COMPARING THE NEW FRACTIONAL DERIVATIVE OPERATORS INVOLVING EXPONENTIAL AND MITTAG-LEFFLER KERNEL

Mehmet Yavuz*

Faculty of Science, Department of Mathematics-Computer Sciences
Necmettin Erbakan University
Konya, 42090, Turkey

Necati Özdemir

Faculty of Sciences and Arts, Department of Mathematics
Balıkesir University
Balıkesir, 10145, Turkey

Abstract. In this manuscript, we have proposed a comparison based on newly defined fractional derivative operators which are called as Caputo-Fabrizio (CF) and Atangana-Baleanu (AB). In 2015, Caputo and Fabrizio established a new fractional operator by using exponential kernel. After one year, Atangana and Baleanu recommended a different-type fractional operator that uses the generalized Mittag-Leffler function (MLF). Many real-life problems can be modelled and can be solved by numerical-analytical solution methods which are derived with these operators. In this paper, we suggest an approximate solution method for PDEs of fractional order by using the mentioned operators. We consider the Laplace homotopy transformation method (LHTM) which is the combination of standard homotopy technique (SHT) and Laplace transformation method (LTM). In this study, we aim to demonstrate the effectiveness of the aforementioned method by comparing the solutions we have achieved with the exact solutions. Furthermore, by constructing the error analysis, we test the practicability and usefulness of the method.

1. **Introduction.** In modelling real-life problems, some fractional derivative operators have been used up-to-now. For this reason, different-type fractional operators have been established. Among them, classical Liouville-Caputo operator defined by power law, Caputo-Fabrizio operator (CFO) [13] defined with exponential decay law and Atangana-Baleanu operator (ABO) [9] uses the Mittag-Leffler kernel have been used extensively.

These operators are very efficient to model the complex nonlinear fractional dynamical systems and to solve them. Caputo and Fabrizio have given a different perspective to fractional operators by introducing a new fractional operator without singular kernel. This definition comes naturally from the constitutive equation relating the flux and gradient by exponential damping functions. In addition to being a very useful mathematical definition, it is an operator that is highly preferred in terms of physical meaning [17].

Especially in recent years, many important theoretical results and applications have been obtained with the CFO and ABO. Some interesting studies such as [17, 10, 2, 3, 18, 25, 24, 8] have been resulted in the sense of the CFO.

However, some studies have pointed out that the kernel in integral of the CFO is non-singular and non-local. Besides, the mentioned integral has not a fractional structure in the CFO. To avoid from this situation, the ABO [9] was defined by the generalized MLF. In this definition, the kernel has non-singularity and non-locality. In addition to this, the integral has been regarded as fractional. After this definition, many physical, mathematical, chemical, biological problems have been solved by using the ABO. For example, [15, 6, 7, 1, 11, 4, 26] are the other studies based on application of AB fractional derivative.

On the other hand, many studies based on the comparison CFO and ABO have been made in a short time. Koca and Atangana [19] examined the Cattaneo-Hristov model in view of the CFO and ABO. In another study [5], the authors compared the CFO and ABO on Allen Cahn model. [22, 23, 16] are some comparison studies based on the CFO and ABO.

In this paper, the mentioned LHTM for numerical-approximate solutions of FPDEs is considered. In order to show the efficiency and accurateness of the method, it is applied to the several illustrative problems. When looking at the results, it can be seen clearly that the suggested method is very influential and infallible for solving FPDEs.

2. **Some preliminaries.** In this section, we present some important definitions of fractional calculus and Laplace transform.

Definition 2.1. The classical case of Caputo operator is given as [14]

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, \ 0 < \alpha \le 1. \tag{1}$$

Definition 2.2. The Caputo-Fabrizio (CF) time fractional derivative is given by [13]

$$_0^{CF}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau, \tag{2}$$

where $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$.

Definition 2.3. The LT of the CFO $_0^{CF}D_t^\alpha f(t)$ can be written as [13, 12]:

$$\begin{aligned}\mathcal{L}\left\{_0^{CF}D_t^{\alpha+n}f(t)\right\}(s) &= \frac{1}{1-\alpha}\mathcal{L}\left\{f^{(\alpha+n)}(t)\right\}\mathcal{L}\left\{\exp\left[-\frac{\alpha t}{1-\alpha}\right]\right\} \\ &= \frac{s^{n+1}\mathcal{L}\{f(t)\}-s^n f(0)-s^{n-1}f'(0)-\cdots-f^{(n)}(0)}{s+\alpha(1-s)}.\end{aligned} \tag{3}$$

From Definition 2.3, we have the results:

$$\begin{aligned}\mathcal{L}\left\{_0^{CF}D_t^\alpha f(t)\right\}(s) &= \frac{s\mathcal{L}\{f(t)\}-f(0)}{s+\alpha(1-s)}, \ \ n=0, \\ \mathcal{L}\left\{_0^{CF}D_t^{\alpha+1}f(t)\right\}(s) &= \frac{s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)}{s+\alpha(1-s)}, \ \ n=1.\end{aligned} \tag{4}$$

Definition 2.4. The ABO in the sense of Liouville-Caputo is given as [9]:

$$_b^{ABC}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_b^t f'(\tau) E_\alpha\left[-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right] d\tau, \tag{5}$$

where $f \in H^1(a,b), \quad b > a, \quad \alpha \in [0,1]$.

