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# New Fixed-Point Theorems on an $S$ -metric Space via Simulation Functions

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**Abstract:** In this paper, we prove new fixed-point theorems using the set of simulation functions on an  $S$ -metric space with some illustrative examples. Our results are stronger than some known fixed-point results. Furthermore, we give an application to the fixed-circle problem with respect to a simulation function.

**Keywords:** fixed point; fixed disc;  $S$ -metric space; simulation function

**MSC:** Primary 54H25; Secondary 47H09, 47H10

## 1. Introduction

Showing the existence and uniqueness of a fixed point has many applications, in different fields, such as computer sciences, engineering, etc.; see [1]. Using the technique of iterations to prove the existence and uniqueness of a fixed point for a self-mapping on a metric space was first introduced by Banach in [2]. Most of the work after was basically a generalization of the work of Banach. These generalizations include more general metric spaces, or more general contractions, etc. (see [3,4]), which are important due to the fact that the more general the metric space, the larger the class, which implies that the obtained results can be applied in more different fields to solve unsolved problems. Moreover, these generalizations do not just include metric spaces; they include contractions, as well.

In this manuscript, we work with  $S$ -metric spaces, which are recent generalizations of metric spaces. In the next section, we provide the reader with a background about this space along with some lemmas.

## 2. Preliminaries

Fixed-point theory has been extensively studied using different approaches, and that is due to the fact that it has many applications in many fields. For example, the concept of a simulation function was defined to obtain new fixed-point theorems as follows:

**Definition 1.** We say that the function  $\zeta : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is a simulation function [5], if it satisfies the following conditions: ( $\zeta_1$ )  $\zeta(0,0) = 0$ , ( $\zeta_2$ )  $\zeta(\Omega, \nu) < \nu - \Omega$  for all  $s, t > 0$ , ( $\zeta_3$ ) If  $\{\Omega_n\}, \{\nu_n\}$  are sequences in  $\mathbb{R}_+$  such that:

$$\lim_{n \rightarrow \infty} \Omega_n = \lim_{n \rightarrow \infty} \nu_n > 0,$$

then

$$\limsup_{n \rightarrow \infty} \zeta(\Omega_n, \nu_n) < 0.$$

The set of all simulation functions is denoted by  $\mathcal{Z}$  [5]. There are some examples of simulation functions in [5,6] and the references therein.

On the other hand, some generalized metric spaces were defined to obtain some generalized fixed-point results. For example,  $G$ -metric spaces and  $S$ -metric spaces were introduced as a generalization of a metric space, respectively in [7,8], as follows:

**Definition 2.** Let  $X$  be a nonempty set and  $\Omega : X^3 \rightarrow [0, \infty)$  be a function such that for all  $v, \mu, \xi, a \in X$ ,

(G1)  $\Omega(v, \mu, \xi) = 0$  if  $v = \mu = \xi$ ,

(G2)  $0 < \Omega(v, v, \mu)$  if  $v \neq \mu$ ,

(G3)  $\Omega(v, v, \mu) \leq \Omega(v, \mu, \xi)$  if  $\mu \neq \xi$ ,

(G4) The symmetry on three variables:

$$\Omega(v, \mu, \xi) = \Omega(v, \xi, \mu) = \Omega(\mu, v, \xi) = \Omega(\mu, \xi, v) = \Omega(\xi, v, \mu) = \Omega(\xi, \mu, v),$$

(G5) The rectangular inequality:

$$\Omega(v, \mu, \xi) \leq \Omega(v, a, a) + \Omega(a, \mu, \xi).$$

Then,  $\Omega$  is called a  $G$ -metric on  $X$ , and the pair  $(X, \Omega)$  is called a  $G$ -metric space [7].

**Definition 3.** Consider the set  $X \neq \emptyset$  and  $\mathcal{S} : X^3 \rightarrow [0, \infty)$  to be a function satisfying the following conditions for all  $v, \mu, \xi, a \in X$ .

