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Some Fixed-Circle Theorems on Metric Spaces

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Abstract The fixed-point theory and its applications to various areas of science are well known. In this paper, we present some existence and uniqueness theorems for fixed circles of self-mappings on metric spaces with geometric interpretation. We verify our results by illustrative examples.

Keywords Fixed circle · The existence theorem · The uniqueness theorem

Mathematics Subject Classification 47H10 · 54H25 · 55M20 · 37E10

1 Introduction

The existence of fixed points of functions has been extensively studied which satisfy certain conditions since the time of Stefan Banach. At first, we recall the Banach contraction principle as follows:

Theorem 1.1 [3] *Let (X, d) be a complete metric space and a self-mapping $T : X \to X$ be a contraction, that is, there exists some $h \in [0, 1)$ such that*

$$d(Tx, Ty) \leq hd(x, y),$$

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for any $x, y \in X$. *Then there exists a unique fixed point* $x_0 \in X$ *of* T.

Since then many authors have been studied new contractive conditions for fixed-point theorems. For example, Caristi gave the following fixed-point theorem.

Theorem 1.2 [2] *Let* (X, d) *be a complete metric space and* $T : X \to X$. *If there exists a lower semicontinuous function* φ *mapping* X *into the nonnegative real numbers*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \tag{1.1}$$

$x \in X$ *then* T *has a fixed point.*

In [10], Rhoades defined the following condition (which is called Rhoades' condition):

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$ with $x \neq y$.

In a very recent work, the usage of the fixed-point theory can be found in various research areas (see [7]).

In some special metric spaces, mappings with fixed points have been used in neural networks as activation functions. For example, Möbius transformations have been used for this purpose. It is known that a Möbius transformation is a rational function of the form

$$T(z) = \frac{az + b}{cz + d}, \tag{1.2}$$

where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. A Möbius transformation has at most two fixed points (see [5] for more details about Möbius transformations). In [6], Mandic identified the activation function of a neuron and a single-pole all-pass digital filter section as Möbius transformations. He observed that the fixed points of a neural network were determined by the fixed points of the employed activation function. So, the existence of the fixed points of an activation function was guaranteed by the underlying Möbius transformation (one or two fixed points).

On the other hand, there are some examples of functions which fix a circle. For example, let $\mathbb{C}$ be the metric space with the usual metric

$$d(z, w) = |z - w|,$$

for all $z, w \in \mathbb{C}$. Let the mapping T be defined as

$$Tz = \frac{1}{\overline{z}},$$

for all $z \in \mathbb{C} \setminus \{0\}$. The mapping T fixes the unit circle $C_{0,1}$. In [9], Özdemir, İskender and Özgür used new types of activation functions which fix a circle for a complex-valued neural network (CVNN). The usage of these types of activation functions leads us to guarantee the existence of fixed points of the complex-valued Hopfield neural network (CVHNN).

Therefore, it is important the notions of "fixed circle" and "mappings with a fixed circle." It will be an interesting problem to study some fixed-circle theorems on general spaces (metric spaces or normed spaces).

Motivated by the above studies, our aim in this paper is to examine some fixed-circle theorems for self-mappings on metric spaces. Also we determine the uniqueness conditions of these theorems. In Sect. 2 we introduce the notion of a fixed circle and prove three theorems for the existence of fixed circles of self-mappings on metric spaces. Also we give some necessary examples for obtained fixed-circle theorems. In Sect. 3 we present some self-mappings which have at least two fixed circles. Hence, we give three uniqueness theorems for the fixed-circle theorems obtained in Sect. 2. In Sect. 4 we give an application of our results to discontinuous activation functions.

2 Existence of the Self-Mappings with Fixed Circles

In this section, we give fixed-circle theorems under some conditions on metric spaces and obtain some examples of mappings which have or not fixed circles. At first, we give the following definition.

Definition 2.1 Let (X, d) be a metric space and $C_{x_0,r} = \{x \in X : d(x_0, x) = r\}$ be a circle. For a self-mapping $T : X \to X$, if $Tx = x$ for every $x \in C_{x_0,r}$ then we call the circle $C_{x_0,r}$ as the fixed circle of T.

Now we give the following existence theorem for a fixed circle using the inequality (1.1).