Definition 2.5. The LT of the ABO $_0^{ABC}D_t^\alpha f(t)$ is given by [9]

$$\mathcal{L}\left\{_0^{ABC}D_t^\alpha f(t)\right\}(s) = \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha \mathcal{L}\{f(t)\}(s) - s^{\alpha-1}f(0)}{s^\alpha + \frac{\alpha}{1-\alpha}}. \tag{6}$$

3. **Solution method which is described with the CFO.** Consider the following model constructed with the CFO [20]:

$$_0^{CF}D_t^\alpha u(x,t) + \eta(x)\frac{\partial u(x,t)}{\partial x} + \gamma(x)\frac{\partial^2 u(x,t)}{\partial x^2} + \varphi(x)u(x,t) = \tau(x,t), \tag{7}$$

where $(x,t) \in [0,1] \times [0,T]$, with the initial conditions

$$\frac{\partial^k u}{\partial t^k}(x,0) = f_k(x), \qquad k = 0,1,...,m-1, \tag{8}$$

and the boundary conditions

$$u(0,t) = g_0(t), \qquad u(1,t) = g_1(t), \qquad t \geq 0, \tag{9}$$

where f_k, $k = 0,1,...,m-1$, τ, g_0, g_1, η, γ and φ are known functions and $T > 0$ is a real number. In this part of the study, we achieve the solution method to solve problem (7)-(9).

The Laplace transform of the Caputo-Fabrizio fractional derivative is satisfied as

$$\mathcal{L}\left\{_0^{CF}D_t^{\alpha+n}u(x,t)\right\} = \frac{s^{n+1}\mathcal{L}\{u(x,t)\} - s^n u(x,0) - s^{n-1}u'(x,0) - \cdots - u^{(n)}(x,0)}{s + \alpha(1-s)}. \tag{10}$$

In Eq. (10), $s \geq 0$ and let we define the $\mathcal{L}\{u(x,t)\}(s) = \chi(x,s)$ for Eq. (7), then we can write

$$\chi(x,s) = \left(\frac{\alpha(s-1)-s}{s^{n+1}}\right)\left[\eta(x)\frac{\partial}{\partial x} + \gamma(x)\frac{\partial^2}{\partial x^2} + \varphi(x)\right]\chi(x,s) \\ + \frac{1}{s^{n+1}}\left[s^n u_0(x) + s^{n-1}u_1(x) + \cdots + u_n(x)\right] + \frac{s+\alpha(1-s)}{s^{n+1}}\tilde{\tau}(x,s). \tag{11}$$

Then we get the homotopy for Eq. (11) as:

$$\chi(x,s) = z\left(\frac{\alpha(s-1)-s}{s^{n+1}}\right)\left[\eta(x)\frac{\partial}{\partial x} + \gamma(x)\frac{\partial^2}{\partial x^2} + \varphi(x)\right]\chi(x,s) \\ + \frac{1}{s^{n+1}}\left[s^n u_0(x) + s^{n-1}u_1(x) + \cdots + u_n(x)\right] + \frac{s+\alpha(1-s)}{s^{n+1}}\tilde{\tau}(x,s), \tag{12}$$

where $\chi(x,s) = \mathcal{L}\{u(x,t)\}$ and $\tilde{\tau}(x,s) = \mathcal{L}\{\tau(x,t)\}$. Also taking the LTs of the boundary conditions we have:

$$\chi(0,s) = \mathcal{L}\{g_0(t)\}, \qquad \chi(1,s) = \mathcal{L}\{g_1(t)\}, \qquad s \geq 0. \tag{13}$$

Then we obtain the solution

$$\chi(x,s) = \sum_{m=0}^{\infty} z^m \chi_m(x,s), \qquad m = 0,1,2,.... \tag{14}$$

Substituting the Eq. (14) in Eq. (12), we get

$$\sum_{m=0}^{\infty} z^m \chi_m(x,s) \\ = z\left(\frac{\alpha(s-1)-s}{s^{n+1}}\right)\left[\eta(x)\frac{\partial}{\partial x} + \gamma(x)\frac{\partial^2}{\partial x^2} + \varphi(x)\right]\sum_{m=0}^{\infty} z^m \chi_m(x,s) \\ + \frac{1}{s^{n+1}}\left[s^n u_0(x) + s^{n-1}u_1(x) + \cdots + u_n(x)\right] + \frac{s+\alpha(1-s)}{s^{n+1}}\tilde{\tau}(x,s). \tag{15}$$

As the last step, we have the following homotopies:

$$
\begin{aligned}
z^0 : \chi_0(x,s) &= \tfrac{1}{s^{n+1}} \left(s^n u_0(x) + s^{n-1} u_1(x) + \cdots + u_n(x) \right) + \left(\tfrac{s+\alpha(1-s)}{s^{n+1}} \right) \tilde{\tau}(x,s), \\
z^1 : \chi_1(x,s) &= -\left(\tfrac{s+\alpha(1-s)}{s^{n+1}} \right) \left[\eta(x) \tfrac{\partial}{\partial x} + \gamma(x) \tfrac{\partial^2}{\partial x^2} + \varphi(x) \right] \chi_0(x,s), \\
z^2 : \chi_2(x,s) &= -\left(\tfrac{s+\alpha(1-s)}{s^{n+1}} \right) \left[\eta(x) \tfrac{\partial}{\partial x} + \gamma(x) \tfrac{\partial^2}{\partial x^2} + \varphi(x) \right] \chi_1(x,s), \\
&\vdots \\
z^{n+1} : \chi_{n+1}(x,s) &= -\left(\tfrac{s+\alpha(1-s)}{s^{n+1}} \right) \left[\eta(x) \tfrac{\partial}{\partial x} + \gamma(x) \tfrac{\partial^2}{\partial x^2} + \varphi(x) \right] \chi_n(x,s).
\end{aligned}
\tag{16}
$$