1.  $\mathcal{S}(v, \mu, \xi) = 0$  if and only if  $v = \mu = \xi$ ,

2.  $\mathcal{S}(v, \mu, \xi) \leq \mathcal{S}(v, v, a) + \mathcal{S}(\mu, \mu, a) + \mathcal{S}(\xi, \xi, a)$ .

Then,  $\mathcal{S}$  is called an  $S$ -metric on  $X$ , and the pair  $(X, \mathcal{S})$  is called an  $S$ -metric space [8].

It is well known that the class of all  $G$ -metrics and the class of all  $S$ -metrics are distinct, that is not every  $G$ -metric is an  $S$ -metric and also not every  $S$ -metric is a  $G$ -metric (see [4] for more details).

On the other hand, the relationships between a metric and an  $S$ -metric are given as follows:

**Lemma 1.** Let  $(X, \Lambda)$  be a metric space [9]. Then, the following properties are satisfied:

1.  $\mathcal{S}_\Lambda(v, \mu, \xi) = \Lambda(v, \xi) + \Lambda(\mu, \xi)$  for all  $v, \mu, \xi \in X$  is an  $S$ -metric on  $X$ .

2.  $v_n \rightarrow v$  in  $(X, \Lambda)$  if and only if  $v_n \rightarrow v$  in  $(X, \mathcal{S}_\Lambda)$ .

3.  $\{v_n\}$  is Cauchy in  $(X, \Lambda)$  if and only if  $\{v_n\}$  is Cauchy in  $(X, \mathcal{S}_\Lambda)$ .

4.  $(X, \Lambda)$  is complete if and only if  $(X, \mathcal{S}_\Lambda)$  is complete.

The metric  $\mathcal{S}_\Lambda$  was called the  $S$ -metric generated by  $\Lambda$  [10]. Some examples of an  $S$ -metric that is not generated by any metric are known (see [9,10] for more details).

Furthermore, Gupta claimed that every  $S$ -metric on  $X$  defines a metric  $d_S$  on  $X$  as follows:

$$d_S(x, y) = \mathcal{S}(x, x, y) + \mathcal{S}(y, y, x), \tag{1}$$

for all  $x, y \in X$  [11]. However, since the triangle inequality is not satisfied for all elements of  $X$  everywhere, the function  $d_S(x, y)$  defined in (1) does not always define a metric (see [10]).

In the following example, we see an example of an  $S$ -metric that is not generated by any metric.

**Example 1.** Let  $K = \mathbb{R}$  and the function  $\mathcal{S} : K^3 \rightarrow [0, \infty)$  be defined as:

$$\mathcal{S}(v, \mu, \xi) = |v - \xi| + |v + \xi - 2\mu|,$$

for all  $\nu, \mu, \xi \in \mathbb{R}$ . Then,  $\mathcal{S}$  is an  $S$ -metric that is not generated by any metric, and the pair  $(K, \mathcal{S})$  is an  $S$ -metric space [10].

The following lemma is used in the proofs of the obtained results.

**Lemma 2.** Let  $(X, \mathcal{S})$  be an  $S$ -metric space [8]. Then, we have:

$$\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x).$$

Motivated by the above results, in this paper, we prove some fixed-point theorems using the set of simulation functions on an  $S$ -metric space. To do this, we are inspired by the idea given in [5,12] using the simulation function approach on a metric and a  $G$ -metric space, respectively. Given the fact that not every  $S$ -metric space is generated by a metric space, since the classes of  $S$ -metric spaces and  $G$ -metric spaces are different, it is interesting to study new fixed-point results on  $S$ -metric spaces using the set  $\mathcal{Z}$ . In Section 1, we recall some necessary definitions and properties. In Section 3, we define a new contractive condition called a  $\mathcal{Z}_S$ -contraction using simulation function  $\zeta$  on an  $S$ -metric space. Using this contraction, we prove some basic lemmas and a fixed-point theorem with an illustrative example and give some remarks. In Section 4, we give an application to the fixed-circle problem, which is a recent geometric approach to the fixed-point theory, modifying the notion of a  $\mathcal{Z}_S$ -contraction on  $S$ -metric spaces.

