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Pata Zamfirescu Type Fixed-Disc Results with a Proximal Application

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Abstract

This paper concerns with the geometric study of fixed points of a self-mapping on a metric space. We establish new generalized contractive conditions which ensure that a self-mapping has a fixed disc or a fixed circle. We introduce the notion of a best proximity circle and explore some proximal contractions for a non-self-mapping as an application. Necessary illustrative examples are presented to highlight the importance of the obtained results.

Keywords Fixed disc · Pata Zamfirescu type x_0-mapping · Proximity point · Proximity circle

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1 Introduction and Motivation

Fixed-point theory has an important role due to solutions of the equation $Tx = x$ where T is a self-mapping on a metric (resp. some generalized metric) space. This theory has been extensively studied with various applications in diverse research areas such as integral equations, differential equations, engineering, statistics, and economics. Some questions have been arisen for the existence and uniqueness of fixed points. Some fixed-point problems are as follows:

(1) Is there always a solution of the equation $Tx = x$?
(2) What are the existence conditions for a fixed point of a self-mapping?

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(3) What are the uniqueness conditions if there is a fixed point of a self-mapping?
(4) Can the number of fixed points be more than one?
(5) If the number of fixed points is more than one, is there a geometric interpretation of these points?

Considering the above questions, many researchers have been studied on fixed-point theory with different aspects.

Some generalized contractive conditions have been investigated to guarantee the existence and uniqueness of a fixed point of a self-mapping. For example, in [22], an existence theorem was given for a generalized contraction mapping. In [16], a refinement of the classical Banach contraction principle was obtained. A new generalization of these results was derived by using both of the above contractive conditions in [3]. In [2], a survey of various variants of fixed point results for single- and multivalued mappings under the Pata-type conditions was given (for more results, see [3–5,17] and the references therein).

In the cases in which the fixed-point equation $Tx = x$ has no solution, the notion of "best proximity point" has been appeared as an approximate solution x such that the error $d(x, Tx)$ is minimum. For example, the existence of best proximity point was investigated using the Pata-type proximal mappings in [3]. These results are the generalizations of ones obtained in [16]. In [17], some generalized best proximity point and optimal coincidence point results were proved for new Pata-type contractions.

If a fixed point is not unique, then the geometry of the fixed points of a self-mapping is an attractive problem. For this purpose, a recent approach called "fixed-circle problem" (resp. "fixed-disc problem") has been studied by various techniques (see [1,6–15,18–21] and the references therein). For example, in [10], some fixed-disc results have been obtained using the set of simulation functions on a metric space.

In this paper, mainly, we focus on the geometric study of fixed points of a self-mapping on a metric space. New generalized contractive conditions are established for a self-mapping to have a fixed disc or a fixed circle with some illustrative examples. As an application, we introduce the notion of a best proximity circle and explore some proximal contractions for a non-self-mapping.

2 Main Results

Throughout the section, we assume that (X, d) is a metric space, $T: X \to X$ is a self-mapping and $D[x_0, r]$ is a disc defined as

$$D[x_0, r] = \{u \in X: d(u, x_0) \leq r\}.$$

If the self-mapping T fixes all of the points in the disc $D[x_0, r]$, that is, $Tu = u$ for all $u \in D[x_0, r]$, then $D[x_0, r]$ is called as the fixed disc of T (see [10,20] and the references therein).

To obtain new fixed-disc results, we modify the notion of a Zamfirescu mapping on metric spaces (see [22] for more details).

Definition 2.1 The self-mapping T is called a Zamfirescu type x_0 -mapping if there exist $x_0 \in X$ and $a, b \in [0, 1)$ such that

$$d(Tu, u) > 0 \Longrightarrow d(Tu, u) \leq \max\left\{ad(u, x_0), \frac{b}{2}\left[d(Tx_0, u) + d(Tu, x_0)\right]\right\},$$

for all $u \in X$.