Then the solution for the problem (7)-(9) is presented by the following sum

$$
H_n(x,s) = \sum_{\varpi=0}^{n} \chi_\varpi(x,s). \tag{17}
$$

Using the inverse LT of Eq. (17), we obtain the approximate solution of Eq. (7),

$$
u_{approx}(x,t) \cong u_n(x,t) = \mathcal{L}^{-1}\{H_n(x,s)\}. \tag{18}
$$

4. **Solution method which is described with the ABO.** Consider the mentioned problem with Eq. (7) with ABO in the clasiccal Caputo sense:

$$
{}_0^{ABC}D_t^\alpha u(x,t) + \eta(x) \frac{\partial u(x,t)}{\partial x} + \gamma(x) \frac{\partial^2 u(x,t)}{\partial x^2} + \varphi(x) u(x,t) = \tau(x,t), \tag{19}
$$

with the same initial and boundary conditions as in Eqs. (8) and (9), respectively.

By using the LT of the ABO as we defined in Eq. (6), we can obtain

$$
\mathcal{L}\left\{ {}_0^{ABC}D_t^\alpha u(x,t) \right\} = \frac{B(\alpha)}{\alpha + s^\alpha(1-\alpha)} (s^\alpha \mathcal{L}\{u(x,t)\} - s^{\alpha-1}u(x,0)), \quad s > 0. \tag{20}
$$

We use the fact that defined previous $\mathcal{L}\{u(x,t)\}(s) = \chi(x,s)$ for Eq. (19), so we can see

$$
\begin{aligned}
\chi(x,s) = &-\left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha} \right) \left[\eta(x) \tfrac{\partial}{\partial x} + \gamma(x) \tfrac{\partial^2}{\partial x^2} + \varphi(x) \right] \chi(x,s) \\
&+ \tfrac{1}{s^\alpha} \left[s^{\alpha-1} u_0(x) \right] + \left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha} \right) \tilde{\tau}(x,s).
\end{aligned}
\tag{21}
$$

Now we can construct the homotopy for Eq. (21) as follows:

$$
\begin{aligned}
\chi(x,s) = &-z\left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha} \right) \left[\eta(x) \tfrac{\partial}{\partial x} + \gamma(x) \tfrac{\partial^2}{\partial x^2} + \varphi(x) \right] \chi(x,s) \\
&+ \tfrac{1}{s^\alpha} \left[s^{\alpha-1} u_0(x) \right] + \left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha} \right) \tilde{\tau}(x,s).
\end{aligned}
\tag{22}
$$

Then we can solve Eq. (22) with the following sum:

$$
\chi(x,s) = \sum_{m=0}^{\infty} z^m \chi_m(x,s), \qquad m = 0,1,2,\ldots. \tag{23}
$$

Substituting the Eq. (23) in Eq. (22), we have

$$
\begin{aligned}
&\sum_{m=0}^{\infty} z^m \chi_m(x,s) \\
&= -z\left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha} \right) \left[\eta(x) \tfrac{\partial}{\partial x} + \gamma(x) \tfrac{\partial^2}{\partial x^2} + \varphi(x) \right] \sum_{m=0}^{\infty} z^m \chi_m(x,s) \\
&\quad + \tfrac{1}{s^\alpha} \left[s^{\alpha-1} u_0(x) \right] + \left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha} \right) \tilde{\tau}(x,s).
\end{aligned}
\tag{24}
$$

The homotopies are obtained in the similar way:

$$
\begin{aligned}
z^0 &: \chi_0\left(x,s\right) = \frac{1}{s^\alpha}\left[s^{\alpha-1}u_0\left(x\right)\right] + \left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\tilde{\tau}\left(x,s\right),\\
z^1 &: \chi_1\left(x,s\right) = -\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left[\eta\left(x\right)\frac{\partial}{\partial x} + \gamma\left(x\right)\frac{\partial^2}{\partial x^2} + \varphi\left(x\right)\right]\chi_0\left(x,s\right),\\
z^2 &: \chi_2\left(x,s\right) = -\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left[\eta\left(x\right)\frac{\partial}{\partial x} + \gamma\left(x\right)\frac{\partial^2}{\partial x^2} + \varphi\left(x\right)\right]\chi_1\left(x,s\right),\\
&\vdots\\
z^{n+1} &: \chi_{n+1}\left(x,s\right) = -\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left[\eta\left(x\right)\frac{\partial}{\partial x} + \gamma\left(x\right)\frac{\partial^2}{\partial x^2} + \varphi\left(x\right)\right]\chi_n\left(x,s\right).
\end{aligned}
\tag{25}
$$