### 3. Main Results

Throughout the paper, we assume that  $(X, \mathcal{S})$  is an  $S$ -metric space,  $T : X \rightarrow X$  is a self-mapping, and  $\zeta \in \mathcal{Z}$ . Recall that  $T$  is called a contraction if there exists a constant  $L \in [0, 1)$  such that:

$$\mathcal{S}(Tx, Tx, Ty) \leq L\mathcal{S}(x, x, y),$$

for all  $x, y \in X$  [8].

**Definition 4.** If  $T$  satisfies the following condition:

$$\zeta(\mathcal{S}(Tx, Tx, Ty), \mathcal{S}(x, x, y)) \geq 0,$$

for all  $x, y \in X$ , then  $T$  is called a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta$ .

**Example 2.** Let  $T$  be a contraction on  $(X, \mathcal{S})$ . If we take  $L \in [0, 1)$  and  $\zeta(t, s) = Ls - t$  for all  $t, s \in [0, \infty)$ , then a contraction  $T$  is a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta$ . Indeed, let  $t = \mathcal{S}(Tx, Tx, Ty)$  and  $s = \mathcal{S}(x, x, y)$ . Since  $T$  is a contraction, we have:

$$\begin{aligned} \mathcal{S}(Tx, Tx, Ty) &\leq L\mathcal{S}(x, x, y) \implies L\mathcal{S}(x, x, y) - \mathcal{S}(Tx, Tx, Ty) \geq 0 \\ \implies \zeta(\mathcal{S}(Tx, Tx, Ty), \mathcal{S}(x, x, y)) &\geq 0, \end{aligned}$$

for all  $x, y \in X$ . Therefore,  $T$  is a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta$ .

We note that every  $\mathcal{Z}_S$ -contraction is a contraction, Therefore, it is continuous (see [8]). Using the  $\mathcal{Z}_S$ -contractive property and the condition  $(\zeta_2)$ , we get:

$$\mathcal{S}(Tx, Tx, Ty) < \mathcal{S}(x, x, y),$$

for all distinct  $x, y \in X$ .

**Lemma 3.** If  $T$  is a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta$  and  $T$  has a fixed point, then the fixed point is unique.

**Proof.** Let  $x \in X$  be a fixed point of  $T$ . Let  $y \in X$  be another fixed point of  $T$  such that  $x \neq y$ . Using the  $\mathcal{Z}_S$ -contractive property and the condition  $(\zeta_2)$ , we get:

$$\begin{aligned} 0 &\leq \zeta(\mathcal{S}(Tx, Tx, Ty), \mathcal{S}(x, x, y)) = \zeta(\mathcal{S}(x, x, y), \mathcal{S}(x, x, y)) \\ &< \mathcal{S}(x, x, y) - \mathcal{S}(x, x, y) = 0, \end{aligned}$$

a contradiction. It should be  $x = y$ .  $\square$

We recall that a self-mapping  $T$  is called asymptotically regular at the point  $x \in X$  if  $\lim_{n \rightarrow \infty} \mathcal{S}(T^n x, T^n x, T^{n+1} x) = 0$  [13].