Theorem 2.1 *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let us define the mapping*

$$\varphi : X \to [0, \infty), \varphi(x) = d(x, x_0), \tag{2.1}$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying

(C1) $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ *and*

(C2) $d(Tx, x_0) \geq r$, *for each $x \in C_{x_0,r}$, then the circle $C_{x_0,r}$ is a fixed circle of T.*

Proof Let us consider the mapping φ defined in (2.1). Let $x \in C_{x_0,r}$ be any arbitrary point. We show that $Tx = x$ whenever $x \in C_{x_0,r}$. Using the condition (C1), we obtain

$$\begin{aligned} d(x, Tx) &\leq \varphi(x) - \varphi(Tx) = d(x, x_0) - d(Tx, x_0) \\ &= r - d(Tx, x_0). \end{aligned} \tag{2.2}$$

Because of the condition (C2), the point Tx should be lies on or exterior of the circle $C_{x_0,r}$. Then we have two cases. If $d(Tx, x_0) > r$, then using (2.2) we have a contradiction. Therefore, it should be $d(Tx, x_0) = r$. In this case, using (2.2) we get

$$d(x, Tx) \leq r - d(Tx, x_0) = r - r = 0$$

and so $Tx = x$.

Hence, we obtain $Tx = x$ for all $x \in C_{x_0,r}$. Consequently, the self-mapping T fixes the circle $C_{x_0,r}$. □

Remark 2.1 1. We note that Theorem 1.2 guarantees the existence of a fixed point, while Theorem 2.1 guarantees the existence of a fixed circle. In the cases where the circle $C_{x_0,r}$ has only one element (see Example 2.10 for an example), Theorem 2.1 is a special case of Theorem 1.2.

2. Notice that the condition (C1) guarantees that Tx is not in the exterior of the circle $C_{x_0,r}$ for each $x \in C_{x_0,r}$. Similarly, the condition (C2) guarantees that Tx is not in the interior of the circle $C_{x_0,r}$ for each $x \in C_{x_0,r}$. Consequently, $Tx \in C_{x_0,r}$ for each $x \in C_{x_0,r}$ and so we have $T(C_{x_0,r}) \subset C_{x_0,r}$ (see Fig. 1 for the geometric interpretation of the conditions (C1) and $(C2)$).

Now we give a fixed-circle example.

Example 2.1 Let (X, d) be a metric space and α be a constant such that

$$d(\alpha, x_0) > r.$$

Let us consider a circle $C_{x_0,r}$ and define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} x; & x \in C_{x_0,r} \\ \alpha; & \text{otherwise} \end{cases},$$

for all $x \in X$. Then it can be easily seen that the conditions (C1) and (C2) are satisfied. Clearly $C_{x_0,r}$ is a fixed circle of T.

Now, in the following examples, we give some examples of self-mappings which satisfy the condition (C1) and do not satisfy the condition (C2).

Example 2.2 Let (X, d) be any metric space, $C_{x_0,r}$ be any circle on X, and the self-mapping $T : X \to X$ be defined as

$$Tx = x_0,$$

for all $x \in X$. Then the self-mapping T satisfies the condition (C1) but does not satisfy the condition (C2). Clearly T does not fix the circle $C_{x_0,r}$.

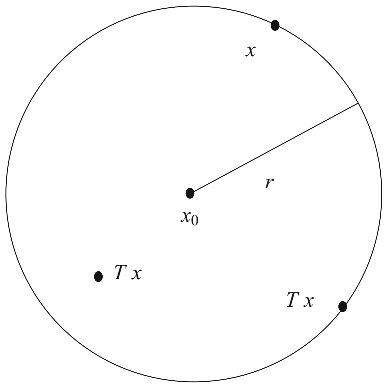
Example 2.3 Let $(\mathbb{R}, d)$ be the usual metric space. Let us consider the circle $C_{1,2}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} 1; & x \in C_{1,2} \\ 2; & \text{otherwise} \end{cases},$$

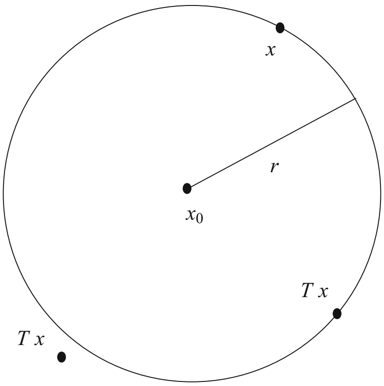
for all $x \in \mathbb{R}$. Then the self-mapping T satisfies the condition (C1) but does not satisfy the condition (C2). Clearly T does not fix the circle $C_{1,2}$ (or any circle).

In the following examples, we give some examples of self-mappings which satisfy the condition (C2) and do not satisfy the condition (C1).