When $z \to 1$, we see that Eq. (25) gives the approximate solution for the problem (19) and the wanted solution can be obtained as

$$
H_n\left(x,s\right) = \sum_{\varpi=0}^{n}\chi_\varpi\left(x,s\right).
\tag{26}
$$

By applying the inverse LT of Eq. (26), we obtain the approximate solution of Eq. (19),

$$
u_{approx}\left(x,t\right) \cong u_n\left(x,t\right) = \mathcal{L}^{-1}\left\{H_n\left(x,s\right)\right\}.
\tag{27}
$$

As the last work, we determine the stability situation of the solution by applying the proposed method to some illustrative examples. If we consider $u_n\left(x,t\right) = \mathcal{L}^{-1}\left\{H_n\left(x,s\right)\right\}$, that is the nth partial sum in Eq. (27), the inaccuracy rate $ER\left(\%\right)$ is evaluated as:

$$
ER\left(\%\right) = \left|\frac{u_n\left(x,t\right) - u_{exact}\left(x,t\right)}{u_{exact}\left(x,t\right)}\right| \times 100.
\tag{28}
$$

5. **Numerical examples.** In this subsection of the paper, we discuss the proposed method via the CFO and ABO senses.

5.1. **Example.** Consider the following well-known Burgers equation of fractional order [21]

$$
\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma\left(3-\alpha\right)} + 2(x-1), \quad t>0, \quad x \in R, \quad 0<\alpha\le 1,
\tag{29}
$$

with the initial condition

$$
u\left(x,0\right) = x^2.
\tag{30}
$$

Firstly, we apply the Laplace homotopy method in Caputo-Fabrizio sense to the problem (29)-(30).

$$
\tilde{\tau}\left(x,s\right) = \mathcal{L}\left\{\tau\left(x,t\right)\right\} = \mathcal{L}\left\{\frac{2t^{2-\alpha}}{\Gamma\left(3-\alpha\right)} + 2x - 2\right\} = \frac{2}{s^{3-\alpha}} + \frac{2x-2}{s},
$$

Now we get the following recurrence relations:

$$
\begin{aligned}
z^0 &: \chi_0\left(x,s\right) = \frac{1}{s}u\left(x,0\right) + \left(\frac{s+\alpha(1-s)}{s}\right)\left(\frac{2}{s^{3-\alpha}} + \frac{2x-2}{s}\right)\\
&= \frac{x^2}{s} + \frac{2}{s^{3-\alpha}} + \frac{2\alpha(1-s)}{s^{4-\alpha}} + \frac{2x-2}{s} + \frac{2(x-1)\alpha(1-s)}{s^2},\\
z^1 &: \chi_1\left(x,s\right) = \left(\frac{s+\alpha(1-s)}{s}\right)\left[\frac{\partial^2\chi_0(x,s)}{\partial x^2} - \frac{\partial\chi_0(x,s)}{\partial x}\right]\\
&= -2\left(\frac{(s+\alpha(1-s))(xs+\alpha(1-s))}{s^3}\right),
\end{aligned}
\tag{31}
$$

$$z^2 : \chi_2\,(x,s) = \left(\tfrac{s+\alpha(1-s)}{s}\right)\left[\tfrac{\partial^2 \chi_1(x,s)}{\partial x^2} - \tfrac{\partial \chi_1(x,s)}{\partial x}\right]$$
$$= 2\tfrac{(s+\alpha(1-s))^2}{s^3},$$

$$\vdots$$

From the last iterations, we have

$$H_n\,(x,s) = \sum_{\varpi=0}^{n} \chi_\varpi\,(x,s)$$
$$= \tfrac{x^2}{s} + \tfrac{2}{s^{3-\alpha}} + \tfrac{2\alpha(1-s)}{s^{4-\alpha}} + \tfrac{2x-2}{s} + \tfrac{2(x-1)\alpha(1-s)}{s^2} \qquad (32)$$
$$- 2\left(\tfrac{(s+\alpha(1-s))(xs+\alpha(1-s))}{s^3}\right) + 2\tfrac{(s+\alpha(1-s))^2}{s^3}.$$

Taking the inverse LT of Eq. (32), the approximate solution of (29)-(30) is presented as:

$$u\,(x,t) \approx u_n\,(x,t) = \mathcal{L}^{-1}\left\{H_n\,(x,s)\right\}$$
$$= x^2 + 2t^{2-\alpha}\left(\tfrac{\alpha t}{\Gamma(4-\alpha)} + \tfrac{1-\alpha}{\Gamma(3-\alpha)}\right). \qquad (33)$$

For the special case $\alpha = 1$, the exact solution is given by $u\,(x,t) = x^2 + t^2$.