**Lemma 4.** *If  $T$  is a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta$ , then  $T$  is asymptotically regular at every point  $x \in X$ .*

**Proof.** Let  $x \in X$ . If we have  $T^m x = T^{m-1} x$ , that is,  $Ta = a$  where  $a = T^{m-1} x$  for some  $m \in \mathbb{N}$ , then:

$$T^n a = T^{n-1} T a = T^{n-1} a = \dots = T a = a,$$

for all  $n \in \mathbb{N}$ . Therefore, we get:

$$\begin{aligned} \mathcal{S}(T^n x, T^n x, T^{n+1} x) &= \mathcal{S}(T^{n-m+1} T^{m-1} x, T^{n-m+1} T^{m-1} x, T^{n-m+2} T^{m-1} x) \\ &= \mathcal{S}(T^{n-m+1} a, T^{n-m+1} a, T^{n-m+2} a) \\ &= \mathcal{S}(a, a, a) = 0, \end{aligned}$$

whence:

$$\lim_{n \rightarrow \infty} \mathcal{S}(T^n x, T^n x, T^{n+1} x) = 0.$$

Now, we suppose that  $T^n x \neq T^{n+1} x$  for all  $n \in \mathbb{N}$ . Using the  $\mathcal{Z}_S$ -contractive property and the condition  $(\zeta_2)$ , we obtain:

$$\begin{aligned} 0 &\leq \zeta(\mathcal{S}(T^{n+1} x, T^{n+1} x, T^n x), \mathcal{S}(T^n x, T^n x, T^{n-1} x)) \\ &= \zeta(\mathcal{S}(TT^n x, TT^n x, TT^{n-1} x), \mathcal{S}(T^n x, T^n x, T^{n-1} x)) \\ &< \mathcal{S}(T^n x, T^n x, T^{n-1} x) - \mathcal{S}(T^{n+1} x, T^{n+1} x, T^n x) \end{aligned}$$

and so:

$$\mathcal{S}(T^{n+1} x, T^{n+1} x, T^n x) < \mathcal{S}(T^n x, T^n x, T^{n-1} x),$$

that is,  $\{\mathcal{S}(T^n x, T^n x, T^{n-1} x)\}$  is a monotonically-decreasing sequence of nonnegative real numbers. Therefore, it should be convergent. Let  $\lim_{n \rightarrow \infty} \mathcal{S}(T^n x, T^n x, T^{n+1} x) = \mu \geq 0$ . If  $\mu > 0$ , then using the  $\mathcal{Z}_S$ -contractive property and the condition  $(\zeta_3)$ , we have:

$$0 \leq \limsup_{n \rightarrow \infty} \zeta(\mathcal{S}(T^{n+1} x, T^{n+1} x, T^n x), \mathcal{S}(T^n x, T^n x, T^{n-1} x)) < 0,$$

a contradiction. It should be  $\mu = 0$ , that is,

$$\lim_{n \rightarrow \infty} \mathcal{S}(T^n x, T^n x, T^{n+1} x) = 0.$$

Consequently,  $T$  is asymptotically regular at every point  $x \in X$ .  $\square$

**Lemma 5.** *If  $T$  is a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta$ , then the Picard sequence  $\{x_n\}$  generated by  $T$  such that  $Tx_{n-1} = x_n$  for all  $n \in \mathbb{N}$  with initial value  $x_0 \in X$  is a bounded sequence.*

**Proof.** Let  $x_0 \in X$  and  $\{x_n\}$  be the Picard sequence. Now, we show that  $\{x_n\}$  is a bounded sequence. On the contrary, we suppose that  $\{x_n\}$  is not bounded. Let  $x_{n+m} \neq x_n$  for all  $n, m \in \mathbb{N}$ . Since  $\{x_n\}$  is not bounded, there exists a subsequence  $\{x_{n_k}\}$  such that  $n_1 = 1$ , and for each  $k \in \mathbb{N}$ ,  $n_{k+1}$  is the minimum integer such that:

$$\mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k}) > 1$$

and:

$$\mathcal{S}(x_m, x_m, x_{n_k}) \leq 1 \text{ for } n_k \leq m \leq n_{k+1} - 1.$$

Therefore, using the condition (S2), we obtain:

$$\begin{aligned} 1 &< \mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k}) \\ &\leq \mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}-1}) + \mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}-1}) + \mathcal{S}(x_{n_k}, x_{n_k}, x_{n_{k+1}-1}) \\ &\leq 2\mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}-1}) + 1. \end{aligned}$$

