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The New Derivation for Wreath Products of Monoids

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Abstract. We first define a new consequence of the (restricted) wreath product for arbitrary two monoids. After that we give a generating and relator set for this new wreath product. Then we denote some finite and infinite applications about it. At the final part of this paper we show that this product satisfies the periodicity and regularity under some conditions.

1. Introduction and Preliminaries

Throughout this paper A and B will always denote arbitrary monoids unless stated otherwise.

In [7, Theorem 2.2], it has been defined a standard presentation for the wreath product of A by B in the meaning of restricted. Also, in [14, Theorem 7.1], it has been showed that the wreath product of semigroups satisfies the periodicity when these semigroups are periodic. In here, we purpose to introduce a new derivation for the wreath product of A and B. Also, we aim to give a presentation for this new type of wreath products. Finally, we will show that this product satisfies the property of periodicity (as in [14]) and regularity (as in [12, 15]) under some conditions.

For the monoids A and B, it is well known that while $A^{\times B}$ denotes the cartesian product of the number of B copies of the monoid A, the set $A^{\oplus B}$ defines the corresponding direct product. Recall that $A^{\oplus B}$ can be thought as the set of whole functions f with finite support (in other words, functions with the property $(x)f = 1_A$ for all but finitely many x in B). Then the (un)restricted wreath product of A by B is defined on the set $A^{\oplus B} \times B$ (or the set $A^{\times B} \times B$ for unrestricted case) with the operation $(f, b)(g, b') = (f\,{}^{b}g, bb')$ such that ${}^{b}g : B \to A$ is given by $(x)^{b}g = (xb)g$ where $x \in B$. With the identity $(\overline{1}, 1_B)$, where $(x)\overline{1} = 1_A$ for all $x \in B$, it is not hard to show that wreath products are monoids. Throughout this paper we will assume restricted when we refer the term wreath products. For more preliminaries and properties over these products, we may refer [4, 8, 11, 13, 14].

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2. A new type of wreath products over monoids

Let A and B be monoids. We recall that $A^{\oplus B}$ and $B^{\oplus A}$ are the sets of all functions having finite support. Now to use in our calculations at the rest of this paper, for $a \in A$ and $b \in B$, let us define $\overline{a_b} : B \to A$ by

$$c\overline{a_b} = \begin{cases} a\,; & \text{if } c = b \\ 1_A\,; & \text{otherwise} \end{cases} . \tag{1}$$

Let us consider the classical operation $P_1P_2 = (a_1a_2, b_1b_2)$ of any two elements $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$ in $A \times B$. The new consequence (or type) for the wreath product of A and B, notated by $A \bowtie B$, is defined on the set $A^{\oplus B} \times (A \times B) \times B^{\oplus A}$ with the multiplication $(f, P_1, g)(h, P_2, k) = (f\,{}^{b_1}h, P_1P_2, g^{a_2}k)$, where ${}^{b_1}h : B \to A$ and $g^{a_2} : A \to B$ are defined by $(y)^{b_1}h = (yb_1)h$ $(y \in B)$ and $(x)g^{a_2} = (a_2x)g$ $(x \in A)$. Actually it is not a big deal to show that $A \bowtie B$ is a monoid with the identity element $(\overline{1}, (1_A, 1_B), \widetilde{1})$, where $\overline{1}$ and $\widetilde{1}$ are defined by $(b)\overline{1} = 1_A$ and $(a)\widetilde{1} = 1_B$, respectively, for all $b \in B$ and $a \in A$. Now, in the following, we will state and proof a generating set (Lemma 2.1 below) and a relator set (Theorem 2.2 below) of the product $A \bowtie B$ as one of the results in this paper.

Lemma 2.1. *Assume that the sets X and Y generate the monoids A and B, respectively. Also, for each $a \in A$ and $b \in B$, let us denote*

$$\overline{X_b} = \{(\overline{x_b}, (1_A, 1_B), \widetilde{1}) : x \in X\}\,,\ \widetilde{Y_a} = \{(\overline{1}, (1_A, 1_B), \widetilde{y_a}) : y \in Y\} \quad \textit{and} \quad P = \{(\overline{1}, (c, d), \widetilde{1}) : c \in A,\ d \in B\}\,.$$

Therefore the monoid $A \bowtie B$ is generated by the set $(\bigcup_{b\in B} \overline{X_b}) \cup (\bigcup_{a\in A} \widetilde{Y_a}) \cup P$.