The following Figure (1) and Figure (2) show the numerical computations of Eq. (33) in the CFO and ABO sense, respectively.
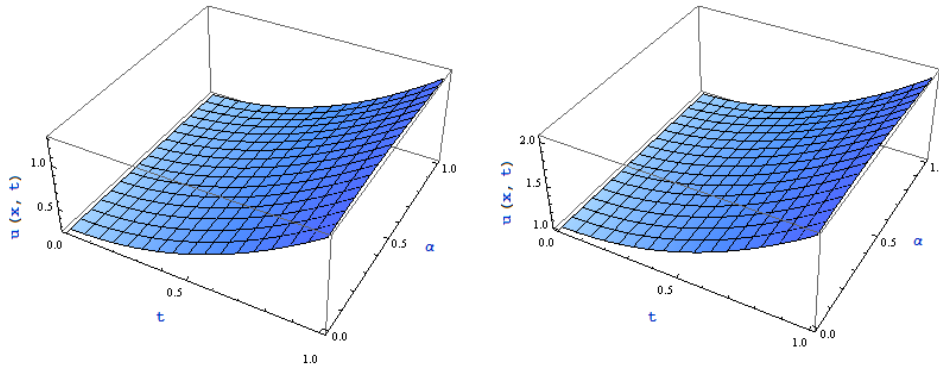

FIGURE 1. The solution function of (29) in the CFO sense for $x = 0.5$ (left) and $x = 1$ (right).

Secondly, we solve the problem (29)-(30) by using the Laplace homotopy method in the Atangana-Baleanu sense.

$$\tilde{\tau}\,(x,s) = \mathcal{L}\left\{\tau\,(x,t)\right\} = \mathcal{L}\left\{\frac{2t^{2-\alpha}}{\Gamma\,(3-\alpha)} + 2x - 2\right\} = \frac{2}{s^{3-\alpha}} + \frac{2x-2}{s},$$

Then we can write the followings:

$$z^0 : \chi_0\,(x,s) = \tfrac{1}{s}u\,(x,0) + \left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left(\tfrac{2}{s^{3-\alpha}} + \tfrac{2x-2}{s}\right)$$
$$= \tfrac{x^2}{s} + 2\left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left(\tfrac{1}{s^{3-\alpha}} + \tfrac{x-1}{s}\right), \qquad (34)$$

$$z^1 : \chi_1\,(x,s) = \left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left[\tfrac{\partial^2 \chi_0(x,s)}{\partial x^2} - \tfrac{\partial \chi_0(x,s)}{\partial x}\right]$$
$$= 2\left(\tfrac{(1-\alpha)s^\alpha+\alpha}{s^{\alpha+1}}\right)\left(1 - x - \tfrac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right),$$

$$z^2 : \chi_2(x,s) = \left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)\left[\frac{\partial^2 \chi_1(x,s)}{\partial x^2} - \frac{\partial \chi_1(x,s)}{\partial x}\right]$$
$$= \frac{2}{s}\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)^2,$$

$$\vdots$$

By using the above iterations, we have

$$H_n(x,s) = \sum_{\varpi=0}^{n} \chi_\varpi(x,s)$$
$$= \frac{x^2}{s} + 2\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^3}\right). \tag{35}$$

Taking the inverse LT of Eq. (35), we get the approximate solution of (29)-(30) as follows:

$$u(x,t) \approx u_n(x,t) = \mathcal{L}^{-1}\{H_n(x,s)\}$$
$$= x^2 + t^{2-\alpha}\left(\frac{\alpha t^\alpha \Gamma(3-\alpha) - 2\alpha + 2}{\Gamma(3-\alpha)}\right). \tag{36}$$

For the special case $\alpha = 1$, the exact solution is offered by $u(x,t) = x^2 + t^2$ which is in settlement with the result in the CFO sense.
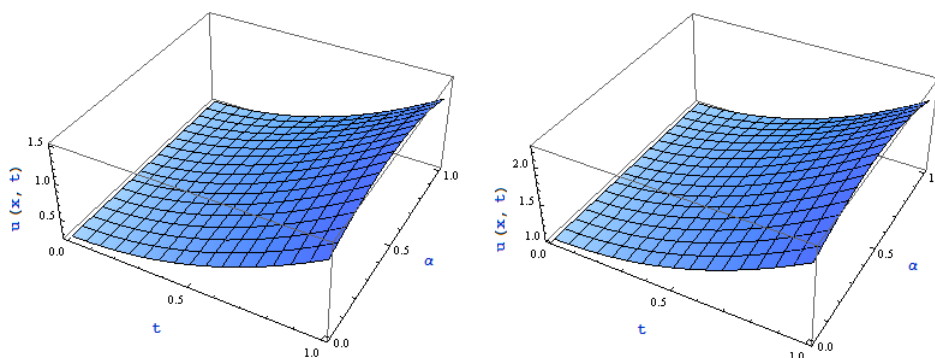

FIGURE 2. The solution function of (29) in the ABO sense for $x = 0.5$ (left) and $x = 1$ (right).

5.2. **Example.** We take the following time-fractional PDE

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u, \quad t > 0, \quad x \in R, \quad 0 < \alpha \le 1, \tag{37}$$

subject to the initial condition

$$u(x,0) = x, \tag{38}$$

and the boundary conditions

$$u_x(x,0) = 1, \qquad u(0,t) = 0. \tag{39}$$

Let apply the LHTM in the Caputo-Fabrizio sense to the problem:

Because the equation is homogeneous, $\tilde{\tau}(x,s) = \mathcal{L}\{\tau(x,t)\} = 0$.