If we take a limit for  $k \rightarrow \infty$ , then using Lemma 4, we get:

$$\lim_{k \rightarrow \infty} \mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k}) = 1.$$

By the  $\mathcal{Z}_S$ -contractive property, we have:

$$\mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k}) \leq \mathcal{S}(x_{n_{k+1}-1}, x_{n_{k+1}-1}, x_{n_k-1})$$

and so, using the condition (S2), we get:

$$\begin{aligned} 1 &< \mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k}) \leq \mathcal{S}(x_{n_{k+1}-1}, x_{n_{k+1}-1}, x_{n_k-1}) \\ &= \mathcal{S}(x_{n_k-1}, x_{n_k-1}, x_{n_{k+1}-1}) \\ &\leq 2\mathcal{S}(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + \mathcal{S}(x_{n_{k+1}-1}, x_{n_{k+1}-1}, x_{n_k}) \\ &\leq 2\mathcal{S}(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + 1. \end{aligned}$$

If we take a limit for  $k \rightarrow \infty$ , then using Lemma 4, we have:

$$\lim_{k \rightarrow \infty} \mathcal{S}(x_{n_{k+1}-1}, x_{n_{k+1}-1}, x_{n_k-1}) = 1.$$

Using the  $\mathcal{Z}_S$ -contractive property and the condition ( $\zeta_3$ ), we get:

$$\begin{aligned} 0 &\leq \limsup_{k \rightarrow \infty} \zeta (\mathcal{S}(Tx_{n_{k+1}-1}, Tx_{n_{k+1}-1}, Tx_{n_k-1}), \mathcal{S}(x_{n_{k+1}-1}, x_{n_{k+1}-1}, x_{n_k-1})) \\ &= \limsup_{k \rightarrow \infty} \zeta (\mathcal{S}(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k}), \mathcal{S}(x_{n_{k+1}-1}, x_{n_{k+1}-1}, x_{n_k-1})) < 0, \end{aligned}$$

a contradiction. Consequently,  $\{x_n\}$  is bounded.  $\square$

**Theorem 1.** Let  $(X, \mathcal{S})$  be a complete  $S$ -metric space and  $T : X \rightarrow X$  be a self-mapping. If  $T$  is a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta$ , then  $T$  has a unique fixed point  $a \in X$ , and the Picard sequence  $\{x_n\}$  converges to the fixed point  $a$ .

**Proof.** Let  $x_0 \in X$  and the Picard sequence  $\{x_n\}$  be defined as:

$$Tx_{n-1} = x_n,$$

for all  $n \in \mathbb{N}$ . Now, we show that  $\{x_n\}$  is a Cauchy sequence. To do this, let:

$$S_n = \sup \{ \mathcal{S}(x_i, x_i, x_j) : i, j \geq n \}.$$

It is not difficult to see that  $\{S_n\}$  is a monotonically-decreasing sequence. Moreover, by Lemma 5,  $\{x_n\}$  is a bounded sequence. Thus,  $S_n < \infty$  for all  $n \in \mathbb{N}$ , which implies that  $\{S_n\}$  is a monotonic bounded sequence, whence it is convergent. Thus, there exists  $s \geq 0$  such that  $\lim_{n \rightarrow \infty} S_n = s$ . We claim that  $s = 0$ . On the contrary, we assume that  $s > 0$ . By the definition of  $S_n$ , for all  $k \in \mathbb{N}$ , there exists  $n_k, m_k$  such that  $m_k > n_k \geq k$  and:

$$S_k - \frac{1}{k} < \mathcal{S}(x_{m_k}, x_{m_k}, x_{n_k}) \leq S_k.$$

Therefore, we get:

$$\lim_{k \rightarrow \infty} \mathcal{S}(x_{m_k}, x_{m_k}, x_{n_k}) = s.$$

Using the  $\mathcal{Z}_S$ -contractive property, Lemma 4, and the condition (S2), we have:

$$\begin{aligned} \mathcal{S}(x_{m_k}, x_{m_k}, x_{n_k}) &\leq \mathcal{S}(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) \\ &\leq 2\mathcal{S}(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + \mathcal{S}(x_{n_k-1}, x_{n_k-1}, x_{m_k}) \\ &\leq 2\mathcal{S}(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + 2\mathcal{S}(x_{n_k-1}, x_{n_k-1}, x_{n_k}) \\ &\quad + \mathcal{S}(x_{m_k}, x_{m_k}, x_{n_k}). \end{aligned}$$