Proposition 2.2 *If T is a Zamfirescu type x_0-mapping with $x_0 \in X$, then we have $Tx_0 = x_0$.*

Proof Let T be a Zamfirescu type x_0-mapping with $x_0 \in X$. Assume that $Tx_0 \neq x_0$. Then, we have $d(Tx_0, x_0) > 0$ and using the Zamfirescu type x_0-mapping hypothesis, we get

$$\begin{aligned} d(Tx_0, x_0) &\leq \max\left\{ad(x_0, x_0), \frac{b}{2}\left[d(Tx_0, x_0) + d(Tx_0, x_0)\right]\right\} \\ &= \max\left\{0, bd(Tx_0, x_0)\right\} = bd(Tx_0, x_0), \end{aligned}$$

a contradiction because of $b \in [0, 1)$. Consequently, T fixes the point $x_0 \in X$, that is, $Tx_0 = x_0$. □

Let the number r be defined as follows:

$$r = \inf\left\{d(Tu, u) : Tu \neq u, u \in X\right\}. \tag{2.1}$$

Theorem 2.3 *If T is a Zamfirescu type x_0-mapping with $x_0 \in X$ and $d(Tu, x_0) \leq r$ for each $u \in D(x_0, r) - \{x_0\}$, then $D[x_0, r]$ is a fixed disc of T.*

Proof Suppose that $r = 0$. Then, we get $D[x_0, r] = \{x_0\}$. By Proposition 2.2, we have $Tx_0 = x_0$ whence $D[x_0, r]$ is a fixed disc of T.

Now assume that $r > 0$ and $u \in D[x_0, r] - \{x_0\}$ is any point such that $Tu \neq u$. Then, we have $d(Tu, u) > 0$. Using the Zamfirescu type x_0-mapping property, the hypothesis $d(Tu, x_0) \leq r$ and Proposition 2.2, we get

$$\begin{aligned} d(Tu, u) &\leq \max\left\{ad(u, x_0), \frac{b}{2}\left[d(Tx_0, u) + d(Tu, x_0)\right]\right\} \\ &\leq \max\left\{ar, br\right\}. \end{aligned} \tag{2.2}$$

Without loss of generality we can assume $a \geq b$. Then, using the inequality (2.2), we obtain

$$d(Tu, u) \leq ar,$$

which is a contradiction with the definition of r because of $a \in [0, 1)$. Consequently, it should be $Tu = u$ and so $D[x_0, r]$ is a fixed disc of T. □

From now on, Θ denotes the class of all increasing functions $\Psi\colon [0,1] \to [0,\infty)$ with $\Psi(0) = 0$. Modifying the notion of a Pata-type contraction (see [16]) and using this class Θ, we give the following definition that exclude the continuity hypothesis on Ψ.

Definition 2.4 Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0,\alpha]$ be any constants. Then, T is called a Pata type x_0-mapping if there exist $x_0 \in X$ and $\Psi \in \Theta$ such that

$$d(Tu,u) > 0 \Longrightarrow d(Tu,u) \leq \frac{1-\varepsilon}{2}\|u\| + \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\left[1 + \|u\| + \|Tu\|\right]^{\beta},$$

for all $u \in X$ and each $\varepsilon \in [0,1]$, where $\|u\| = d(u,x_0)$.

Proposition 2.5 *If T is a Pata type x_0-mapping with $x_0 \in X$, then we have $Tx_0 = x_0$.*

Proof Let T be a Pata type x_0-mapping with $x_0 \in X$. Assume that $Tx_0 \neq x_0$. Then, we have $d(Tx_0,x_0) > 0$. Using the Pata type x_0-mapping hypothesis, we get

$$\begin{aligned} d(Tx_0,x_0) &\leq \frac{1-\varepsilon}{2}\|x_0\| + \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\left[1 + \|x_0\| + \|Tx_0\|\right]^{\beta} \\ &= \frac{1-\varepsilon}{2}d(x_0,x_0) + \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\left[1 + d(x_0,x_0) + d(Tx_0,x_0)\right]^{\beta} \\ &= \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\left[1 + d(Tx_0,x_0)\right]^{\beta}. \end{aligned} \tag{2.3}$$