Proof. Let us consider a function $\overline{x_b}$ from B to A as defined in (1), and with a similar approach let us also define a function $\widetilde{y_a} : A \to B$ by

$$c\widetilde{y_a} = \begin{cases} y\,; & \text{if } a = c \\ 1_B\,; & \text{otherwise} \end{cases} .$$

For $x, x' \in X$, $y, y' \in Y$, $a_1, a_2 \in A$, $b_1, b_2 \in B$, P_1, $P_2 \in A \times B$, we can easily show that the proof follows from the equalities

$$(\overline{x_{b_1}}, (1_A, 1_B), \widetilde{1})(\overline{x'_{b_2}}, (1_A, 1_B), \widetilde{1}) = (\overline{x_{b_1}}\,{}^{1_B}\overline{x'_{b_2}}, (1_A, 1_B), \widetilde{1}^{1_A}\,\widetilde{1}) = (\overline{x_{b_1}}\overline{x'_{b_2}}, (1_A, 1_B), \widetilde{1}), \tag{2}$$

$$(\overline{1}, (1_A, 1_B), \widetilde{y_{a_1}})(\overline{1}, (1_A, 1_B), \widetilde{y_{a_2}}) = (\overline{1}\,{}^{1_B}\overline{1}, (1_A, 1_B), \widetilde{y_{a_1}}^{1_A}\,\widetilde{y_{a_2}}) = (\overline{1}, (1_A, 1_B), \widetilde{y_{a_1}}\,\widetilde{y_{a_2}}), \tag{3}$$

$$(\overline{1}, P_1, \widetilde{1})(\overline{1}, P_2, \widetilde{1}) = (\overline{1}\,{}^{1_B}\overline{1}, P_1P_2, \widetilde{1}^{1_A}\,\widetilde{1}) = (\overline{1}, P_1P_2, \widetilde{1}), \tag{4}$$

$$(\overline{x_b}, (1_A, 1_B), \widetilde{1})(\overline{1}, (c, d), \widetilde{1})(\overline{1}, (1_A, 1_B), \widetilde{y_a}) = (\overline{x_b}, P, \widetilde{y_a})\,,$$

as required. □

We then prove the following result.

Theorem 2.2. *Assume $[X; R]$ and $[Y; S]$ are presentations of A and B, respectively. For any elements $a \in A$ and $b \in B$, let $X_b = \{x_b : x \in X\}$ and $Y_a = \{y_a : y \in Y\}$ be the corresponding copies of the sets X and Y whereas R_b and S_a be the corresponding copies of the sets R and S, respectively. Then the product $A \bowtie B$ is defined by generators*

$$Z = (\bigcup_{b\in B} X_b) \cup (\bigcup_{a\in A} Y_a) \cup \{z_{c,d} : c \in A, d \in B\}$$

and relations

$$R_b\ (b \in B),\ S_a\ (a \in A); \tag{5}$$

$$x_b x'_e = x'_e x_b \quad (x, x' \in X, b, e \in B,\ b \neq e); \tag{6}$$
$$y_a y'_c = y'_c y_a \quad (y, y' \in Y, a, c \in A,\ a \neq c); \tag{7}$$
$$x_b y_a = y_a x_b \quad (x \in X, y \in Y, a \in A,\ b \in B); \tag{8}$$

$$z_{c,d} x_b = (\prod_{m' \in bd^{-1}} x_{m'}) z_{c,d} \quad (x \in X,\ c \in A,\ b, d \in B); \tag{9}$$
$$y_a z_{c,d} = z_{c,d} (\prod_{n' \in c^{-1}a} y_{n'}) \quad (y \in Y,\ a, c \in A,\ d \in B). \tag{10}$$

Before giving the proof we first recall that, for a set of alphabet Z, the monoid of all words in Z is notated by Z^*. Now, for just simplicity, let us denote the set $\{m' \in B : b = m'd\}$ by bd^{-1} and the set $\{n' \in A : a = cn'\}$ by $c^{-1}a$, where $b, d \in B$ and $a, c \in A$.

Proof. Let us consider a monoid homomorphism $\theta : Z^* \longrightarrow A \bowtie B$ defined by $(x_b)\theta = (\overline{x_b}, (1_A, 1_B), \widetilde{1})$ $(x \in X,\ b \in B)$, $(y_a)\theta = (\overline{1}, (1_A, 1_B), \widetilde{y_a})$ $(y \in Y, a \in A)$ and $(z_{c,d})\theta = (\overline{1}, (c, d), \widetilde{1})$ $(c \in A, d \in B)$. In fact θ is onto by Lemma 2.1. Now we need to show that $A \bowtie B$ satisfies relations from (5) to (10). However, it is clear that relations (5), (6) and (7) follow from (2), (3) and (4).