As the second step of CF derivative, the homotopies can be given as:

$$z^0 : \chi_0(x,s) = \frac{1}{s}u(x,0) = \frac{x}{s}, \tag{40}$$

$$z^1 : \chi_1(x,s) = \left(\frac{s+\alpha(1-s)}{s}\right)\left[\frac{\partial^2 \chi_0(x,s)}{\partial x^2} + x\frac{\partial \chi_0(x,s)}{\partial x} + \chi_0(x,s)\right]$$
$$= \frac{2x(s+\alpha(1-s))}{s^2},$$

$$z^2 : \chi_2(x,s) = \left(\frac{s+\alpha(1-s)}{s}\right)\left[\frac{\partial^2 \chi_1(x,s)}{\partial x^2} + x\frac{\partial \chi_1(x,s)}{\partial x} + \chi_1(x,s)\right]$$
$$= \frac{4x(s+\alpha(1-s))^2}{s^3},$$

$$\vdots$$

$$z^{n+1} : \chi_{n+1}(x,s) = \left(\frac{s+\alpha(1-s)}{s}\right)\left[\frac{\partial^2 \chi_n(x,s)}{\partial x^2} + x\frac{\partial \chi_n(x,s)}{\partial x} + \chi_n(x,s)\right]$$
$$= \frac{2^{n+1}x(s+\alpha(1-s))^{n+1}}{s^{n+2}},$$

So, the approximate solution is

$$H_n(x,s) = \sum_{\varpi=0}^{n} \chi_\varpi(x,s) \tag{41}$$
$$= \frac{x}{s} + x\sum_{\varpi=1}^{n} \frac{2^\varpi(s+\alpha(1-s))^\varpi}{s^{\varpi+1}}.$$

If we consider the inverse LT of Eq. (41) and, when $n \to \infty$ we get the approximate solution of problem (37)-(39) as follows:

$$u(x,t) \approx u_n(x,t) = \mathcal{L}^{-1}\{H_n(x,s)\} \tag{42}$$
$$= \frac{xe^{\frac{2\alpha t}{2\alpha-1}}}{2\alpha-1}.$$

For the special case $\alpha = 1$, the exact solution is given by $u(x,t) = xe^{2t}$.

The following Figure (3) shows the LHTM solution of Eq. (37) with the conditions (38)-(39) in the Caputo-Fabrizio sense for various values of α and t.
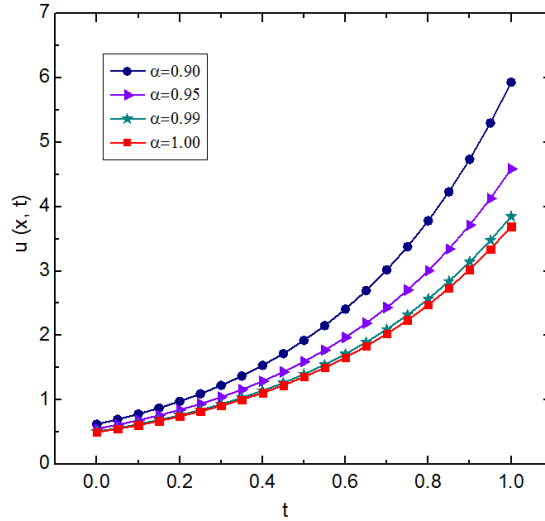

FIGURE 3. The solution of Eq. (37) in the CFO sense for various values of α.

Now, we apply the LHTM in the ABO sense to the problem (37)-(39):
Similarly, we set the homotopies of the series with respect to the ABO as:

$$z^0 : \chi_0(x,s) = \frac{1}{s}u(x,0) = \frac{x}{s}, \tag{43}$$

$$z^1 : \chi_1(x,s) = \left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left[\frac{\partial^2 \chi_0(x,s)}{\partial x^2} + x\frac{\partial \chi_0(x,s)}{\partial x} + \chi_0(x,s)\right]$$
$$= \frac{2x}{s}\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right),$$

$$z^2 : \chi_2(x,s) = \left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)\left[\frac{\partial^2 \chi_1(x,s)}{\partial x^2} + x\frac{\partial \chi_1(x,s)}{\partial x} + \chi_1(x,s)\right]$$
$$= \frac{4x}{s}\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)^2,$$

$$\vdots$$

$$z^{n+1} : \chi_{n+1}(x,s) = \left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)\left[\frac{\partial^2 \chi_n(x,s)}{\partial x^2} + x\frac{\partial \chi_n(x,s)}{\partial x} + \chi_n(x,s)\right]$$
$$= \frac{2^{n+1}x}{s}\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)^{n+1},$$

After that, we have the following sum

$$H_n(x,s) = \sum_{\varpi=0}^{n} \chi_\varpi(x,s)$$
$$= \frac{x}{s} + \frac{x}{s}\sum_{\varpi=1}^{n}\left(\frac{2((1-\alpha)s^\alpha + \alpha)}{s^\alpha}\right)^\varpi. \tag{44}$$

Getting the inverse LT for Eq. (44), we have the approximate solution of (37)-(39) as:

$$u(x,t) \approx u_n(x,t) = L^{-1}\{H_n(x,s)\}$$
$$= x + 2x\left(1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\right) + 4x\left((\alpha-1)^2 - \frac{2t^\alpha \alpha(\alpha-1)}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}\alpha^2}{\Gamma(2\alpha+1)}\right)$$
$$+ \frac{8x\left(-3t^{2\alpha}\alpha^2(\alpha-1)\Gamma(3\alpha+1) + \Gamma(2\alpha+1)\left(t^{3\alpha}\alpha^3 - (\alpha-1)^3\Gamma(3\alpha+1)\right)\right)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}$$
$$+ \frac{24xt^\alpha\alpha(\alpha-1)^2}{\Gamma(\alpha+1)} + \cdots$$
$$= \frac{x}{2\alpha-1}E_\alpha\left(\frac{2\alpha t^\alpha}{2\alpha-1}\right). \tag{45}$$

where $E_\alpha(z)$ is the Mittag-Leffler function in one parameter. In Eq. (45) if we use the special case $\alpha = 1$, the exact solution of the problem is given by $u(x,t) = xe^{2t}$.