If we take a limit for  $k \rightarrow \infty$ , then we get:

$$\lim_{k \rightarrow \infty} \mathcal{S}(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = s.$$

Using the  $\mathcal{Z}_S$ -contractive property and the condition  $(\zeta_3)$ , we have:

$$0 \leq \limsup_{k \rightarrow \infty} \zeta (\mathcal{S}(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), \mathcal{S}(x_{m_k}, x_{m_k}, x_{n_k})) < 0,$$

a contradiction. Then, it should be  $s = 0$ , and so,  $\{x_n\}$  is Cauchy. Since  $(X, \mathcal{S})$  is a complete  $S$ -metric space, there exists  $a \in X$  such that  $\lim_{n \rightarrow \infty} x_n = a$ . Now, we prove that  $a$  is a fixed point of  $T$ . If  $Ta \neq a$ , then  $\mathcal{S}(a, a, Ta) = \mathcal{S}(Ta, Ta, a) > 0$ . Using the  $\mathcal{Z}_S$ -contractive property and the conditions  $(\zeta_2)$  and  $(\zeta_3)$ , we have:

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \zeta (\mathcal{S}(Tx_n, Tx_n, Ta), \mathcal{S}(x_n, x_n, a)) \\ &\leq \limsup_{n \rightarrow \infty} [\mathcal{S}(x_n, x_n, a) - \mathcal{S}(Tx_n, Tx_n, Ta)] \\ &= -\mathcal{S}(a, a, Ta), \end{aligned}$$

a contradiction. It should be  $\mathcal{S}(a, a, Ta) = 0$ , that is  $Ta = a$ . Therefore,  $a$  is a fixed point of  $T$ . The uniqueness part of this theorem can be easily seen from Lemma 3.  $\square$

We give the following example to show the validity of Theorem 1.

**Example 3.** Let  $X = [0, \frac{1}{4}]$  and  $(X, \mathcal{S})$  be the  $S$ -metric space defined in Example 1. Then,  $(X, \mathcal{S})$  is a complete  $S$ -metric space. Let us define the self-mapping  $T : X \rightarrow X$  as:

$$Tx = \frac{x}{x + 1},$$

for all  $x \in X$ . Then, clearly,  $T$  is a continuous function. However,  $T$  is not a contraction. Indeed, we get:

$$\mathcal{S}(Tx, Tx, Ty) = \frac{2|x - y|}{(x + 1)(y + 1)} \leq 2|x - y| = \mathcal{S}(x, x, y),$$

a contradiction with the definition of a contraction. Now, we show that  $T$  is a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta$  defined as:

$$\zeta(t, s) = \frac{s}{s + \frac{1}{4}} - t,$$

for all  $t, s \in [0, \infty)$ . Indeed, for all  $x, y \in X$ , we have:

$$\begin{aligned} \zeta(\mathcal{S}(Tx, Tx, Ty), \mathcal{S}(x, x, y)) &= \frac{\mathcal{S}(x, x, y)}{\mathcal{S}(x, x, y) + \frac{1}{4}} - \mathcal{S}(Tx, Tx, Ty) \\ &= \frac{2|x - y|}{2|x - y| + \frac{1}{4}} - \frac{2|x - y|}{(x + 1)(y + 1)} \geq 0 \end{aligned}$$

and so, by Theorem 1,  $T$  has a unique fixed point  $a = 0$ .

