper, we study the commutator subgroups $H'(p_1, \cdots, p_n)$ and $\overline{H}'(p_1, \cdots, p_n)$ of the generalized Hecke group $H(p_1, \cdots, p_n)$ and the extended generalized Hecke group $\overline{H}(p_1, \cdots, p_n)$. We give the generators and the signatures of $H'(p_1, \cdots, p_n)$ and $\overline{H}'(p_1, \cdots, p_n)$.

Key words: Generalized Hecke groups, extended generalized Hecke groups, commutator subgroups

1. Introduction

Let $p_1, \cdots, p_n$ be integers where $n \geq 2$ and each $p_i \geq 2$. Let us consider the linear fractional transformations

$$X_i(z) = -\frac{1}{z + \lambda_i} \ ,$$

where $\lambda_i = 2\cos(\frac{\pi}{p_t})$ for $p_i \geq 2$ is an integer. Generalized Hecke groups $H(p_1, \cdots, p_n)$ are generated by $X_i's$ and have the presentation

$$H(p_1, \cdots, p_n) = < X_i : X_i^{p_i} = I > \equiv C_{p_1} * \cdots * C_{p_n}.$$

and the signature $(0;\ p_1, \cdots, p_n, \infty)$, [8] and [9]. Extended generalized Hecke groups $\overline{H}(p_1, \cdots, p_n)$ can be defined by adding the reflection $R(z) = 1/\overline{z}$ to the generators of $H(p_1, \cdots, p_n)$. Hence the extended generalized Hecke groups $\overline{H}(p_1, \cdots, p_n)$ have a presentation

$$\overline{H}(p_1, \cdots, p_n) = < X_i, R : X_i^{p_i} = R^2 = I, \ \ RX_i = X_i^{-1}R >,$$

or

$$\overline{H}(p_1, \cdots, p_n) = < X_i, R : X_i^{p_i} = R^2 = (X_iR)^2 = I > \cong D_{p_1} *_{\mathbb{Z}_2} \cdots *_{\mathbb{Z}_2} D_{p_n}, [8].$$

Notice that the generalized Hecke group $H(2,3)$ is the modular group $\Gamma = PSL(2,\mathbb{Z})$. The modular group is the discrete subgroup of $PSL(2,\mathbb{R})$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad S(z) = -\frac{1}{z+1}.$$

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Then the modular group Γ has a presentation

$$\Gamma = < T, S \mid T^2 = S^3 = I \ > \cong C_2 * C_3.$$

Also, if $q \geq 3$ is an integer, then the generalized Hecke group $H(2,q)$ is the Hecke group $H(\lambda_q)$ ([2], [7], [10], [11], [12]). If $p_1 = p$ and $p_2 = q$ are integers where $2 \leq p \leq q$ and $p+q > 4$, then generalized Hecke group $H(p,q)$ is the generalized Hecke group $H_{p,q}$ ([5], [15], [18]).

On the other hand, the extended generalized Hecke group $\overline{H}(2,3)$ is the extended modular group $\overline{\Gamma}$ (or Π) ([23], [24]). We know that the extended modular group $\Pi = PGL(2,\mathbb{Z})$ is defined by adding the reflection $R(z) = 1/\ \overline{z}$ to the generators of the modular group Γ. The extended modular group Π has a presentation

$$\Pi = < T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I > \cong D_2 *_{\mathbb{Z}_2} D_3.$$

Also, if $q \geq 3$ is an integer, then the extended generalized Hecke group $\overline{H}(2,q)$ is the extended Hecke group $\overline{H}(\lambda_q)$, ([4], [26], [27]). Finally, if $p_1 = p$ and $p_2 = q$ are integers where $2 \leq p \leq q$ and $p+q > 4$, then the extended generalized Hecke group $\overline{H}(p,q)$ is the extended generalized Hecke group $\overline{H}_{p,q}$, ([6]).