In Figure (4), we present the graphs of Eq. (45) in the ABO sense for various values of α, x, and t.
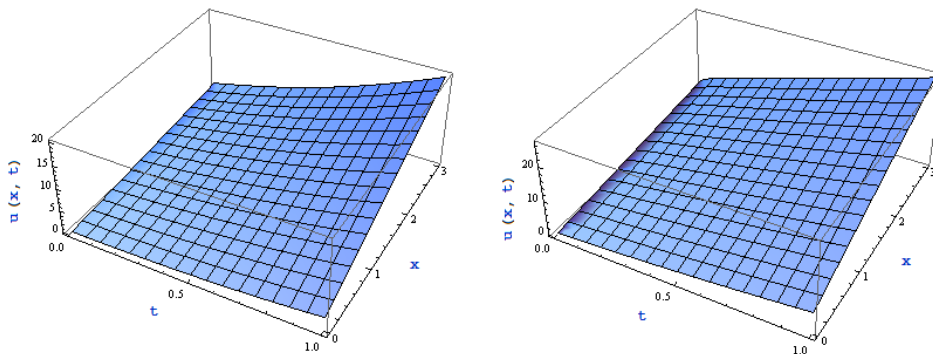

FIGURE 4. The solution function of (45) in the ABO sense for various values of $\alpha = 0.7$ (left) and $\alpha = 0.9$ (right).

6. **Stability and convergence analysis of the mentioned method.** In this subsection of the study, we analyze the convergence and the stability of the method. If the series (17) and (26) converge where $\chi(x, s)$ is occurred by Eq. (14) and Eq. (23), they have to be the solutions of Eq. (7) and Eq. (19), respectively. Besides, the solution results declare that the mentioned method is stable. Our suggested method provides a good convergence area of the solution by generative functions (15) and (22). Furthermore the approximate results get with the homotopy are good agreement with the accurate solutions. In order to verify the convergent and stability of the proposed method defined in Sects. (3) and (4), the error rates $ER(\%)$ are obtained for some values of space variable x and time variable t. In Figure (5), we compare the numerical solution in Eq. (41), for $n = 10$, with the exact solution.
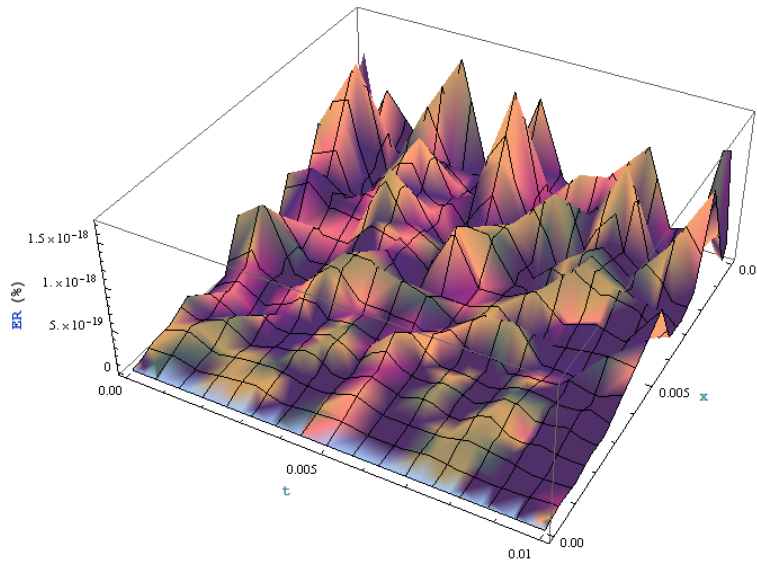

FIGURE 5. Inaccuracy rates $ER(\%)$ of the mentioned method

Concluding remarks. Two newly defined fractional operators with respect to the exponential and the GML functions have been used in this paper to obtain the solutions of FPDEs with the LHTM. Two illustrative problems have been solved numerically-approximately and the results have been presented for different values of the fractional parameter α. Besides the convergence and stability analysis have been constructed of the model. The results obtained in this study show that the suggested method has a good stability and they have verified the validity and effectiveness of the LHTM in both the CFO and ABO senses.

REFERENCES

[1] T. Abdeljawad and D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, *J. Nonlinear Sci. Appl.,* **10** (2017), 1098–1107, arXiv:1607.00262.

[2] B. S. T. Alkahtani and A. Atangana, Controlling the wave movement on the surface of shallow water with the Caputo–Fabrizio derivative with fractional order, *Chaos, Solitons & Fractals*, **89** (2016), 539–546.