For $\varepsilon = 0$, using the inequality (2.3), we obtain

$$d(Tx_0,x_0) \leq 0,$$

whence it should be $Tx_0 = x_0$. □

Theorem 2.6 *If T is a Pata type x_0-mapping with $x_0 \in X$, then $D[x_0,r]$ is a fixed disc of T.*

Proof Suppose that $r = 0$. Then, we get $D[x_0,r] = \{x_0\}$. By Proposition 2.5, we have $Tx_0 = x_0$ whence $D[x_0,r]$ is a fixed disc of T. Now assume that $r > 0$ and $u \in D[x_0,r] - \{x_0\}$ is any point such that $Tu \neq u$. Then, we have $d(Tu,u) > 0$. Using the Pata type x_0-mapping property, we get

$$d(Tu,u) \leq \frac{1-\varepsilon}{2}\|u\| + \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\left[1 + \|u\| + \|Tu\|\right]^{\beta}. \tag{2.4}$$

For $\varepsilon = 0$, using the inequality (2.4), we obtain

$$d(Tu,u) \leq \frac{\|u\|}{2} = \frac{d(u,x_0)}{2} \leq \frac{r}{2},$$

a contradiction with the definition of r. Consequently, it should be $Tu = u$, that is, $D[x_0,r]$ is a fixed disc of T. □

Combining the notions of a Zamfirescu type x_0-mapping and of a Pata type x_0-mapping, we define the following notion inspiring the concept of a Pata-type Zamfirescu mapping [3].

Definition 2.7 If there exist $x_0 \in X$ and $\Psi \in \Theta$ such that

$$\begin{aligned} d(Tu,u) > 0 \Longrightarrow d(Tu,u) \leq \frac{1-\varepsilon}{2} M(u,x_0) + \\ \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|u\| + \|Tu\| + \|Tx_0\|\right]^{\beta}, \end{aligned}$$

for all $u \in X$ and each $\varepsilon \in [0,1]$, where $\|u\| = d(u,x_0)$, $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0,\alpha]$ are constants and

$$M(u,v) = \max\left\{d(u,v), \frac{d(Tu,u)+d(Tv,v)}{2}, \frac{d(Tv,u)+d(Tu,v)}{2}\right\},$$

then T is called a Pata Zamfirescu type x_0-mapping with respect to $\Psi \in \Theta$.

Now we compare a Zamfirescu type x_0-mapping and a Pata Zamfirescu type x_0-mapping. Let $\gamma = \max\{a,b\}$ in Definition 2.1 and let us consider the Bernoulli's inequality $1 + rt \leq (1+t)^r$, $r \geq 1$ and $t \in [-1,\infty)$. Then, we have

$$\begin{aligned} d(Tu,u) > 0 \Longrightarrow d(Tu,u) &\leq \max\left\{ad(u,x_0), \frac{b}{2}\left[d(Tx_0,u) + d(Tu,x_0)\right]\right\} \\ &\leq \gamma \max\left\{d(u,x_0), \frac{d(Tx_0,u)+d(Tu,x_0)}{2}\right\} \\ &\leq \gamma \max\left\{d(u,x_0), \frac{d(Tu,u)+d(Tx_0,x_0)}{2}, \frac{d(Tx_0,u)+d(Tu,x_0)}{2}\right\} \\ &\leq \frac{1-\varepsilon}{2} \max\left\{d(u,x_0), \frac{d(Tu,u)+d(Tx_0,x_0)}{2}, \frac{d(Tx_0,u)+d(Tu,x_0)}{2}\right\} \\ &\quad + \left(\gamma + \frac{\varepsilon-1}{2}\right)\left[1 + \max\left\{\|u\|, \frac{\|u\| + \|Tu\| + \|Tx_0\|}{2}\right\}\right] \\ &\leq \frac{1-\varepsilon}{2} \max\left\{d(u,x_0), \frac{d(Tu,u)+d(Tx_0,x_0)}{2}, \frac{d(Tx_0,u)+d(Tu,x_0)}{2}\right\} \\ &\quad + \gamma\left(1 + \frac{\varepsilon-1}{\gamma}\right)\left[1 + \|u\| + \|Tu\| + \|Tx_0\|\right] \\ &\leq \frac{1-\varepsilon}{2} M(u,x_0) + \gamma \varepsilon^{\frac{1}{\gamma}}\left[1 + \|u\| + \|Tu\| + \|Tx_0\|\right] \\ &\leq \frac{1-\varepsilon}{2} M(u,x_0) + \gamma \varepsilon \varepsilon^{\frac{1-\gamma}{\gamma}}\left[1 + \|u\| + \|Tu\| + \|Tx_0\|\right]. \end{aligned}$$