The motivation of this paper is to study the commutator subgroups of the generalized Hecke groups $H(p_1, \cdots, p_n)$ and the extended generalized Hecke groups $\overline{H}(p_1, \cdots, p_n)$. If $n = 2$, then the commutator subgroups of $H(p_1, p_2)$ and $\overline{H}(p_1, p_2)$ was studied by many authors in [1], [3], [13], [14], [15], [17], [19], [21], [22], [25], [29].

Here, our aim is to generalize the results given in [14] in the case $p_1 = p$ and $p_2 = q$ where $2 \leq p \leq q$ and $p+q > 4$, to the case $p_1, \cdots, p_n$ are integers where $n \geq 2$ and each $p_i \geq 2$. To do this, we use the Reidemeister–Schreier method, the permutation method (see, [28]) and the extended Riemann–Hurwitz condition (see, [16]). Here we give the generators and the signatures of the commutator subgroups of $H(p_1, \cdots, p_n)$ and $\overline{H}(p_1, \cdots, p_n)$. Of course, if we take $n = 2$, $p_1 = p$ and $p_2 = q$, then our results coincide with the results given in [14] for $H_{p,q}$ and $\overline{H}_{p,q}$.

2. Commutator subgroups of $H(p_1, ..., p_n)$ and $\overline{H}(p_1, ..., p_n)$

First we study the commutator subgroup $H'(p_1, \cdots, p_n)$ of the generalized Hecke group $H(p_1, \cdots, p_n)$.)

Theorem 2.1 *Let $p_1, \cdots, p_n$ be integers where $n \geq 2$ and each $p_i \geq 2$. Then*

i) $|H(p_1, \cdots, p_n) : H'(p_1, \cdots, p_n)| = p_1 \cdot p_2 \cdot \cdots \cdot p_n$.

ii) The commutator subgroup $H'(p_1, \cdots, p_n)$ of $H(p_1, \cdots, p_n)$ is a free group of rank $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (p_i - 1) \cdot (p_j - 1) + 2 \cdot \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} (p_i - 1) \cdot (p_j - 1) \cdot (p_k - 1) + \cdots + (n-1) \cdot \sum_{i=1}^{1} \sum_{j=2}^{2} \cdots \sum_{s=n}^{n} (p_i - 1) \cdot (p_j - 1) \cdots (p_s - 1)$.

Proof *i)* To obtain the quotient group $H(p_1, \cdots, p_n)/H'(p_1, \cdots, p_n)$, we add the relations $X_i X_j = X_j X_i$ where $i, j = 1, \cdots, n$ for $i \neq j$ to the relations of $H(p_1, \cdots, p_n)$. Hence we get

$$H(p_1, \cdots, p_n)/H'(p_1, \cdots, p_n) = < X_i : \ X_i^{p_i} = I, \ X_i X_j = X_j X_i > \cong C_{p_1} \times \cdots \times C_{p_n}.$$

Therefore we find the index as $|H(p_1,\cdots,p_n):H'(p_1,\cdots,p_n)| = p_1\cdot p_2\cdot\cdots\cdot p_n$.

ii) Now we can use the Reidemeister–Schreier method for the generators of $H'(p_1,\cdots,p_n)$. First we choose a Schreier transversal Σ for $H'(p_1,\cdots,p_n)$. Here Σ consists of the identity element I; $\sum\limits_{i=1}^{n}(p_i-1)$ elements of the form $X_i^{a_i}$ where $1\le i\le n$ and $1\le a_i\le p_i-1$; $\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n}(p_i-1)(p_j-1)$ elements of the form $X_i^{a_i}X_j^{a_j}$ where $1\le i<j\le n$ and for $t=i,j,\ 1\le a_t\le p_t-1$; $\sum\limits_{i=1}^{n-2}\sum\limits_{j=i+1}^{n-1}\sum\limits_{k=j+1}^{n}(p_i-1)(p_j-1)(p_k-1)$ elements of the form $X_i^{a_i}X_j^{a_j}X_k^{a_k}$ where $1\le i<j<k\le n$ and for $t=i,j,k,\ 1\le a_t\le p_t-1$; $\cdots$; $\sum\limits_{i=1}^{1}\sum\limits_{j=2}^{2}\cdots\sum\limits_{s=n}^{n}(p_i-1)(p_j-1)\cdots(p_s-1)$ elements of the form $X_1^{a_1}X_2^{a_2}\cdots X_n^{a_n}$ where $1\le t\le n,\ 1\le a_t\le p_t-1$.