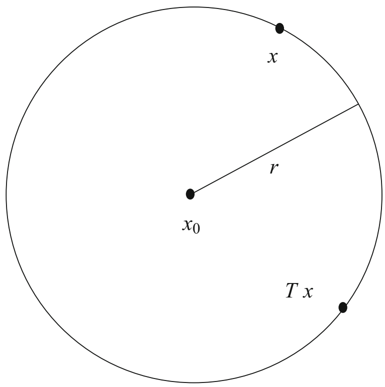
Fig. 1 The conditions (C1) and (C2). **a** The geometric interpretation of the condition (C1), **b** the geometric interpretation of the condition (C2), **c** the geometric interpretation of the condition $(C1) \cap (C2)$



(A)



(B)



(C)

Example 2.4 Let (X, d) be any metric space and $C_{x_0,r}$ be any circle on X. Let α be chosen such that $d(\alpha, x_0) = \rho > r$ and consider the self-mapping $T : X \to X$ defined by

$$Tx = \alpha,$$

for all $x \in X$. Then the self-mapping T satisfies the condition (C2) but does not satisfy the condition (C1). Clearly T does not fix the circle $C_{x_0,r}$.

Example 2.5 Let $(\mathbb{C}, d)$ be the usual complex metric space and $C_{0,1}$ be the unit circle on $\mathbb{C}$. Let us consider the self-mapping $T : \mathbb{C} \to \mathbb{C}$ defined by

$$Tz = \begin{cases} \frac{1}{z}; & z \neq 0 \\ 0; & z = 0 \end{cases},$$

for all $z \in \mathbb{C}$. Then the self-mapping T satisfies the condition (C2) but does not satisfy the condition (C1). Clearly T does not fix the circle $C_{0,1}$ (or any circle). Notice that T fixes only the points -1 and 1 on the unit circle.

Now we give another existence theorem for fixed circles.

Theorem 2.2 *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let the mapping φ be defined as (2.1). If there exists a self-mapping $T : X \to X$ satisfying*

*(C1)** $d(x, Tx) \leq \varphi(x) + \varphi(Tx) - 2r$ *and*
*(C2)** $d(Tx, x_0) \leq r$*, for each* $x \in C_{x_0,r}$*, then* $C_{x_0,r}$ *is a fixed circle of* T.

Proof We consider the mapping φ defined in (2.1). Let $x \in C_{x_0,r}$ be any arbitrary point. Using the condition $(C1)^*$, we obtain

$$\begin{aligned} d(x, Tx) &\leq \varphi(x) + \varphi(Tx) - 2r = d(x, x_0) + d(Tx, x_0) - 2r \\ &= d(Tx, x_0) - r. \end{aligned} \tag{2.3}$$

Because of the condition $(C2)^*$, the point Tx should be lies on or interior of the circle $C_{x_0,r}$. Then we have two cases. If $d(Tx, x_0) < r$, then using (2.3) we have a contradiction. Therefore, it should be $d(Tx, x_0) = r$. If $d(Tx, x_0) = r$, then using (2.3) we get

$$d(x, Tx) \leq d(Tx, x_0) - r = r - r = 0$$

and so we find $Tx = x$.

Consequently, $C_{x_0,r}$ is a fixed circle of T. □

Remark 2.2 Notice that the condition $(C1)^*$ guarantees that Tx is not in the interior of the circle $C_{x_0,r}$ for each $x \in C_{x_0,r}$. Similarly, the condition $(C2)^*$ guarantees that Tx is not in the exterior of the circle $C_{x_0,r}$ for each $x \in C_{x_0,r}$. Consequently, $Tx \in C_{x_0,r}$ for each $x \in C_{x_0,r}$ and so we have $T(C_{x_0,r}) \subset C_{x_0,r}$ (see Fig. 2 for the geometric interpretation of the conditions $(C1)^*$ and $(C2)^*$).

Now we give some fixed-circle examples.