Now consider again the operation in Section 2 to obtain the remaining relations. For the relation in (8), we have $(\overline{x_b}, (1_A, 1_B), \widetilde{1})(\overline{1}, (1_A, 1_B), \widetilde{y_a}) = (\overline{x_b}, (1_A, 1_B), \widetilde{y_a}) = (\overline{1}, (1_A, 1_B), \widetilde{y_a})(\overline{x_b}, (1_A, 1_B), \widetilde{1})$. On the other hand, to show the existence of relations (9) and (10), we need to use the equalities $(\overline{1}, (c, d), \widetilde{1})(\overline{x}_b, (1_A, 1_B), \widetilde{1}) = ({}^d\overline{x_b}, (c, d), \widetilde{1})$ and $(\overline{1}, (1_A, 1_B), \widetilde{y_a})(\overline{1}, (c, d), \widetilde{1}) = (\overline{1}, (c, d), \widetilde{y_a^c})$. In fact, for each $e \in B$, we can write

$$(e)\ {}^d\overline{x_b} = (ed)\overline{x_b} = \begin{cases} x, & b = ed \\ 1_B, & \text{otherwise} \end{cases} = \begin{cases} x, & e \in bd^{-1} \\ 1_B, & \text{otherwise} \end{cases} = \prod_{m' \in bd^{-1}} e\overline{x_{m'}} = e(\prod_{m' \in bd^{-1}} \overline{x_{m'}}).$$

So we have ${}^d\overline{x_b} = \prod_{m' \in bd^{-1}} \overline{x_{m'}}$. Hence $(\overline{1}, (c, d), \widetilde{1})(\overline{x}_b, (1_A, 1_B), \widetilde{1}) = (\prod_{m' \in bd^{-1}} (\overline{x}_b, (1_A, 1_B), \widetilde{1}))(\overline{1}, (c, d), \widetilde{1})$, for all $x \in X, c \in A, b, d \in B$.

By a similar argument, we also obtain $(\overline{1}, (1_A, 1_B), \widetilde{y_a})(\overline{1}, (c, d), \widetilde{1}) = (\overline{1}, (c, d), \widetilde{1})(\prod_{n' \in c^{-1}a} (\overline{1}, (1_A, 1_B), \widetilde{y_a}))$. Hence we obtain that there exists an epimorphism $\overline{\theta} : M \to A \bowtie B$ induced by θ which is defined by the relations given in (5)-(10). Let us consider a nontrivial word $w \in Z^*$. Using relations from (6) to (10), we can see that there exist some words $w(b)$ in X^* $(b \in B)$, $w(a)$ in Y^* $(a \in A)$ and $w' \in \{z_{c,d} : c \in A, d \in B\}^*$ such that $w = (\prod_{b \in B} (w(b))_b) w' (\prod_{a \in A} (w(a))_a)$ in M. We note that relations from (6) to (10) can be used to show that there exists a set $T_w \subseteq A \times B$ such that $w' = \prod_{(c,d) \in T_w} z_{c,d}$. Depending on that, let us define $P_w = \prod_{(c,d) \in T_w} (c, d)$. As a result of this, for any word $w \in Z^*$, we have

$$\begin{aligned} (w)\theta &= ((\prod_{b \in B} (w(b))_b) w' (\prod_{a \in A} (w(a))_a))\theta = (\prod_{b \in B} \overline{(w(b))}_b, (1_A, 1_B), \widetilde{1})(\overline{1}, P_w, \widetilde{1})(\overline{1}, (1_A, 1_B), \prod_{a \in A} (\widetilde{w(a)})_a) \\ &= (\prod_{b \in B} \overline{(w(b))}_b, P_w, \prod_{a \in A} (\widetilde{w(a)})_a)). \end{aligned}$$