Consequently, we obtain that a Zamfirescu type x_0-mapping is a special case of a Pata Zamfirescu type x_0-mapping with $\Lambda = \gamma$, $\Psi(u) = u^{\frac{1-\gamma}{\gamma}}$ and $\alpha = \beta = 1$.

In the following proposition, we see that the point x_0 in the notion of a Pata Zamfirescu type x_0-mapping is a fixed point of a self-mapping T.

Proposition 2.8 *If T is a Pata Zamfirescu type x_0-mapping with respect to $\Psi \in \Theta$ for $x_0 \in X$, then we have $Tx_0 = x_0$.*

Proof Let T be a Pata Zamfirescu type x_0-mapping with respect to $\Psi \in \Theta$ for $x_0 \in X$. Suppose that $Tx_0 \neq x_0$. Then, we have $d(Tx_0, x_0) > 0$. Using the Pata Zamfirescu type x_0-mapping hypothesis, we obtain

$$\begin{aligned} d(Tx_0, x_0) &\leq \frac{1-\varepsilon}{2} M(x_0, x_0) + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|x_0\| + 2\|Tx_0\|\right]^{\beta} \\ &= \frac{1-\varepsilon}{2} d(Tx_0, x_0) + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|x_0\| + 2\|Tx_0\|\right]^{\beta}. \end{aligned} \tag{2.5}$$

For $\varepsilon = 0$, using the inequality (2.5), we get

$$d(Tx_0, x_0) \leq \frac{d(Tx_0, x_0)}{2},$$

a contradiction. Hence, it should be $Tx_0 = x_0$. □

Using Proposition 2.8, we give the following fixed-disc theorem.

Theorem 2.9 *If T is a Pata Zamfirescu type x_0-mapping with respect to $\Psi \in \Theta$ for $x_0 \in X$ and $d(Tu, x_0) \leq r$ for each $u \in D[x_0, r] - \{x_0\}$, then $D[x_0, r]$ is a fixed disc of T.*

Proof Suppose that $r = 0$. Then, we get $D[x_0, r] = \{x_0\}$. By Proposition 2.8, we have $Tx_0 = x_0$ whence $D[x_0, r]$ is a fixed disc of T. Now assume that $r > 0$ and $u \in D[x_0, r] - \{x_0\}$ is any point such that $Tu \neq u$. Then, we have $d(Tu, u) > 0$. Using the Pata Zamfirescu type x_0-mapping property, the hypothesis $d(Tu, x_0) \leq r$ and Proposition 2.8, we obtain

$$\begin{aligned} d(Tu, u) &\leq \frac{1-\varepsilon}{2} M(u, x_0) + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|u\| + \|Tu\| + \|Tx_0\|\right]^{\beta} \\ &= \frac{1-\varepsilon}{2} \max\left\{d(u, x_0), \frac{d(Tu, u) + d(Tx_0, x_0)}{2}, \frac{d(Tx_0, u) + d(Tu, x_0)}{2}\right\} \\ &\quad + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|u\| + \|Tu\| + \|Tx_0\|\right]^{\beta} \\ &\leq \frac{1-\varepsilon}{2} \max\left\{r, \frac{d(Tu, u)}{2}, r\right\} \\ &\quad + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|u\| + \|Tu\| + \|Tx_0\|\right]^{\beta}. \end{aligned} \tag{2.6}$$