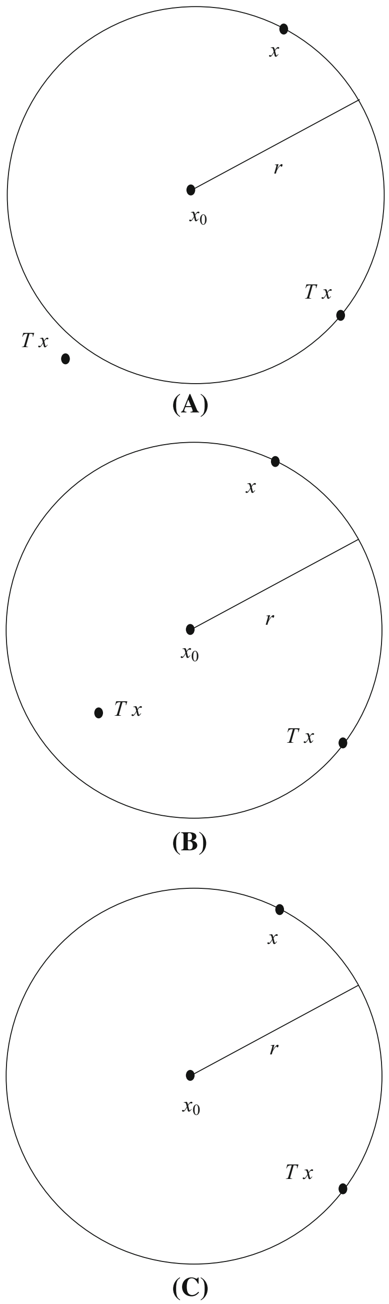


Fig. 2 The conditions $(C1)^*$ and $(C2)^*$. **a** The geometric interpretation of the condition $(C1)^*$, **b** the geometric interpretation of the condition $(C2)^*$, **c** the geometric interpretation of the condition $(C1)^* \cap (C2)^*$

Example 2.6 Let (X, d) be a metric space and α be a constant such that

$$d(\alpha, x_0) < r.$$

Let us consider a circle $C_{x_0,r}$ and define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} x; & x \in C_{x_0,r} \\ \alpha; & \text{otherwise} \end{cases},$$

for all $x \in X$. Then it can be easily checked that the conditions $(C1)^*$ and $(C2)^*$ are satisfied. Clearly $C_{x_0,r}$ is a fixed circle of the self-mapping T.

Example 2.7 Let $(\mathbb{R}, d)$ be the usual metric space and $C_{0,1}$ be the unit circle on $\mathbb{R}$. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} \frac{1}{x}; & x \in C_{0,1} \\ 5; & \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T satisfies the conditions $(C1)^*$ and $(C2)^*$. Hence, $C_{0,1}$ is the fixed circle of T. Notice that the fixed circle $C_{0,1}$ is not unique. $C_{3,2}$ and $C_{2,3}$ are also fixed circles of T. It can be easily verified that T satisfies the conditions $(C1)^*$ and $(C2)^*$ for the circles $C_{3,2}$ and $C_{2,3}$.

In the following example, we give an example of a self-mapping which satisfies the condition $(C2)^*$ and does not satisfy the condition $(C1)^*$.

Example 2.8 Let (X, d) be any metric space and $C_{x_0,r}$ be any circle on X. Let α be chosen such that $d(\alpha, x_0) = \rho < r$ and consider the self-mapping $T : X \to X$ defined by

$$Tx = \alpha,$$

for all $x \in X$. Then the self-mapping T satisfies the condition $(C2)^*$ but does not satisfy the condition $(C1)^*$. Clearly T does not fix the circle $C_{x_0,r}$.

In the following example, we give an example of a self-mapping which satisfies the condition $(C1)^*$ and does not satisfy the condition $(C2)^*$.

Example 2.9 Let $(\mathbb{R}, d)$ be the usual metric space and $C_{0,1}$ be the unit circle on $\mathbb{R}$. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} -5; & x = -1 \\ 5; & x = 1 \\ 10; & \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T satisfies the condition $(C1)^*$ but does not satisfy the condition $(C2)^*$. Clearly T does not fix the circle $C_{0,1}$ (or any circle).

Using the inequality (1.1), we give another existence fixed-circle theorem on a metric space.

Theorem 2.3 *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let the mapping φ be defined as (2.1). If there exists a self-mapping $T : X \to X$ satisfying*

$(C1)^{**}$ $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$, *and*
$(C2)^{**}$ $hd(x, Tx) + d(Tx, x_0) \geq r$, *for each* $x \in C_{x_0,r}$ *and some* $h \in [0, 1)$, *then* $C_{x_0,r}$ *is a fixed circle of* T.

Proof We consider the mapping φ defined in (2.1). Assume that $x \in C_{x_0,r}$ and $Tx \neq x$. Then using the conditions $(C1)^{**}$ and $(C2)^{**}$, we obtain

$$\begin{aligned} d(x, Tx) &\leq \varphi(x) - \varphi(Tx) = d(x, x_0) - d(Tx, x_0) \\ &= r - d(Tx, x_0) \\ &\leq hd(x, Tx) + d(Tx, x_0) - d(Tx, x_0) \\ &= hd(x, Tx), \end{aligned}$$

which is a contradiction with our assumption since $h \in [0, 1)$. Therefore, we get $Tx = x$ and $C_{x_0,r}$ is a fixed circle of T. □

Remark 2.3 Notice that the condition $(C1)^{**}$ guarantees that Tx is not in the exterior of the circle $C_{x_0,r}$ for each $x \in C_{x_0,r}$. The condition $(C2)^{**}$ implies that Tx can be lies on or exterior or interior of the circle $C_{x_0,r}$. Consequently, Tx should be lies on or interior of the circle $C_{x_0,r}$ (see Fig. 3 for the geometric interpretation of the conditions $(C1)^{**}$ and $(C2)^{**}$).