Using the Reidemeister–Schreier method, after some calculations, we have the generators of $H'(p_1,\cdots,p_n)$ as follows:

There are $\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n}(p_i-1)(p_j-1)$ generators of the form $[X_i^a,\ X_j^b]$ where $1\le i<j\le n$ and for $t=i,j,\ 1\le a_t\le p_t-1$.

There are $2\sum\limits_{i=1}^{n-2}\sum\limits_{j=i+1}^{n-1}\sum\limits_{k=j+1}^{n}(p_i-1)(p_j-1)(p_k-1)$ generators of the form $[X_i^{a_i},X_j^{a_j}X_k^{a_k}]$ or $[X_i^{a_i}X_j^{a_j},X_k^{a_k}]$ (for the difference, please see the place of the comma) where $1\le i<j<k\le n$ and for $t=i,j,k$, $1\le a_t\le p_t-1$.

If we continue similarly, then we find that there are

$$(n-1)\sum_{i=1}^{1}\sum_{j=2}^{2}\cdots\sum_{s=n}^{n}(p_i-1)(p_j-1)\cdots(p_s-1)$$

generators of the form $[X_1^{a_1},X_2^{a_2}\cdots X_n^{a_n}]$ or $[X_1^{a_1}X_2^{a_2},\cdots,X_n^{a_n}]$ or $[X_1^{a_1}X_2^{a_2}\cdots,X_n^{a_n}]$ where $1\le t\le n$, $1\le a_t\le p_t-1$. Indeed, from [20], the generators are

$$1+p_1p_2\cdots p_n\{-1+\sum_{i=1}^{n}(1-\frac{1}{p_i})\}.$$

Also, using the Riemann–Hurwitz formula and the permutation method, we find the signature of $H'(p_1,\cdots,p_n)$ as

$$(1+\frac{((n-1)-\sum\limits_{i=1}^{n}\frac{1}{p_i})\cdot p_1\cdot p_2\cdots p_n-\frac{p_1\cdot p_2\cdots p_n}{lcm(p_1,p_2,\cdots,p_n)}}{2};\infty^{(\frac{p_1\cdot p_2\cdots p_n}{lcm(p_1,p_2,\cdots,p_n)})}).$$

□

Example 2.2 *Let us consider the generalized Hecke group $H(2,3,4)$. Then we have the index $|H(2,3,4):H'(2,3,4)| = 24$. Now, we set up a Schreier transversal Σ for $H'(2,3,4)$. Σ consists of I; $\sum_{i=1}^{3}(p_i-1)=6$ elements of the form $X_1,\ X_2,\ X_2^2,\ X_3,\ X_3^2, X_3^3$; $\sum_{i=1}^{2}\sum_{j=i+1}^{3}(p_i-1)(p_j-1)=11$ elements of the form $X_1X_2,\ X_1X_2^2,\ X_1X_3,$ $X_1X_3^2,\ X_1X_3^3,\ X_2X_3,\ X_2X_3^2,\ X_2X_3^3,\ X_2^2X_3,\ X_2^2X_3^2,\ X_2^2X_3^3$ and $\sum_{i=1}^{1}\sum_{j=2}^{2}\sum_{k=3}^{3}(p_i-1)(p_j-1)(p_k-1)=6$ elements of the form $X_1X_2X_3,\ X_1X_2X_3^2,\ X_1X_2X_3^3,\ X_1X_2^2X_3,\ X_1X_2^2X_3^2,\ X_1X_2^2X_3^3$. Using the Reidemeister–Schreier method, we find the generators of $H'(2,3,4)$ as follows: $\sum_{i=1}^{2}\sum_{j=i+1}^{3}(p_i-1)(p_j-1)=11$ generators of the form $[X_1,X_2],\ [X_1,X_2^2],\ [X_1,X_3],\ [X_1,X_3^2],\ [X_1,X_3^3],\ [X_2,X_3],\ [X_2,X_3^2],\ [X_2,X_3^3],\ [X_2^2,X_3],$ $[X_2^2,X_3^2],\ [X_2^2,X_3^3]$; $2\sum_{i=1}^{1}\sum_{j=2}^{2}\sum_{k=3}^{3}(p_i-1)(p_j-1)(p_k-1)=12$ generators of the form $[X_1X_2,X_3],\ [X_1X_2,X_3^2],$ $[X_1X_2,X_3^3],\ [X_1X_2^2,X_3],\ [X_1X_2^2,\ X_3^2],\ [X_1X_2^2,X_3^3],\ [X_1,X_2X_3],\ [X_1,X_2X_3^2],\ [X_1,X_2X_3^3],\ [X_1,X_2^2X_3],\ [X_1,X_2^2X_3^2],$ $[X_1,X_2^2X_3^3]$. Finally, using the Riemann–Hurwitz formula, we find the signature of $H'(2,3,4)$ as $(11;\ \infty^{(2)})$.*