For $\varepsilon = 0$, using the inequality (2.6), we get

$$d(Tu, u) \leq \frac{1}{2} \max\left\{r, \frac{d(Tu, u)}{2}\right\}.$$

Hence, we get two cases as follows:

Case 1 If $\max\left\{r, \frac{d(Tu,u)}{2}\right\} = r$, then we have

$$d(Tu, u) \leq \frac{r}{2},$$

a contradiction with the definition of r.
Case 2 If $\max\left\{r, \frac{d(Tu,u)}{2}\right\} = \frac{d(Tu,u)}{2}$, then we find

$$d(Tu, u) \leq \frac{d(Tu, u)}{2},$$

a contradiction.

Consequently, it should be $Tu = u$ and so T fixes the disc $D[x_0, r]$. □

We give some illustrative examples to show the validity of our obtained results.

Example 2.10 Let $X = \mathbb{R}$ be the usual metric space with the metric $d(u, v) = |u - v|$ for all $u, v \in \mathbb{R}$. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tu = \begin{cases} u & \text{if} \quad u \in [-4, 4] \\ u + 1 & \text{if} \quad u \in (-\infty, -4) \cup (4, \infty) \end{cases},$$

for all $u \in \mathbb{R}$. Then,

• The self-mapping T is a Zamfirescu type x_0-mapping with $x_0 = 0$, $a = \frac{1}{2}$ and $b = 0$. Indeed, we get

$$d(Tu, u) = 1 > 0,$$

for all $u \in (-\infty, -4) \cup (4, \infty)$. Hence, we find

$$d(Tu, u) = 1 \leq \frac{|u|}{2} = \max\left\{ad(u, 0), \frac{b}{2}\left[d(0, u) + d(u + 1, 0)\right]\right\}.$$

• The self-mapping T is a Pata type x_0-mapping with $x_0 = 0$, $\Lambda = \alpha = \beta = 1$ and

$$\Psi(u) = \begin{cases} 0 & \text{if} \quad u = 0 \\ \frac{1}{2} & \text{if} \quad u \in (0, 1] \end{cases}.$$

Indeed, we have

$$d(Tu, u) = 1 > 0,$$

for all $u \in (-\infty, -4) \cup (4, \infty)$. So we obtain

$$\begin{aligned} d(Tu, u) = 1 &\leq \frac{|u|}{2} + \frac{\varepsilon}{2} + \frac{\varepsilon\,|u + 1|}{2} \\ &= \frac{1 - \varepsilon}{2}\,\|u\| + \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\left[1 + \|u\| + \|Tu\|\right]^{\beta}. \end{aligned}$$

• The self-mapping T is a Pata Zamfirescu type x_0-mapping with $x_0 = 0$, $\Lambda = \alpha = \beta = 1$ and

$$\Psi(u) = \begin{cases} 0 & \text{if} \quad u = 0 \\ \frac{1}{2} & \text{if} \quad u \in (0, 1] \end{cases} .$$

Indeed, we get

$$d(Tu, u) = 1 > 0,$$

for all $u \in (-\infty, -4) \cup (4, \infty)$ and

$$M(u, 0) = \max\left\{|u|, \frac{1}{2}, \frac{|u| + |u+1|}{2}\right\} = \max\left\{|u|, \frac{|u| + |u+1|}{2}\right\}.$$