Example 2.10 Let $X = \mathbb{R}$ and the mapping $d : X^2 \to [0, \infty)$ be defined as

$$d(x, y) = \left| e^x - e^y \right|,$$

for all $x \in \mathbb{R}$. Then $(\mathbb{R}, d)$ be a metric space. Let us consider the circle $C_{0,1} = \{\ln 2\}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} \ln 2; & x \in C_{0,1} \\ 1; & \text{otherwise} \end{cases},$$

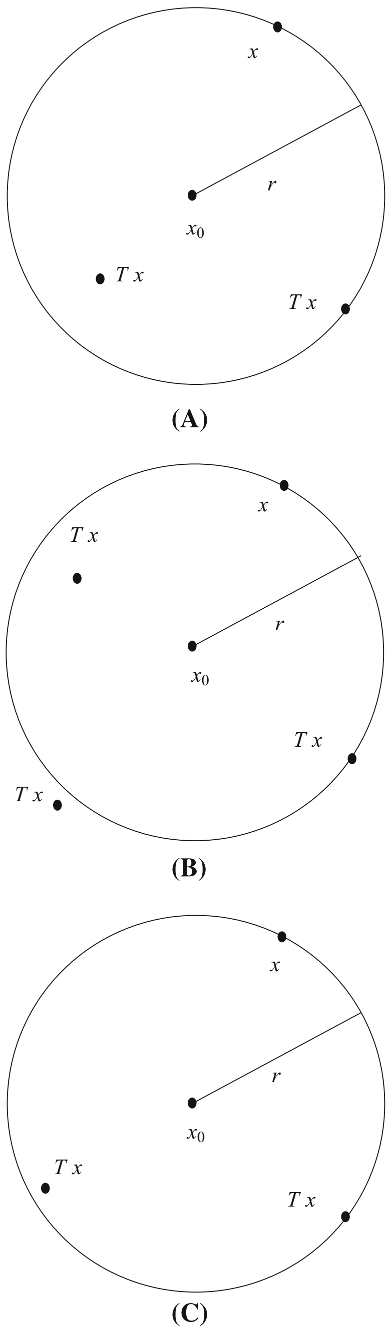
for all $x \in \mathbb{R}$. Then it can be easily checked that the conditions $(C1)^{**}$ and $(C2)^{**}$ are satisfied. Hence, the unit circle $C_{0,1}$ is a fixed circle of T.

In the following example, we give an example of a self-mapping which satisfies the condition $(C1)^{**}$ and does not satisfy the condition $(C2)^{**}$.

Example 2.11 Let $(\mathbb{R}, d)$ be the usual metric space. Let us consider the circle $C_{2,4} = \{-2, 6\}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} 2; & x \in C_{2,4} \\ 6; & \text{otherwise} \end{cases},$$

Fig. 3 The conditions $(C1)^{**}$ and $(C2)^{**}$. **a** The geometric interpretation of the condition $(C1)^{**}$, **b** the geometric interpretation of the condition $(C2)^{**}$, **c** the geometric interpretation of the condition $(C1)^{**} \cap (C2)^{**}$



for all $x \in \mathbb{R}$. Then the self-mapping T satisfies the condition $(C1)^{**}$ but does not satisfy the condition $(C2)^{**}$. Clearly T does not fix the circle $C_{2,4}$ (or any circle).

In the following example, we give an example of a self-mapping which satisfies the condition $(C2)^{**}$ and does not satisfy the condition $(C1)^{**}$.

Example 2.12 Let $(\mathbb{R}, d)$ be the usual metric space. Let us consider the circle $C_{0,2}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = 2,$$

for all $x \in \mathbb{R}$. Then the self-mapping T satisfies the condition $(C2)^{**}$ but does not satisfy the condition $(C1)^{**}$. Clearly T does not fix the circle $C_{0,2}$ (or any circle).

Example 2.13 Let $X = \mathbb{R}$ and the mapping $d : X^2 \to [0, \infty)$ be defined as

$$d(x, y) = \begin{cases} 0; & x = y \\ |x| + |y|; & x \neq y \end{cases},$$

for all $x \in \mathbb{R}$. Then $(\mathbb{R}, d)$ be a metric space. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} \frac{1}{2}; & x \in \{-1, 1\} \\ 0; & \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T does not satisfy the condition $(C1)^{*}$ but satisfies the condition $(C2)^{*}$ for the circle $C_{1,2}$. Hence, T does not fix the circle $C_{1,2}$. On the other hand, it can be easily checked that T satisfies both the conditions $(C1)^{*}$ and $(C2)^{*}$ for the circle $C_{1,1}$ and so fixes $C_{1,1}$. Actually notice that T fixes all of the circles centered at $x_0 = a \in \mathbb{R}^{+}$ with radius a.