We investigate some relationships between some known selected contractive conditions and a  $\mathcal{Z}_S$ -contractive condition as follows:

**Remark 1.** (i) Let  $T$  be a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta_1 \in \mathcal{Z}$  defined by:

$$\zeta_1(t, s) = Ls - t, L \in [0, 1) \tag{2}$$

for all  $s, t \in [0, \infty)$ . Then, we get:

$$\begin{aligned} 0 &\leq \zeta_1(\mathcal{S}(Tx, Tx, Ty), \mathcal{S}(x, x, y)) = L\mathcal{S}(x, x, y) - \mathcal{S}(Tx, Tx, Ty) \\ &\implies \mathcal{S}(Tx, Tx, Ty) \leq L\mathcal{S}(x, x, y). \end{aligned} \tag{3}$$

This shows that  $T$  is a contraction. The inequality (3) can be considered as the Banach-type contractive condition on  $S$ -metric spaces (see [2,8]). Furthermore, using the inequality (3), we obtain:

$$\mathcal{S}(Tx, Tx, Ty) \leq L\mathcal{S}(x, x, y) < \mathcal{S}(x, x, y), \tag{4}$$

which can be considered as the Nemystkii-Edelstein type contractive condition on an  $S$ -metric space (see [8,14–16]). From the inequality (4), we get:

$$\begin{aligned} \mathcal{S}(Tx, Tx, Ty) &< \mathcal{S}(x, x, y) \\ &< \max \{ \mathcal{S}(x, x, y), \mathcal{S}(Tx, Tx, x), \mathcal{S}(Ty, Ty, y), \mathcal{S}(Ty, Ty, x), \mathcal{S}(Tx, Tx, y) \}, \end{aligned}$$

which can be considered as the Rhoades-type (S25) contractive condition on an  $S$ -metric space (see [16,17]). Again, if we take the inequality (3), then we get:

$$\begin{aligned} \mathcal{S}(Tx, Tx, Ty) &\leq L\mathcal{S}(x, x, y) \\ &\leq L \max \left\{ \begin{array}{l} \mathcal{S}(x, x, y), \mathcal{S}(Tx, Tx, x), \mathcal{S}(Ty, Ty, y), \\ \mathcal{S}(Ty, Ty, x), \mathcal{S}(Tx, Tx, y) \end{array} \right\}, \end{aligned}$$

which can be considered as the Ćirić-type contractive condition on S-metric spaces (see [3,18]). On the other hand, using the inequality (4), we obtain:

$$\begin{aligned} \mathcal{S}(Tx, Tx, Ty) &< \mathcal{S}(x, x, y) \\ &< \max \{ \mathcal{S}(x, x, y), \mathcal{S}(Tx, Tx, x), \mathcal{S}(Ty, Ty, y) \}, \end{aligned}$$

which can be considered as the Sehgal-type contractive condition on an S-metric space (see [19]).

(ii) Let T be a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta_2 \in \mathcal{Z}$  defined by:

$$\zeta_2(t, s) = \alpha s - t,$$

for all  $s, t \in [0, \infty)$ , where  $\alpha : (0, \infty) \rightarrow [0, 1)$  is a monotone decreasing function. Then, we get:

$$\begin{aligned} 0 &\leq \zeta_2(\mathcal{S}(Tx, Tx, Ty), \mathcal{S}(x, x, y)) = \alpha \mathcal{S}(x, x, y) - \mathcal{S}(Tx, Tx, Ty) \\ \implies &\mathcal{S}(Tx, Tx, Ty) \leq \alpha \mathcal{S}(x, x, y), \end{aligned} \tag{5}$$

which can be considered as the Rakotch-type contractive condition on S-metric spaces (see [20]). Using the similar arguments given in (i), the inequality (5) can be generalized to the Nemyskii-Edelstein type (resp. Rhoades-type (S25) and Sehgal-type) contractive condition on S-metric spaces.