For each $w \in X^* \cup Y^*$ and for each $c \in A$, $d \in B$, we then have $d\overline{w_b} = \begin{cases} w; & \text{if } d = b \\ \iota; & \text{otherwise} \end{cases}$ and $c\widetilde{w}_a = \begin{cases} w; & \text{if } c = a \\ \iota; & \text{otherwise} \end{cases}$, where ι denotes empty word. Hence $d(\prod_{b \in B} \overline{(w(b))}_b) = \prod_{b \in B} d\overline{(w(b))}_b = w(d)$ and $c(\prod_{a \in A} (\widetilde{w(a)})_a) = \prod_{a \in A} c(\widetilde{w(a)})_a = w(c)$. Therefore, for some $w_1, w_2 \in Z^*$, if $(w_1)\theta = (w_2)\theta$ then, by the equality of these components, we deduce that $w_1(d) = w_2(d)$ in A for every d in B, $w_1(c) = w_2(c)$ in B for every c in A, and $P_{w_1} = P_{w_2}$. Relations in (5) imply that $w_1(d) = w_2(d)$ and $w_1(c) = w_2(c)$ hold in M, so that $w_1 = w_2$ holds as well. Therefore $\overline{\theta}$ is injective. These complete the proof. □

3. Some other applications

As an application of the Theorem 2.2, our aim in this section is to give an explicit presentation for this new type of wreath product while A and B are some special monoids.

3.1. Case I: A finite example

In this case, we actually will consider out new product on finite cyclic (monogenic) monoids in which some examples, applications and algebraic structures about these monoids can be found, for instance, in [2]. So let A and B be two such monoids having presentations $\mathcal{P}_A = [x\,;\, x^k = x^l (k > l)]$ and $\mathcal{P}_B = [y\,;\, y^s = y^t (s > t)]$, respectively. Therefore we have the following result as an application of Theorem 2.2.

Corollary 3.1. *The product $A \bowtie B$ has a presentation*

$$\begin{aligned}
\mathcal{P}'_{A\bowtie B} \;=\; & [x^{(i)},\, y^{(j)},\, z_{x^m,y^n}\,;\, x^{(i)}x^{(p)} = x^{(p)}x^{(i)}\ (i < p),\ y^{(j)}y^{(q)} = y^{(q)}y^{(j)}\ (j < q),\\
& x^{(i)^k} = x^{(i)^l},\ y^{(j)^s} = y^{(j)^t}, x^{(i)}y^{(j)} = y^{(j)}x^{(i)}\ (0 \le i,n,p \le s-1, 0 \le j,q,m \le k-1),\\
& z_{x^m,y^n}x^{(i)} = x^{(i-n)}z_{x^m,y^n}\ (0 \le n \le i \le t-1),\\
& z_{x^m,y^n}x^{(t+i)} = x^{(s+i-n)}z_{x^m,y^n}\ (i = 0,1,\cdots,s-t-1), z_{x^m,y^n}x^{(i)} = z_{x^m,y^n}\ (0 \le i \le t-1 < n),\\
& y^{(j)}z_{x^m,y^n} = z_{x^m,y^n}y^{(j-m)}\ (0 \le m \le j \le l-1),\\
& y^{(l+j)}z_{x^m,y^n} = z_{x^m,y^n}y^{(k+j-m)}\ (j = 0,1,\cdots,k-l-1), y^{(j)}z_{x^m,y^n} = z_{x^m,y^n}\ (0 \le j \le l-1 < m)].
\end{aligned}$$

Proof. Now let us consider the relators (9) and (10) in Theorem 2.2. For the sake of simplicity, let us label y_{x^q} by $y^{(q)}$, where each x^q is the representative element A, and label x_{y^p} by $x^{(p)}$, where each y^p is the representative element B such that $0 \le q \le k-1$ and $0 \le p \le s-1$.

We note that since $d \in B$ in (9), we can take it as y^n in this case. So, for $0 \le n \le i \le t-1$, let us think the relator $z_{c,d}x_b = (\prod_{m'\in bd^{-1}} x_{m'})z_{c,d}$, where $c = x^m, d = y^n$ and $b = y^i$. Since we have $m' \in bd^{-1}$ such that $b = m'd$, we get $y^i = m'y^n$. So we have $m' = y^{i-n}$. Thus we obtain the relator $z_{x^m,y^n}x^{(i)} = x^{(i-n)}z_{x^m,y^n}$. Moreover, for the monoid B, since we have $y^s = y^t$ in $\mathcal{P}_B$ as a relator, we have $y^{s+i} = y^{t+i}$ where $i = 0,1,\cdots,s-t-1$. Let us think $y^{t+i} = y^{s+i-n}y^n$. In here, if we take $b = y^{t+i}$ and $m = y^{s+i-n}$ then we certainly have $z_{x^m,y^n}x^{(t+i)} = x^{(s+i-n)}z_{x^m,y^n}$. Also let us consider the elements $\prod\limits_{m'\in bd^{-1}} x_{m'}$, where $0 \le i \le t-1 < n$. In fact, we do not have any $b = m'd$, since we do not have any element m' that satisfies $y^i = m'y^n$. Thus the element $\prod\limits_{m'\in bd^{-1}} x_{m'}$ actually represents identity. So we only have $z_{x^m,y^n}x^{(i)} = z_{x^m,y^n}$. Furthermore, for $c = x^m, d = y^n$ and $a = x^j$, by considering the relator $y_a z_{c,d} = z_{c,d}(\prod\limits_{n'\in c^{-1}a} y_{n'})$ and then applying similar argument as in the above paragraph, we get the remaining relations in $\mathcal{P}'_{A\bowtie B}$, as required. □