Let $I_X : X \to X$ be the identity map defined as $I_X(x) = x$ for all $x \in X$. Notice that the identity map satisfies the conditions (C1) and (C2) (resp. $(C1)^{*}$ and $(C2)^{*}$, $(C1)^{**}$ and $(C2)^{**}$) in Theorem 2.1 (resp. Theorems 2.2, 2.3). Now we investigate a condition which excludes I_X in Theorems 2.1, 2.2 and 2.3. We give the following theorem.

Theorem 2.4 *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let the mapping φ be defined as (2.1). If a self-mapping $T : X \to X$ satisfies the condition*

$$(I_d) \qquad d(x, Tx) \leq \frac{\varphi(x) - \varphi(Tx)}{h},$$

for all $x \in X$ and some $h > 1$ then $T = I_X$ and $C_{x_0,r}$ is a fixed circle of T.

Proof Let $x \in X$ and $Tx \neq x$. Then using the inequality (I_d) and the triangle inequality, we get

$$\begin{aligned} hd(x, Tx) &\leq \varphi(x) - \varphi(Tx) \\ &= d(x, x_0) - d(Tx, x_0) \\ &\leq d(x, Tx) + d(Tx, x_0) - d(Tx, x_0) \\ &= d(x, Tx) \end{aligned}$$

and so

$$(h-1)d(x, Tx) \leq 0,$$

which is a contradiction since $h > 1$. Hence, we obtain $Tx = x$ and $T = I_X$. Consequently, $C_{x_0,r}$ is a fixed circle of T. □

Notice that the converse statement of this theorem is also true. Hence, if a self-mapping T in Theorem 2.1 (resp. Theorems 2.2, 2.3) does not satisfy the condition (I_d) given in Theorem 2.4 then T cannot be the identity map.

Considering the above examples, we see that our existence theorems are depending on the given circle (and so the metric on X). Also fixed circle should not to be unique as seen in Example 2.7. Therefore, it is necessary and important to determine some uniqueness theorems for fixed circles.

3 Some Uniqueness Theorems

In this section, we investigate the uniqueness of the fixed circles in theorems obtained in Sect. 2. Notice that the fixed circle $C_{x_0,r}$ is not necessarily unique in Theorem 2.1 (resp. Theorems 2.2, 2.3). We can give the following result.

Proposition 3.1 *Let (X, d) be a metric space. For any given circles $C_{x_0,r}$ and $C_{x_1,\rho}$, there exists at least one self-mapping T of X such that T fixes the circles $C_{x_0,r}$ and $C_{x_1,\rho}$.*

Proof Let $C_{x_0,r}$ and $C_{x_1,\rho}$ be any circles on X. Let us define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} x; & x \in C_{x_0,r} \cup C_{x_1,\rho} \\ \alpha; & \text{otherwise} \end{cases}, \tag{3.1}$$

for all $x \in X$, where α is a constant satisfying $d(\alpha, x_0) \neq r$ and $d(\alpha, x_1) \neq \rho$. Let us define the mappings $\varphi_1, \varphi_2 : X \to [0, \infty)$ as

$$\varphi_1(x) = d(x, x_0)$$

and

$$\varphi_2(x) = d(x, x_1),$$

for all $x \in X$. Then it can be easily checked that the conditions (C1) and (C2) are satisfied by T for the circles $C_{x_0,r}$ and $C_{x_1,\rho}$ with the mappings $\varphi_1(x)$ and $\varphi_2(x)$, respectively. Clearly $C_{x_0,r}$ and $C_{x_1,\rho}$ are the fixed circles of T by Theorem 2.1. □

Notice that the circles $C_{x_0,r}$ and $C_{x_1,\rho}$ do not have to be disjoint (see Example 2.7).

Remark 3.1 Let (X, d) be a metric space and $C_{x_0,r}$, $C_{x_1,\rho}$ be two circles on X. If we consider the self-mapping T defined in (3.1), then the conditions $(C1)^{*}$ and $(C2)^{*}$ are satisfied by T for the circles $C_{x_0,r}$ and $C_{x_1,\rho}$ with the mappings $\varphi_1(x)$ and $\varphi_2(x)$, respectively. Clearly $C_{x_0,r}$ and $C_{x_1,\rho}$ are the fixed circles of T by Theorem 2.2. Similarly, the self-mapping T in (3.1) satisfies the conditions $(C1)^{**}$ and $(C2)^{**}$ for the circles $C_{x_0,r}$ and $C_{x_1,\rho}$ with the mappings $\varphi_1(x)$ and $\varphi_2(x)$, respectively.