Now, we study the commutator subgroups $\overline{H}'(p_1,\cdots,p_n)$ of the extended generalized Hecke groups $\overline{H}(p_1,\cdots,p_n)$. To do this, firstly, we rename the generators X_i of the extended generalized Hecke group $\overline{H}(p_1,\cdots,p_n)$. Let s,t and u be the number of the generators X_i of order 2, of even order ≥ 4 and of odd order ≥ 3 in $\overline{H}(p_1,\cdots,p_n)$, respectively. Let us denote the generators of order 2 by $A_1,A_2,\cdots,A_s$; generators of even order ≥ 4 by $B_1,B_2,\cdots,B_t$; and generators of odd order ≥ 3 by $C_1,C_2,\cdots,C_u$. Also, let q_k and r_l be the orders of the generators B_k and C_l, respectively, where $0\leq k\leq t$ and $0\leq l\leq u$. Then, the extended generalized Hecke group $\overline{H}(p_1,\cdots,p_n)$ has a presentation

$$< A_j,B_k,C_l,R : A_j^2=B_k^{q_k}=C_l^{r_l}=R^2=I,\quad RA_j=A_jR,\ RB_k=B_k^{-1}R,\ RC_l=C_l^{-1}R > .$$

Therefore, we can write as

$$\overline{H}(p_1,\cdots,p_n)=\overline{H}(\underbrace{2,2,\cdots,2}_{s\text{ times}},q_1,q_2,\cdots,q_t,r_1,r_2,\cdots,r_u)$$

Thus, we can give the following result:

Theorem 2.3 *Let $p_1,\cdots,p_n$ be integers for $n\geq 2$ and $p_i\geq 2$. Then*

i) $\left[\overline{H}(p_1,\cdots,p_n):\ \overline{H}'(p_1,\cdots,p_n)\right]=2^{s+t+1}$.

ii) The commutator subgroup $\overline{H}'(p_1,\cdots,p_n)$ is a group of generators $\sum_{j=2}^{s}(j-1)\binom{s}{j}+\sum_{k=1}^{t}(2k-1)\binom{t}{k}+\sum_{j=1}^{s}\sum_{k=1}^{t}(2k+j-1)\binom{s}{j}\binom{t}{k}+2^{s+t}u$.