3.2. Case II: Infinite examples

For a free abelian group of rank 2, say A, and a finite cyclic monoid B, let $\mathcal{P}_A = [x_1, x_2\,;\, x_1x_2 = x_2x_1]$ and $\mathcal{P}_B = [y\,;\, y^s = y^t\ (s > t)]$ be their monoid presentations, respectively. For a representative element y^n in the monoid B, let us label x_{y^n} by $x^{(n)}$ where $0 \le n \le s-1$ (as in the previous section) and for a representative element $x_1^k x_2^l$ in the monoid A, let us label $y_{x_1^k x_2^l}$ by $y^{(k,l)}$ where $0 \le k, l$. Again, by considering Theorem 2.2, we then have the following corollary which can be proved quite similarly as in Corollary 3.1.

Corollary 3.2. *The product $A \bowtie B$ has a presentation with generators $x_1^{(i_1)}$, $x_2^{(i_2)}$, $y^{(j_1,j_2)}$, $z_{x_1^k x_2^l, y^n}$ $(0 \leq i_1, i_2, n \leq s-1, 0 \leq j_1, j_2)$ and relators*

$$x_p^{(i_1)} x_q^{(i_2)} = x_q^{(i_2)} x_p^{(i_1)} \quad (0 \leq i_1 < i_2 \leq s-1,\ p,q \in \{1,2\}),\ y^{(j_1,j_2)} y^{(j_3,j_4)} = y^{(j_3,j_4)} y^{(j_1,j_2)} \quad (j_1, j_2) < (j_3, j_4),$$
$$y^{(j_1,j_2)^s} = y^{(j_1,j_2)^t} \quad (j_1, j_2 \geq 0),\ x^{(i_1)} y^{(j_1,j_2)} = y^{(j_1,j_2)} x^{(i_1)} \quad (0 \leq i_1 \leq s-1), (0 \leq j_1, j_2)$$
$$z_{x_1^k x_2^l, y^n} x_1^{(i_1)} = x_1^{(i_1-n)} z_{x_1^k x_2^l, y^n} \quad (0 \leq n \leq i_1 \leq t-1),\ z_{x_1^k x_2^l, y^n} x_1^{(t+i_1)} = x_1^{(s+i_1-n)} z_{x_1^k x_2^l, y^n} \quad (i_1 = 0, 1, \cdots, s-t-1)$$
$$z_{x_1^k x_2^l, y^n} x_1^{(i_1)} = z_{x_1^k x_2^l, y^n} (0 \leq i_1 \leq t-1 < n),\ z_{x_1^k x_2^l, y^n} x_2^{(i_2)} = x_2^{(i_2-n)} z_{x_1^k x_2^l, y^n} \quad (0 \leq n \leq i_2 \leq t-1)$$
$$z_{x_1^k x_2^l, y^n} x_2^{(t+i_2)} = x_2^{(s+i_2-n)} z_{x_1^k x_2^l, y^n} \quad (i_2 = 0, 1, \cdots, s-t-1),\ z_{x_1^k x_2^l, y^n} x_2^{(i_2)} = z_{x_1^k x_2^l, y^n} \quad (0 \leq i_2 \leq t-1 < n)$$
$$y^{(j_1,j_2)} z_{x_1^k x_2^l, y^n} = z_{x_1^k x_2^l, y^n} y^{(j_1-k,\, j_2-l)} \quad (0 \leq k \leq j_1,\ 0 \leq l \leq j_2),\ y^{(j_1,j_2)} z_{x_1^k x_2^l, y^n} = z_{x_1^k x_2^l, y^n} \quad (0 \leq j_1 < k \text{ or } 0 \leq j_2 < l).$$

In fact the above corollary can be generalized for the free abelian group A rank $n > 2$.