Corollary 3.1 *Let (X, d) be a metric space. For any given circles $C_{x_1,r_1}, \ldots, C_{x_n,r_n}$, there exists at least one self-mapping T of X such that T fixes the circles $C_{x_1,r_1}, \ldots, C_{x_n,r_n}$.*

Example 3.1 Let (X, d) be a metric space and $C_{x_1,r_1}, \ldots, C_{x_n,r_n}$ be any circles on X. Let α be a constant such that

$$d(\alpha, x_i) \neq r_i \quad (1 \leq i \leq n).$$

Let us define the self-mapping $T : X \to X$ by

$$Tx = \begin{cases} x; & x \in \bigcup_{i=1}^{n} C_{x_i,r_i} \\ \alpha; & \text{otherwise} \end{cases},$$

for all $x \in X$ and the mappings $\varphi_i : X \to [0, \infty)$ as

$$\varphi_i(x) = d(x, x_i) \quad (1 \leq i \leq n).$$

Then it can be easily checked that the conditions (C1) and (C2) are satisfied by T for the circles $C_{x_1,r_1}, \ldots, C_{x_n,r_n}$, respectively. Consequently, $C_{x_1,r_1}, \ldots, C_{x_n,r_n}$ are fixed circles of T by Theorem 2.1. Notice that these circles do not have to be disjoint.

Therefore, it is important to investigate the uniqueness of the fixed circles. Now we determine the uniqueness conditions for the fixed circles in Theorem 2.1.

Theorem 3.1 *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let $T : X \to X$ be a self-mapping satisfying the conditions (C1) and (C2) given in Theorem 2.1. If the contraction condition*

$$(C3) \qquad d(Tx, Ty) \leq hd(x, y), \tag{3.2}$$

is satisfied for all $x \in C_{x_0,r}$, $y \in X \backslash C_{x_0,r}$ and some $h \in [0, 1)$ by T, then $C_{x_0,r}$ is the unique fixed circle of T.

Proof Assume that there exist two fixed circles $C_{x_0,r}$ and $C_{x_1,\rho}$ of the self-mapping T, that is, T satisfies the conditions (C1) and $(C2)$ for each circles $C_{x_0,r}$ and $C_{x_1,\rho}$.

Let $u \in C_{x_0,r}$ and $v \in C_{x_1,\rho}$ be arbitrary points. We show that $d(u, v) = 0$ and hence $u = v$. Using the condition $(C3)$, we have

$$d(u, v) = d(Tu, Tv) \leq hd(u, v),$$

which is a contradiction since $h \in [0, 1)$. Consequently, $C_{x_0,r}$ is the unique fixed circle of T. □

Notice that the self-mapping T given in the proof of Proposition 3.1 does not satisfy the contraction condition $(C3)$.

We give a uniqueness condition for the fixed circles in Theorem 2.2.

Theorem 3.2 *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let $T : X \to X$ be a self-mapping satisfying the conditions $(C1)^*$ and $(C2)^*$ given in Theorem 2.2. If the contraction condition $(C3)$ defined in (3.2) is satisfied for all $x \in C_{x_0,r}$, $y \in X \backslash C_{x_0,r}$ and some $h \in [0, 1)$ by T, then $C_{x_0,r}$ is the unique fixed circle of T.*

Proof It can be easily seen by the same arguments used in the proof of Theorem 3.1. □

Finally, we give a uniqueness condition for the fixed circles in Theorem 2.3.

Theorem 3.3 *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let $T : X \to X$ be a self-mapping satisfying the conditions $(C1)^{**}$ and $(C2)^{**}$ given in Theorem 2.3. If the contraction condition*

$$(C3)^{**} \quad d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

is satisfied for all $x \in C_{x_0,r}$, $y \in X \backslash C_{x_0,r}$ by T, then $C_{x_0,r}$ is the unique fixed circle of T.