Proof i) The quotient group $\overline{H}(p_1,\cdots,p_n)/\overline{H}'(p_1,\cdots,p_n)$ is the group obtained by adding the relations $RB_k=B_kR,\ RC_l=C_lR$, where $0\leq k\leq t$ and $0\leq l\leq u$ to the relations in $\overline{H}(p_1,\cdots,p_n)$. Thus the quotient group $\overline{H}(p_1,\cdots,p_n)/\overline{H}'(p_1,\cdots,p_n)$ is

$$\begin{aligned}&< A_j, B_k, C_l, R : A_j^2 = B_k^{q_k} = C_l^{r_l} = R^2 = I,\ \ RA_j = A_jR,\ RB_k = B_k^{-1}R,\\ &\qquad\qquad\qquad\qquad RC_l = C_l^{-1}R\ ,\ RB_k = B_kR,\ RC_l = C_lR >\\ &\cong < A_j, B_k, R : A_j^2 = B_k^{q_k} = C_l^{r_l} = R^2 = (A_jR)^2 = (B_kR)^2 = (C_lR)^2 = I > .\end{aligned}$$

From the relations $RB_k = B_k^{-1}R;\ RB_k = B_kR$ and $RC_l = C_l^{-1}R;\ RC_l = C_lR$, we find $B_k^2 = C_l^2 = I$. Also from the relations $B_k^{q_k} = C_l^{r_l} = B_k^2 = C_l^2 = I$, we have $B_k^2 = I$ and $C_l = I$ since q_k is even number and r_l is odd number. Thus, we get

$$\begin{aligned}\overline{H}(p_1,\cdots,p_n)/\overline{H}'(p_1,\cdots,p_n) &= < A_j, B_k, R : A_j^2 = B_k^2 = R^2 = (A_jR)^2 = (B_kR)^2 = I >\\ &= \underbrace{C_2 \times \cdots \times C_2}_{s+t \text{ times}} \times C_2. \qquad (2.1)\end{aligned}$$

Therefore, we find the index as

$$\left[\overline{H}(p_1,\cdots,p_n) : \overline{H}'(p_1,\cdots,p_n)\right] = 2^{s+t+1}.$$

ii) Now, we can determine the Schreier transversal Σ. To do this, we use the set $M = \{A_1,\cdots,A_s, B_1,\cdots,B_t\}$. It is clear that there are $2^{s+t}-1$ subsets of M, except null set. Using the elements of these subsets, we can obtain the elements of Σ. For example, if $\{A_2,\ A_3,\ A_5,\ B_2,\ B_3\}$ is a subset of M, then $A_2A_3A_5B_2B_3$ is an element of Σ, (since the quotient group is abelian, these elements can be written as alphabetically and numerically ordered). Thus, we can obtain $2^{s+t}-1$ elements in Σ. If we multiply these $2^{s+t}-1$ elements by R (for example, $A_2A_3A_5B_2B_3R$), then we have $2^{s+t}-1$ new elements of Σ. Also, the elements I and R are in Σ. Consequently there are $2^{s+t}-1+2^{s+t}-1+2 = 2^{s+t+1}$ elements in Σ. Notice that if $s=t=0$, then Σ consists of only the elements I and R. Using the Reidemeister-Schreier method, after required calculations, we get the generators of $\overline{H}'(p_1,\cdots,p_n)$ as follows: Notice that $A_j^{-1} = A_j$ and $B_k^{-1} \neq B_k$. If $s \geq 2$, then there are $\binom{s}{2}$ generators of the form $A_dA_eA_dA_e$ where $1 \leq d < e \leq s$; $2\cdot\binom{s}{3}$ generators of the form $A_dA_eA_fA_d(A_eA_f)^{-1}$ or $A_dA_eA_fA_e(A_dA_f)^{-1}$; $\cdots$ $(s-1)\cdot\binom{s}{s}$ generators of the form $A_1A_2\cdots A_sA_1(A_2\cdots A_s)^{-1}$, or $A_1A_2\cdots A_sA_2(A_1A_3\cdots A_s)^{-1}$, or $\cdots$ or $A_1A_2\cdots A_sA_{s-1}(A_1\cdots A_{s-2}A_s)^{-1}$. If $t \geq 1$, then there are $1\cdot\binom{t}{1}$ generators of the form B_g^2 where $1 \leq g \leq t$; $3\cdot\binom{t}{2}$ generators of the form $B_gB_hB_gB_h^{-1}$, or $B_gB_hB_g^{-1}B_h^{-1}$ or $B_gB_h^2B_g^{-1}$ where $1 \leq g < h \leq t$; $5\cdot\binom{t}{3}$ generators of the form $B_gB_hB_mB_g(B_hB_m)^{-1}$, or $B_gB_hB_mB_g^{-1}(B_hB_m)^{-1}$, or $B_gB_hB_mB_h(B_gB_m)^{-1}$, or $B_gB_hB_mB_h^{-1}(B_gB_m)^{-1}$, or $B_gB_hB_m^2(B_gB_m)^{-1}$ where $1 \leq g < h < m \leq t$; $\cdots$, $(2t-1)\cdot\binom{t}{t}$ generators of the form $B_1B_2\cdots B_tB_1(B_2\cdots Y_t)^{-1}$ or $B_1B_2\cdots B_tB_1^{-1}(B_2\cdots Y_t)^{-1}$ or $B_1B_2\cdots B_tB_2(B_1B_3\cdots B_t)^{-1}$ or $B_1B_2\cdots B_tB_2^{-1}(B_1B_3\cdots B_t)^{-1}$ or $\cdots$ or $B_1B_2\cdots B_tB_{t-1}(B_1\cdots B_{t-2}B_t)^{-1}$