Another application of Theorem 2.2 is the following. Let A be the free group with a presentation $\mathcal{P}_A = [x;\ ]$ and let B be the direct product monoid $Z_s \times Z_m$ with a presentation $\mathcal{P}_B = [y_1, y_2\ ;\ y_1 y_2 = y_2 y_1,\ y_1^s = y_1^t,\ y_2^m = y_2^n\ (s > t, m > n)]$. For a representative element $y_1^k y_2^l$ in the monoid B, let us label $x_{y_1^k y_2^l}$ by $x^{(k,l)}$ where $0 \leq k \leq s-1$, $0 \leq l \leq m-1$. Then we have a generating set $\{x^{(i_1,i_2)}, y_1^{(j_1)}, y_2^{(j_2)}, z_{x^r, y_1^k y_2^l}\}$ for the monoid $A \bowtie B$. Therefore, applying suitable changes in Theorem 2.2, the following corollary is obtained.

Corollary 3.3. *For the monoids A and B as given above, the set of relators for the monoid $A \bowtie B$ is*

$$\begin{aligned}
\{y_1^{(j_1)^s} = y_1^{(j_1)^t}, y_2^{(j_2)^m} = y_2^{(j_2)^n}, &\qquad x^{(i_1,i_2)} x^{(i_3,i_4)} = x^{(i_3,i_4)} x^{(i_1,i_2)} \quad ((i_1, i_2) < (i_3, i_4)),\\
y_p^{(j_1)} y_q^{(j_2)} = y_q^{(j_2)} y_p^{(j_1)}, &\qquad (0 \leq i_1, i_3 \leq s-1,\ 0 \leq i_2, i_4 \leq m-1,\ 0 \leq j_1 < j_2),\\
z_{x^r, y_1^k y_2^l} x^{(i_1,i_2)} = x^{(i_1-k,\, i_2-l)} z_{x^r, y_1^k y_2^l} &\qquad (0 \leq k \leq i_1 \leq s-1,\ 0 \leq l \leq i_2 \leq m-1),\\
z_{x^r, y_1^k y_2^l} x^{(t+i_1, n+i_2)} = x^{(s+i_1-k,\, m+i_2-l)} z_{x^r, y_1^k y_2^l} &\qquad (i_1 = 0, 1, \cdots, s-t-1, i_2 = 0, 1, \cdots, m-n-1),\\
z_{x^r, y_1^k y_2^l} x^{(i_1,i_2)} = z_{x^r, y_1^k y_2^l} &\qquad (0 \leq i_1 < k \quad \text{or} \quad 0 \leq i_2 < l),\\
y_1^{(j_1)} z_{x^{r_1}, y_1^k y_2^l} = z_{x^{r_1}, y_1^k y_2^l} y_1^{(j_1-r_1)} &\qquad (0 \leq r_1 \leq j_1), \quad y_2^{(j_2)} z_{x^{r_2}, y_1^k y_2^l} = z_{x^{r_2}, y_1^k y_2^l} y_2^{(j_2-r_2)} \quad (0 \leq r_2 \leq j_2)\},\\
y_1^{(j_1)} z_{x^{r_1}, y_1^k y_2^l} = z_{x^{r_1}, y_1^k y_2^l} &\qquad (0 \leq j_1 < r_1), \quad y_2^{(j_2)} z_{x^{r_2}, y_1^k y_2^l} = z_{x^{r_2}, y_1^k y_2^l} \quad (0 \leq j_2 < r_2).
\end{aligned}$$

4. Periodicity

In this part of the paper, our aim is to prove that this special wreath product satisfies the periodicity. Recall that a monoid A is called *periodic* if every element $a \in A$ has finite order.

For arbitrary monoids A and B, we can give the following periodicity result for $A \bowtie B$.

Theorem 4.1. *The product $A \bowtie B$ is periodic if and only if both A and B are periodic.*

Proof. ($\Rightarrow$) By the assumption, the element $(\overline{1}, (a, b), \widetilde{1})$ has finite order where $a \in A$ and $b \in B$. Thus there exist $m, n \in \mathbb{N}$ with $m < n$ such that $(\overline{1}, (a, b), \widetilde{1})^m = (\overline{1}, (a, b), \widetilde{1})^n$. By equating first components, we have $a^m = a^n$ and $b^m = b^n$ which gives both A and B are periodic.