Then, we obtain two cases:
Case 1 Let $|u| > \frac{|u|+|u+1|}{2}$. We find $M(u, 0) = |u|$ and

$$\begin{aligned} d(Tu, u) = 1 &\le \frac{|u|}{2} + \frac{\varepsilon}{2} + \frac{\varepsilon|u+1|}{2} \\ &= \frac{1-\varepsilon}{2} M(u, 0) + \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\left[1 + \|u\| + \|Tu\| + \|T0\|\right]^{\beta}. \end{aligned}$$

Case 2 Let $|u| < \frac{|u|+|u+1|}{2}$. We obtain $M(u, 0) = \frac{|u|+|u+1|}{2}$ and

$$\begin{aligned} d(Tu, u) = 1 &\le \frac{|u|}{4} + \frac{\varepsilon|u|}{2} + \frac{|u+1|}{4} + \frac{\varepsilon|u+1|}{2} + \frac{\varepsilon}{2} \\ &= \frac{1-\varepsilon}{2} M(u, 0) + \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\left[1 + \|u\| + \|Tu\| + \|T0\|\right]^{\beta}. \end{aligned}$$

Also, we find

$$r = \inf\{d(Tu, u) \colon Tu \neq u, u \in X\} = 1$$

and

$$d(Tu, 0) = d(u, 0) \le 1,$$

for all $u \in D[0, 1] - \{0\}$. Consequently, from Theorem 2.3 (resp. Theorems 2.6 and 2.9), T fixes the disc $D[0, 1]$.

Example 2.11 Let $X = [0, 1]$ be the usual metric space. Let us define the self-mapping $T \colon X \to X$ as

$$Tu = \begin{cases} u & \text{if} \quad u \in \{0, 1\} \\ 2u & \text{if} \quad u \in (0, 1) \end{cases},$$

for all $u \in X$. Then, T is a Zamfirescu type x_0-mapping with $x_0 = 0$, $a = 0$ and $b = \frac{2}{3}$, but T is not a Zamfirescu type x_0-mapping with $x_0 = 1$. Also we get $r = 0$ and so T fixes the point $x_0 = 0$.

Example 2.12 Let $X = \mathbb{R}$ be the usual metric space. Let us define the self-mapping $T\colon \mathbb{R} \to \mathbb{R}$ as

$$Tu = \begin{cases} u & \text{if} \quad u \in [-2, \infty) \\ u + 1 & \text{if} \quad u \in (-\infty, -2) \end{cases},$$

for all $u \in \mathbb{R}$. Then, T is a Zamfirescu type x_0-mapping with $a = \frac{1}{2}$, $b = 0$, both $x_0 = 0$ and $x_0 = 5$. We obtain $r = 1$ whence by Theorem 2.3 T fixes both of the discs $D[0, 1]$ and $D[5, 1]$.

Example 2.13 Let $X = \mathbb{R}$ be the usual metric space. Let us define the self-mapping $T\colon \mathbb{R} \to \mathbb{R}$ as

$$Tu = \begin{cases} u & \text{if} \quad u \in [-1, 1] \\ 0 & \text{if} \quad u \in (-\infty, -1) \cup (1, \infty) \end{cases},$$

for all $u \in \mathbb{R}$. Then, we have $r = 1$ and T is not a Zamfirescu type x_0-mapping (resp. a Pata type x_0-mapping and a Pata Zamfirescu type x_0-mapping) with any $x_0 \in X$ but T fixes the disc $D[0, 1]$.

Considering the above examples, we conclude the following remarks.

Remark 2.14 (1) The point x_0 satisfying the definition of a Zamfirescu type x_0-mapping (resp. a Pata type x_0-mapping and a Pata Zamfirescu type x_0-mapping) is a fixed point of the self-mapping T. But the converse statement is not always true, that is, a fixed point of T does not always satisfy the definition of a Zamfirescu type x_0 -mapping (resp. a Pata type x_0-mapping and a Pata Zamfirescu type x_0-mapping). For example, if we consider Example 2.11, then T fixes the point $x_0 = 1$, but the point 1 does not satisfy the definition of a Zamfirescu type x_0-mapping.