or $B_1B_2\cdots B_tB_{t-1}^{-1}(B_1\cdots B_{t-2}B_t)^{-1}$ or $B_1B_2\cdots B_t^2(B_1B_2\cdots B_{t-1})^{-1}$. If $s\geq 1$ and $t\geq 1$, then there are $2\cdot\binom{s}{1}\binom{t}{1}$ generators of the form $A_dB_gA_dB_g^{-1}$, or $A_dB_g^2A_d$ where $1\leq d\leq s$ and $1\leq g\leq t$; $3\cdot\binom{s}{2}\binom{t}{1}$ generators of the form $A_dA_eB_gA_d(A_eB_g)^{-1}$ or $A_dA_eB_gA_e(A_dB_g)^{-1}$ or $A_dA_eB_g^2(A_dA_e)^{-1}$ where $1\leq d<e\leq s$ and $1\leq g\leq t$; $4\cdot\binom{s}{1}\binom{t}{2}$ generators of the form $A_dB_gB_hA_d(B_gB_h)^{-1}$ or $A_dB_gB_hB_g(A_dB_h)^{-1}$ or $A_dB_gB_hB_g^{-1}(A_dB_h)^{-1}$ or $A_dB_gB_h^2(A_dB_g)^{-1}$ where $1\leq d\leq s$ and $1\leq g<h\leq t$; $\cdots$; $(2t+s-1)\cdot\binom{s}{s}\binom{t}{t}$ generators of the form

$$\begin{array}{l}
A_1A_2\cdots A_sB_1B_2\cdots B_tA_1(A_2\cdots A_sB_1B_2\cdots B_t)^{-1} \text{ or}\\
A_1A_2\cdots A_sB_1B_2\cdots B_tA_2(A_1A_3\cdots A_sB_1B_2\cdots B_t)^{-1} \text{ or}\\
\vdots\\
A_1A_2\cdots A_sB_1B_2\cdots B_tA_s(A_1A_2\cdots A_{s-1}B_1B_2\cdots B_t)^{-1} \text{ or}\\
A_1A_2\cdots A_sB_1B_2\cdots B_tB_1(A_1A_2\cdots A_sB_2\cdots B_t)^{-1} \text{ or}\\
A_1A_2\cdots A_sB_1B_2\cdots B_tB_1^{-1}(A_1A_2\cdots A_sB_2\cdots B_t)^{-1} \text{ or}\\
A_1A_2\cdots A_sB_1B_2\cdots B_tB_2(A_1A_2\cdots A_sB_1B_3\cdots B_t)^{-1} \text{ or}\\
A_1A_2\cdots A_sB_1B_2\cdots B_tB_2^{-1}(A_1A_2\cdots A_sB_1B_3\cdots B_t)^{-1} \text{ or}\\
\vdots\\
A_1A_2\cdots A_sB_1B_2\cdots B_tB_{t-1}(A_1A_2\cdots A_sB_1\cdots B_{t-2}B_t)^{-1} \text{ or}\\
A_1A_2\cdots A_sB_1B_2\cdots B_tB_{t-1}^{-1}(A_1A_2\cdots A_sB_1\cdots B_{t-2}B_t)^{-1} \text{ or}\\
A_1A_2\cdots A_sB_1B_2\cdots B_t^2(A_1A_2\cdots A_sB_1\cdots B_{t-2}B_{t-1})^{-1}.
\end{array}$$