($\Leftarrow$) Let $(f, (a, b), g)$ be an arbitrary element of $A \bowtie B$. Since A and B are periodic, we may assume that $a = d_1$ and $b = d_2$ are idempotents. It is known that f and g have finite images $X \subseteq A$ and $Y \subseteq B$, respectively, for $f \in A^{\oplus B}$, $g \in B^{\oplus A}$. Since X and Y are finite sets of periodic elements, we may find positive integers $m < n$ such that $x^m = x^n$, for all $x \in X$, and $y^m = y^n$, for all $y \in Y$. Therefore, for all $a' \in A$, $b' \in B$, we have $(b'd_2)f \in X$ and $(d_1 a')g \in Y$, and so

$$(b')(f({}^{d_2}f)^m) = (b')f((b')\,{}^{d_2}f)^m = (b')f((b'd_2)f)^m = (b')f((b'd_2)f)^n = (b')f((b')\,{}^{d_2}f)^n = (b')(f({}^{d_2}f)^n),$$
$$(a')(g(g^{d_1})^m) = (a')g((a')g^{d_1})^m = (a')g((d_1 a')g)^m = (a')g((d_1 a')g)^n = (a')g((a')g^{d_1})^n = (a')(g(g^{d_1})^n).$$

It follows that $(f, (a, b), g)^{m+1} = (f, (a, b), g)^{n+1}$ which proves that $A \bowtie B$ is periodic, as required. $\square$

5. Regularity

In [15], the question of the regularity of the wreath product of monoids has been explained. After that, in [12], it has been investigated the regularity of semidirect products of monoids. In this part we purpose to give necessary and sufficient conditions of $A \bowtie B$ to be regular where both A and B are any monoids. We recall that a monoid M is called *regular* if, for every $a \in M$, there exists $b \in M$ such that $aba = a$ and $bab = b$.

Theorem 5.1. *Let A and B be monoids. The wreath product $A \bowtie B$ is regular if and only if A and B are regular, and also for every $x \in B$, $y \in A$, $f \in A^{\oplus B}$ and $g \in B^{\oplus A}$, there exist $e_1 \in B$ and $e_2 \in A$ such that $e_1^2 = e_1$, $e_2^2 = e_2$ with $(x)f \in A(xe_1)f$ and $(y)g \in (e_2y)gB$.*

Proof. Let us suppose that $A \bowtie B$ is regular. Thus, for $(\overline{1},(a,b),\widetilde{1}) \in A \bowtie B$, there exists $(\overline{1},(c,d),\widetilde{1})$ such that $(\overline{1},(a,b),\widetilde{1}) = (\overline{1},(a,b),\widetilde{1})(\overline{1},(c,d),\widetilde{1})(\overline{1},(a,b),\widetilde{1})$ and $(\overline{1},(c,d),\widetilde{1}) = (\overline{1},(c,d),\widetilde{1})(\overline{1},(a,b),\widetilde{1})(\overline{1},(c,d),\widetilde{1})$. We then have $a = aca$, $c = cac$, $b = bdb$ and $d = dbd$. This implies that both A and B are regular. Moreover, by the assumption, for $(f,(a,b),g) \in A \bowtie B$, we have $(h,(c,d),k) \in A \bowtie B$ such that

$$(f,(a,b),g) = (f,(a,b),g)(h,(c,d),k)(f,(a,b),g) = (f\,{}^{b}h\,{}^{bd}f,(aca,bdb),g^{ca}k^{a}g).$$

Hence, by equating the components, $f = f\,{}^{b}h\,{}^{bd}f$ and $g = g^{ca}k^{a}g$. Clearly we had already obtained $a = aca$ and $b = bdb$ since A and B are regular by (i). These show that, for every $x \in B$ and $y \in A$,

$$(x)f = (x)f\,(x)^{b}h\,(x)^{bd}f = (x)f\,(xb)h\,(xbd)f \in A(xbd)f \text{ and } (y)g = (y)g^{ca}\,(y)k^{a}\,(y)g = (cay)g\,(ay)k\,(y)g \in (cay)gB.$$

If we take $e_1 = bd$ and $e_2 = ca$ then condition (ii) becomes true.