(2) The choice of x_0 is independent from the number r (see Examples 2.10, 2.11 and 2.12).

(3) The radius r can be zero (see Example 2.11).

(4) The number of x_0 satisfying the definition of a Zamfirescu type x_0-mapping (resp. a Pata type x_0-mapping and a Pata Zamfirescu type x_0-mapping) can be more than one (see Example 2.12).

(5) The converse statements of Theorems 2.3, 2.6 and 2.9 are not always true (see Example 2.13).

(6) The obtained fixed-disc results can be also considered as fixed-circle results (resp. fixed-point results).

3 A best proximity circle application

In this section, we define the notion of a best proximity circle on a metric space. At first, we recall the definition of a best proximity point and some basic concepts. Let A, B be two nonempty subsets of a metric space (X, d). We consider the following:

$$\mathsf{d}(A, B) = \inf\{d(u, v) : u \in A \text{ and } v \in B\},$$
$$A_0 = \{u \in A : d(u, v) = \mathsf{d}(A, B) \text{ for some } v \in B\}$$

and

$$B_0 = \{v \in B : d(u, v) = \mathsf{d}(A, B) \text{ for some } u \in A\}.$$

For a mapping $T : A \to B$, the point $u \in A$ is called a best proximity point of T if

$$d(u, Tu) = \mathsf{d}(A, B).$$

If T has more than one best proximity point, then it is an interesting problem to consider the geometric properties of these points. For this purpose we define a circle $C_{x_0,r} = \{u \in A : d(u, x_0) = r\}$ as the best proximity circle of T if

$$d(u, Tu) = \mathsf{d}(A, B),$$

for all $u \in C_{x_0,r}$. We note that the best proximity circle becomes a fixed circle of T if we take $A = B = X$ (the circle $C_{x_0,r}$ is called as the fixed circle of the self-mapping $T : X \to X$ if $Tu = u$ for every $u \in C_{x_0,r}$ [12]). Also if $C_{x_0,r}$ has only one element, then the best proximity circle becomes to a best proximity point or a fixed point of T. Using this notion, we give an application to a Pata type x_0-mapping.

Definition 3.1 Let (X, d) be a metric space and $T : A \to B$ be a mapping. Then, T is called a Pata type proximal x_0-contraction if there exist $x_0 \in A_0$ and $\Psi \in \Theta$ such that

$$d(x, u) \leq \frac{1-\varepsilon}{2} d(x, x_0) + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) [1 + \|x\| + \|u\|]^{\beta},$$

for all $x \in A$ and each $\varepsilon \in [0, 1]$, where $u \in A$ with $d(u, Tx) = \mathsf{d}(A, B)$, $\|x\| = d(x, x_0)$ and $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ are any constants.

Proposition 3.2 *If T is a Pata type proximal x_0-contraction with $x_0 \in A_0$ such that $TA_0 \subset B_0$, then x_0 is a best proximity point of T in A.*

Proof Let $x_0 \in A_0$. Then, we have $Tx_0 \in B_0$ because of $TA_0 \subset B_0$. Hence, there exists $u \in A$ such that

$$d(u, Tx_0) = \mathsf{d}(A, B).$$

Using the Pata type proximal x_0-contractive property, we obtain

$$d(x_0, u) \leq \frac{1-\varepsilon}{2} d(x_0, x_0) + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|x_0\| + \|u\|\right]^{\beta}. \tag{3.1}$$

For $\varepsilon = 0$, using the inequality (3.1), we get

$$d(x_0, u) \leq 0,$$

which implies $x_0 = u$. Consequently, x_0 is a best proximity point of T in A, that is, $d(x_0, Tx_0) = \mathsf{d}(A, B)$. □

Theorem 3.3 *Let $\mu = \inf\{d(x, u) : x, u \in A$ such that $x \neq u\}$. If T is a Pata type proximal x_0-contraction with $x_0 \in A_0$ such that $TA_0 \subset B_0$ and $C_{x_0,\mu} \subset A_0$, then $C_{x_0,\mu}$ is a best proximity circle of T.*