Also there are $u\cdot\binom{s+t}{0}$ generators of the form C_l where $1\leq l\leq u$; $u\cdot\binom{s+t}{1}$ generators of the form $A_dC_lA_d$ or $B_gC_lB_g^{-1}$ where $1\leq d\leq s$, $1\leq g\leq t$ and $1\leq l\leq u$; $u\cdot\binom{s+t}{2}$ generators of the form $A_dA_eC_l(A_dA_e)^{-1}$ or $B_gB_hC_l(B_gB_h)^{-1}$ or $A_dB_gC_l(A_dB_g)^{-1}$ where $1\leq d(<e)\leq s, 1\leq g(<h)\leq t$ and $1\leq l\leq u$; $u\cdot\binom{s+t}{3}$ generators of the form $A_dA_eA_fC_l(A_dA_eA_f)^{-1}$ or $A_dA_eB_gC_l(A_dA_eB_g)^{-1}$ or $A_dB_gB_hC_l(A_dB_gB_h)^{-1}$ or $B_gB_hB_mC_l(B_gB_hB_m)^{-1}$ where $1\leq d(<e(<f))\leq s, 1\leq g(<h(<m))\leq t$ and $1\leq l\leq u$; $\cdots$; $u\cdot\binom{s+t}{s+t}$ generators of the form $A_1A_2\cdots A_sB_1B_2\cdots B_tC_l(A_1A_2\cdots A_sB_1B_2\cdots B_t)^{-1}$ where $1\leq l\leq u$.

Also, using the Riemann–Hurwitz formula and permutation method, the signature of $\overline{H}'(p_1,\cdots,p_n)$ is

$$\begin{cases}
(1+\frac{2^{s+t-1}(s+t-3)}{2};(q_k/2)^{(2^{s+t-1})},{r_l}^{(2^{s+t})},\infty^{(2^{s+t-1})}), & \text{if } s\geq 1 \text{ and } t\geq 1,\\
(0;{r_l}^{(2)},\infty), & \text{if } s=1 \text{ and } t=0,\\
(0;(q_1/2),{r_l}^{(2)},\infty), & \text{if } s=0 \text{ and } t=1,\\
(0;r_l,\infty), & \text{if } s=0 \text{ and } t=0,
\end{cases} \tag{2.2}$$

where $0\leq k\leq t$ and $0\leq l\leq u$. □

Example 2.4 *Let us consider the generalized Hecke group* $\overline{H}(2,2,3,4,5)$. *Since* $s=2$, $t=1$ *and* $u=2$, *we take the generators as* A_1, A_2, B_1, C_1, C_2. *Here, there are the relations* $A_1^2=A_2^2=B_1^4=C_1^3=C_2^5$.