[3] B. S. T. Alkahtani and A. Atangana, Analysis of non-homogeneous heat model with new trend of derivative with fractional order, *Chaos, Solitons & Fractals*, **89** (2016), 566–571.

[4] B. S. T. Alkahtani, Chua's circuit model with Atangana–Baleanu derivative with fractional order, *Chaos, Solitons & Fractals*, **89** (2016), 547–551.

[5] O. J. J. Alkahtani, Comparing the Atangana–Baleanu and Caputo–Fabrizio derivative with fractional order: Allen Cahn model, *Chaos, Solitons & Fractals*, **89** (2016), 552–559.

[6] R. T. Alqahtani, Atangana-Baleanu derivative with fractional order applied to the model of groundwater within an unconfined aquifer, *Journal of Nonlinear Sciences and Applications*, **9** (2016), 3647–3654.

[7] F. A. M. N. Al-Salti and E. Karimov, Initial and boundary value problems for fractional differential equations involving Atangana-Baleanu derivative, preprint, arXiv:1706.00740.

[8] N. A. Asif, Z. Hammouch, M. B. Riaz and H. Bulut, Analytical solution of a Maxwell fluid with slip effects in view of the Caputo-Fabrizio derivative, *The European Physical Journal Plus*, **133** (2018), 272.

[9] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Thermal Science*, **22** (2016), 763–769.

[10] A. Atangana and I. Koca, On the new fractional derivative and application to nonlinear Baggs and Freedman model, *Journal of Nonlinear Sciences and Applications*, **9** (2016), 2467–2480.

[11] A. Atangana and I. Koca, Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order, *Chaos, Solitons & Fractals*, **89** (2016), 447–454.

[12] A. Atangana and B. S. T. Alkahtani, New model of groundwater flowing within a confine aquifer: Application of Caputo-Fabrizio derivative, *Arabian Journal of Geosciences*, **9** (2016), 3647–3654.

[13] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications*, **1** (2015), 1–13.

[14] M. Caputo, Linear models of dissipation whose Q is almost frequency independent I, *Geophysical Journal International*, **13** (1967), 529–539.

[15] J. F. Gómez-Aguilar, R. F. Escobar-Jiménez, M. G. López-López and V. M. Alvarado-Martínez, Atangana-Baleanu fractional derivative applied to electromagnetic waves in dielectric media, *Journal of Electromagnetic Waves and Applications*, **30** (2016), 1937–1952.

[16] J. F. Gómez-Aguilar, V. F. Morales-Delgado, M. A. Taneco-Hernández, D. Baleanu, R. F. Escobar-Jiménez and M. M. Al Qurashi, Analytical solutions of the electrical RLC circuit via Liouville aputo operators with local and non-local kernels, *Entropy*, **18** (2016), 402.

[17] J. Hristov, Transient heat diffusion with a non-singular fading memory: from the Cattaneo constitutive equation with Jeffrey's kernel to the Caputo-Fabrizio time-fractional derivative, *Thermal Science*, **20** (2016), 757–762.

[18] J. Hristov, Steady-state heat conduction in a medium with spatial non-singular fading memory: derivation of Caputo-Fabrizio space-fractional derivative with Jeffrey's kernel and analytical solutions, *Thermal Science*, **21** (2017), 827–839.

[19] I. Koca and A. Atangana, Solutions of Cattaneo-Hristov model of elastic heat diffusion with Caputo-Fabrizio and Atangana-Baleanu fractional derivatives, *Thermal Science*, **21** (2017), 2299–2305.

[20] V. F. Morales-Delgado, J. F. Gómez-Aguilar, H. Yépez-Martínez, D. Baleanu, R. F. Escobar-Jimenez and V. H. Olivares-Peregrino, Laplace homotopy analysis method for solving linear partial differential equations using a fractional derivative with and without kernel singular, *Advances in Difference Equations*, **2016** (2016), Paper No. 164, 17 pp.

[21] Z. Odibat and S. Momani, The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics, *Computers & Mathematics with Applications*, **58** (2009), 2199–2208.

[22] N. A. Sheikh, F. Ali, M. Saqib, I. Khan, S. A. A. Jan, A. S. Alshomrani and M. S. Alghamdi, Comparison and analysis of the Atangana–Baleanu and Caputo–Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction, *Results in Physics*, **7** (2017), 789–800.

[23] N. A. Sheikh, F. Ali, M. Saqib, I. Khan and S. A. A. Jan, A comparative study of Atangana-Baleanu and Caputo-Fabrizio fractional derivatives to the convective flow of a generalized Casson fluid, *The European Physical Journal Plus*, **132** (2017), 54.

[24] J. Singh, D. Kumar, Z. Hammouch and A. Atangana, A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, *Applied Mathematics and Computation*, **316** (2018), 504–515.

[25] M. Yavuz and N. Ozdemir, European vanilla option pricing model of fractional order without singular kernel, *Fractal and Fractional*, **2** (2018), 3.

[26] M. Yavuz, N. Ozdemir and H. M. Baskonus, Solutions of partial differential equations using the fractional operator involving Mittag-Leffler kernel, *The European Physical Journal Plus*, **133** (2018), 215.

E-mail address: mehmetyavuz@erbakan.edu.tr
E-mail address: nozdemir@balikesir.edu.tr