Proof Let $\mu = 0$. Then, we have $C_{x_0,\mu} = \{x_0\}$. From Proposition 3.2, $C_{x_0,\mu}$ is a best proximity circle of T. Now suppose that $\mu > 0$. Let $x \in C_{x_0,\mu}$ be any point. Then, using the hypothesis $C_{x_0,\mu} \subset A_0$, we have $x \in A_0$ and so $Tx \in B_0$. Hence, there exists $u \in A$ such that

$$d(u, Tx) = \mathsf{d}(A, B).$$

Using the Pata type proximal x_0-contractive property, we obtain

$$d(x, u) \leq \frac{1-\varepsilon}{2} d(x, x_0) + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|x\| + \|u\|\right]^{\beta}. \tag{3.2}$$

For $\varepsilon = 0$, using the inequality (3.2), we get

$$d(x, u) \leq \frac{1}{2} d(x, x_0) = \frac{\mu}{2} \leq \frac{d(x, u)}{2},$$

which is a contradiction. It should be $x = u$. Consequently, $C_{x_0,\mu}$ is a best proximity circle of T. □

Corollary 3.4 *Let (X, d) be a metric space and $T : X \to X$ be a mapping which satisfies the conditions of Definition 2.4. Then, T has a fixed circle $C_{x_0,\mu}$ in X.*

Proof The proof can be easily obtained from Theorem 3.3 taking $A = B = X$. In this case, the definition of the number μ coincides with the definition of the number r. □

Notice that Corollary 3.4 is a special case of Theorem 2.6. In fact, T fixes each of the circles $C_{x_0,\rho}$ where $\rho \leq \mu$.

Finally, we give an example to Theorem 3.3 following [3].

Example 3.5 Let $A = \{(0, a): a \in [0, 1]\}$ and $B = \{(1, b): b \in [0, 1]\}$ on $\mathbb{R}^2$ with the metric $d: X \times X \to \mathbb{R}$ defined as $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let us define the mapping $T: A \to B$ as

$$T(0, x) = \left(1, \frac{2x}{3}\right),$$

for all $(0, x) \in A$. Then, T is a Pata type proximal x_0-contraction with $x_0 = (0, 0)$, $\Lambda = \alpha = \beta = 1$ and

$$\Psi(u) = \begin{cases} 0 & \text{if} \quad u = 0 \\ \frac{1}{2} & \text{if} \quad u \in (0, 1] \end{cases}.$$

Indeed, we get $\mathsf{d}(A, B) = 1$ and $\mu = 0$. Now we show that T satisfies the Pata type proximal x_0-contractive property for all $\varepsilon \in [0, 1]$.

Let $\varepsilon = 0$. For all $(0, x) = x' \in A$, we get

$$d(x', u) = d\left((0, x), \left(0, \frac{2x}{3}\right)\right) = \frac{x}{3} \leq \frac{x}{2} = \frac{1}{2} d(x', x_0),$$

where $d(u, Tx) = \mathsf{d}(A, B) = 1$.

Let $\varepsilon \in (0, 1]$. For all $(0, x) = x' \in A$, we get

$$\begin{aligned} d(x', u) &= \frac{x}{3} \\ &\leq \frac{x}{2} + \frac{\varepsilon}{2} \\ &\leq \frac{1 - \varepsilon}{2} d(x', x_0) + \Lambda \varepsilon^{\alpha} \Psi(\varepsilon) \left[1 + \|x'\| + \|u\|\right]^{\beta}, \end{aligned}$$

where $d(u, Tx) = \mathsf{d}(A, B) = 1$.

Hence, T is a Pata type proximal x_0-contraction and there exists a best proximity circle $C_{x_0,\mu} = \{(0, 0)\}$ in A. Also the circle $C_{x_0,\mu}$ can be considered as a best proximity point.

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