Conversely, let us suppose that the monoids A and B satisfy conditions (i) and (ii). For $x, b, d \in B$ and $f, h \in A^{\oplus B}$, consider $(x)f\,(x)^{b}h\,(x)^{bd}f$, where $dbd = d$. By condition (ii), for $a \in A$, we have $(x)f = a(xbd)f$ where $bd = e_1$. Thus

$$(x)f\,(x)^{b}h\,(x)^{bd}f = a(xbd)f\,(x)^{b}h\,(x)^{bd}f = a(x)^{bd}f\,(x)^{b}h\,(x)^{bd}f. \tag{11}$$

Since A is regular, $A^{\oplus B}$ is regular [12]. Thus we can take $h = {}^{d}v$ such that $fvf = f$ and $vfv = v$. Hence (11) becomes $a(x)^{bd}f\,(x)^{b}h\,(x)^{bd}f = a(x)^{bd}f\,(x)^{bd}v\,(x)^{bd}f = a(x)^{bd}(fvf) = a(x)^{bd}f = (x)f$. This implies that $f = f\,{}^{b}h\,{}^{bd}f$. On the other hand, similarly as in the above procedure, we obtain $h\,{}^{d}f\,{}^{db}h = {}^{d}v\,{}^{d}f\,{}^{dbd}v = {}^{d}v\,{}^{d}f\,{}^{d}v = {}^{d}(vfv) = {}^{d}v = h$. Furthermore, by condition (ii), let us take $(y)g = (cay)gb$ where $cac = c$ and $b \in B$ such that $ca = e_2$. Also, let us consider

$$(y)g^{ca}\,(y)k^{a}\,(y)g = (y)g^{ca}\,(y)k^{a}\,(cay)gb = (y)g^{ca}\,(y)k^{a}\,(y)g^{ca}b. \tag{12}$$

Again, by [12], regularity of B implies regularity of $B^{\oplus A}$. Hence we may take $k = u^{c}$ such that $ugu = u$ and $gug = g$. Thus (12) becomes $(y)g^{ca}\,(y)k^{a}\,(y)g^{ca}b = (y)g^{ca}\,(y)u^{ca}\,(y)g^{ca}b = (y)(gug)^{ca}b = (y)g^{ca}b = (y)g$. This conclude that $g^{ca}\,k^{a}\,g = g$. Similarly, we also get $k^{ac}g^{c}k = u^{cac}g^{c}u^{c} = u^{c}g^{c}u^{c} = (ugu)^{c} = u^{c} = k$. Therefore, for every $(f,(a,b),g) \in A \bowtie B$, there exists $(h,(c,d),h) \in A \bowtie B$ such that

$$(f,(a,b),g) = (f\,{}^{b}h\,{}^{bd}f,(aca,bdb),g^{ca}k^{a}g) \text{ and } (h,(c,d),k) = (h\,{}^{d}f\,{}^{db}h,(cac,dbd),k^{ac}g^{c}k)$$

with the equalities obtained above. Hence the result. □

Theorem 5.2. *Let A and B be regular monoids. Then the wreath product $A \bowtie B$ is regular if and only if either A or B is a group.*

Proof. Let $A \bowtie B$ be regular. Now let us assume that A is not a group. (By this assumption we will show that the group B must be a group). So there is an element $t \in A$ such that $At \neq A$, for otherwise every element of A would have an inverse since $1 \in A$. Choose $x \in B$ and define $f_x : B \to A$ (as in the proof of [15, Proposition 3.2]) such that $(u)f_x = \begin{cases} 1, & u = x \\ t, & \text{otherwise} \end{cases}$. By the regularity of $A \bowtie B$, for $(f_x,(a,b),g) \in A \bowtie B$, we have $(h,(c,d),k) \in A \bowtie B$ such that $b = bdb$ and $(u)f(ub)h(ubd)f = (u)f$, for all $u \in B$. Letting $u = x$, we see that this can be only happen if $(xbd)f = 1$ since $1 \notin At$. But, for $e = bd$, this shows that $xe = x$. Thus, by taking $x = 1$, we have $1e = 1$. So $bd = 1$. This implies that B is a group.

The converse part of the proof is clear. Hence the result. □

6. Conclusions and Open Problems

This paper mainly deals with a new monoid obtained by *advanced version* of the standard (restricted) wreath products, and so presents some new results and applications in terms of this subject. Since the unrestricted version of this new product also defines a monoid (see the last paragraph of Section 1), one may generalize the whole results in here for unrestricted case for a future study.

In the light of the idea used in here, in fact there might also be studied such new products not only wreath products based extensions but also, for instance, Zappa-Szep products based monoids (cf. [3, 5, 6, 9, 16, 17]). It is known that this product is also defined on mutual actions between monoids and can be obtained some other interesting results as well.

Finally, by considering the new extension just on groups rather than monoids and also taking into account A and B (in Lemma 2.1, Theorem 2.2 and Corollary 3.1) are maximal subgroups of the Sylow subgroups of a finite group G, it would be worth to study the characterization of the generalized Fitting subgroup of some normal subgroup of G. We may refer [1, 10] for the fundamentals of those classifications.

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