Then the index is 16 *and the Schreier transversal* Σ *is* $\{A_1,\ A_2,\ B_1,\ A_1A_2,\ A_1B_1,\ A_2B_1,\ A_1A_2B_1,\ A_1R,\ A_2R,\ B_1R,\ A_1A_2R,\ A_1B_1R,\ A_2B_1R,\ A_1A_2B_1R,\ I,\ R\}$*. If we use the Reidemeister–Schreier method and make the required calculations, then we get one generator of the form* $A_1A_2A_1A_2$*; one generator of the form* B_1^2*; four generators of the form* $A_1B_1A_1B_1^{-1},\ A_1B_1^2A_1,\ A_2B_1A_2B_1^{-1},\ A_2B_1^2A_2$*; three generators of the form* $A_1A_2B_1A_1(A_2B_1)^{-1},\ A_1A_2B_1A_2(A_1B_1)^{-1},\ A_1A_2B_1^2(A_1A_2)^{-1}$*; two generators of the form* C_1*,* C_2*; six generators of the form* $A_1C_1A_1,\ A_2C_1A_2,\ A_1C_2A_1,\ A_2C_2A_2,\ B_1C_1B_1^{-1},\ B_1C_2B_1^{-1}$*; six generators of the form* $A_1A_2C_1(A_1A_2)^{-1},\ A_1A_2C_2(A_1A_2)^{-1},\ A_1B_1C_1(A_1B_1)^{-1},\ A_1B_1C_2(A_1B_1)^{-1},\ A_2B_1C_1(A_2B_1)^{-1},\ A_2B_1C_2(A_2B_1)^{-1}$ *and two generators of the form* $A_1A_2B_1C_1(A_1A_2B_1)^{-1},\ A_1A_2B_1C_2(A_1A_2B_1)^{-1}$*. Totally, there are* 25 *generators of* $\overline{H}'(2,2,3,4,5)$*. Also, the signature of* $\overline{H}'(2,2,3,4,5)$ *is* $(1;2^{(4)},3^{(8)},5^{(8)},\infty^{(4)})$ *.*

Example 2.5 *Let us consider the generalized Hecke group* $\overline{H}(5,5,7,8)$*. Since* $s=0$*,* $t=1$ *and* $u=3$*, we take the generators as* $B_1,\ C_1,\ C_2,\ C_3$*. Thus we have the relations* $B_1^8=C_1^5=C_2^5=C_3^7$*. Here the index is* 4*. Then we can determine the Schreier transversal* $\Sigma=\{B_1,B_1R,I,R\}$*. If we use the Reidemeister–Schreier method and make the required calculations, then we find the generators of* $\overline{H}'(5,5,7,8)$ *as one generator of the form* B_1^2*; three generators of the form* $C_1,\ C_2,\ C_3$*; and finally, three generators of the form* $B_1C_1B_1^{-1}$*,* $B_1C_2B_1^{-1}$*,* $B_1C_2B_1^{-1}$*. Therefore, there are seven generators of* $\overline{H}'(5,5,7,8)$*. Also, we obtain the signature of* $\overline{H}'(5,5,7,8)$ *as* $(0;5^{(4)},7^{(2)},4,\infty)$*.*

From Eq. (2.1), if $s+t\leq 1$, then the commutator subgroup $\overline{H}'(p_1,\cdots,p_n)$ is isomorphic to the free product of some finite cyclic groups. Thus, we can study the second commutator subgroup $\overline{H}''(p_1,\cdots,p_n)$ using Theorem 1.1:

Corollary 2.6 *i) If* $s=0$ *and* $t=0$*, then* $\overline{H}''(p_1,\cdots,p_n)(\cong H'(p_1,\cdots,p_n))$ *is a free group of rank* $1+r_1\cdot r_2\cdots r_u\{-1+\sum_{l=1}^{u}(1-\frac{1}{r_l})\}$*.*

ii) If $s=1$ *and* $t=0$*, then* $\overline{H}''(p_1,\cdots,p_n)$ *is a free group of rank* $1+r_1^2\cdot r_2^2\cdots r_u^2\{-1+2\sum_{l=1}^{u}(1-\frac{1}{r_l})\}$*.*

iii) If $s=0$ *and* $t=1$*, then* $\overline{H}''(p_1,\cdots,p_n)$ *is a free group of rank* $1+(q_1/2)r_1^2\cdot r_2^2\cdots r_u^2\{-\frac{2}{q_1}+2\sum_{l=1}^{u}(1-\frac{1}{r_l})\}$*.*

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