Proof Suppose that there exist two fixed circles $C_{x_0,r}$ and $C_{x_1,\rho}$ of the self-mapping T, that is, T satisfies the conditions $(C1)^{**}$ and $(C2)^{**}$ for each circles $C_{x_0,r}$ and $C_{x_1,\rho}$. Let $u \in C_{x_0,r}$, $v \in C_{x_1,\rho}$ and $u \neq v$ be arbitrary points. We show that $d(u, v) = 0$ and hence $u = v$. Using the condition $(C3)^{**}$, we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{d(u, v), d(u, Tu), d(v, Tv), d(u, Tv), d(v, Tu)\} \\ &= d(u, v), \end{aligned}$$

which is a contradiction. Consequently, it should be $u = v$ for all $u \in C_{x_0,r}$, $v \in C_{x_1,\rho}$ and so $C_{x_0,r}$ is the unique fixed circle of T. □

Notice that the uniqueness of the fixed circle in Theorems 2.1 and 2.2 can be also obtained using the contraction condition $(C3)^{**}$. Similarly, the uniqueness of the fixed circle in Theorem 2.3 can be also obtained using the contraction condition $(C3)$. More generally it is possible to use appropriate contractive conditions for the uniqueness of the fixed-circle theorems obtained in Sect. 2.

4 An Application of the Fixed-Circle Theory to Discontinuous Activation Functions

There exist some applications of discontinuous activation functions in real-valued and complex-valued neural networks. Such applications have been become important in some recent studies (see [4,8,11] for more details).

In this section, we give an application of Theorem 2.2 given in Sect. 2 to discontinuous functions.

Now we recall the following discontinuity theorem given in [1]. Let

$$M(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}. \tag{4.1}$$

Theorem 4.1 [1] *Let (X,d) be a complete metric space. Let T be a self-mapping on X such that T^2 is continuous and satisfies the conditions;*

1. *$d(Tx,Ty)\leq\phi(M(x,y))$, where $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is such that $\phi(t)<t$ for each $t>0$,*
2. *For a given $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ such that*

$$\varepsilon<M(x,y)<\varepsilon+\delta,$$

implies $d(Tx,Ty)\leq\varepsilon$,

then T has a unique fixed point z and $T^n x\to z$ for each $x\in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x\to z}M(x,z)\neq 0.$$

If we consider Theorem 2.2 together with the set $M(x,y)$ defined in (4.1), then we get the following proposition:

Proposition 4.1 *Let (X,d) be a metric space, T be a self-mapping on X and $C_{x_0,r}$ be a fixed circle of T. Then T is discontinuous at any $x\in C_{x_0,r}$ if and only if $\lim_{z\to x}M(z,x)\neq 0$.*

Now we give an application of Proposition 4.1 to discontinuous activation functions.

In [8], the problem of multistability of competitive neural networks with discontinuous activation functions was studied and a general class of discontinuous activation function was defined by

$$T_i x=\begin{cases} u_i; & -\infty<x<p_i\\ l_{i,1}x+c_{i,1}; & p_i\leq x\leq r_i\\ l_{i,2}x+c_{i,2}; & r_i<x\leq q_i\\ v_i; & q_i<x<+\infty\end{cases}, \tag{4.2}$$

where $p_i, r_i, q_i, u_i, v_i, l_{i,1}, l_{i,2}, c_{i,1}$ and $c_{i,2}$ are constants with

$$\begin{aligned}
&-\infty < p_i < r_i < q_i < +\infty, \\
&l_{i,1} > 0, \quad l_{i,2} < 0, \\
&u_i = l_{i,1} p_i + c_{i,1} = l_{i,2} q_i + c_{i,2}, \\
&l_{i,1} r_i + c_{i,1} = l_{i,2} r_i + c_{i,2}, \\
&v_i > T_i r_i, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

To obtain an application of Proposition 4.1, we take

$$\begin{aligned}
p_i &= -2, r_i = 1, q_i = 4, \\
u_i &= 4, v_i = 7, l_{i,1} = 1, \\
c_{i,1} &= 6, l_{i,2} = -1, c_{i,2} = 8,
\end{aligned}$$

in the above activation function T_i defined in (4.2). Then we get the following discontinuous activation function.

$$Tx = \begin{cases} 4; & -\infty < x < -2 \\ x + 6; & -2 \leq x \leq 1 \\ -x + 8; & 1 < x \leq 4 \\ 9; & 4 < x < +\infty \end{cases}.$$

The function T satisfies the conditions of Theorem 2.2 for the circle $C_{\frac{13}{2},\frac{5}{2}} = \{4, 9\}$ with the center $x_0 = \frac{13}{2}$ and the radius $r = \frac{5}{2}$. Hence, T fixes the circle $C_{\frac{13}{2},\frac{5}{2}}$. We obtain that the function T is discontinuous at any $x \in C_{\frac{13}{2},\frac{5}{2}}$ if and only if $\lim_{x \to z} M(x, z) \neq 0$ by Proposition 4.1. It can be easily seen that T is continuous at the point $x_1 = 9$ but it is discontinuous at $x_2 = 4$.

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