(iii) Let T be a  $\mathcal{Z}_S$ -contraction with respect to  $\zeta_3 \in \mathcal{Z}$  defined by:

$$\zeta_3(t, s) = s - \varphi(s) - t, \tag{6}$$

for all  $s, t \in [0, \infty)$ , where  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a lower semi-continuous function and  $\varphi^{-1}(0) = \{0\}$ . Then, we get:

$$\begin{aligned} 0 &\leq \zeta_3(\mathcal{S}(Tx, Tx, Ty), \mathcal{S}(x, x, y)) \\ &= \mathcal{S}(x, x, y) - \varphi(\mathcal{S}(x, x, y)) - \mathcal{S}(Tx, Tx, Ty) \\ \implies &\mathcal{S}(Tx, Tx, Ty) \leq \mathcal{S}(x, x, y) - \varphi(\mathcal{S}(x, x, y)), \end{aligned} \tag{7}$$

which can be considered as the Rhoades-type contractive condition on an S-metric space (see [21]). In this case, our result is stronger than the original version of Rhoades since there exist some examples of an S-metric that is not generated by any metric (or does not generate any metric) and  $\varphi$  is lower semi-continuous in our result instead of  $\varphi$  being continuous, nondecreasing, and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ .

In the closing of this section, we introduce the notions of an expanding map and a  $\mathcal{Z}_S$ -expanding self-mapping on S-metric spaces modifying the inequality given in Theorem 3.3 in [22], along with a fixed point result.

**Definition 5.** We say that a self-mapping T on an S-metric space X is an expanding map if there exists a constant  $L \in [0, 1)$  such that:

$$L\mathcal{S}(Tx, Tx, Ty) \geq \mathcal{S}(x, x, y),$$

for all  $x, y \in X$ .

**Definition 6.** If T satisfies the following condition:

$$\zeta(\mathcal{S}(x, x, y), \mathcal{S}(Tx, Tx, Ty)) \geq 0,$$

for all  $x, y \in X$ , then T is a  $\mathcal{Z}_S$ -expanding map with respect to  $\zeta$ .



If  $T$  is a  $\mathcal{Z}_S$ -expanding map with respect to  $\zeta$ , then using the condition  $(\zeta_2)$ , we have the following inequality:

$$\mathcal{S}(x, x, y) \neq \mathcal{S}(Tx, Tx, Ty),$$

for all distinct  $x, y \in X$ . Therefore, the notion of a  $\mathcal{Z}_S$ -expanding mapping is different from the notion of a  $\mathcal{Z}_S$ -contraction with respect to any simulation function  $\zeta$ .

**Lemma 6.** *If  $T$  is a surjective self-mapping on a set  $X$ , then there exists a self-mapping  $T^*$  on  $X$  such that for every  $x \in X$ , we have  $(T \circ T^*)x = x$ .*

**Theorem 2.** *If  $T$  is a surjective  $\mathcal{Z}_S$ -expanding self mapping on a set  $X$ , then there exists a unique  $u \in X$  such that  $Tu = u$ .*

**Proof.** By the hypothesis of the theorem, we know that  $T$  is surjective. Thus, by Lemma 6, there exists a self-mapping  $T^*$  on  $X$  such that for every  $x \in X$ , we have  $(T \circ T^*)x = x$ . Let  $x, y \in X$  be any arbitrary points. Suppose that  $z = T^*x$  and  $v = T^*y$ . First, we note that:

$$Tz = T(T^*x) = x \text{ and } Tv = T(T^*y) = y.$$

Now, since  $T$  is  $\mathcal{Z}_S$ -expanding we have:

$$\zeta(\mathcal{S}(z, z, v), \mathcal{S}(Tz, Tz, Tv)) \geq 0,$$

which implies:

$$\zeta(\mathcal{S}(T^*x, T^*x, T^*y), \mathcal{S}(x, x, y)) \geq 0.$$

Thus,  $T^*$  is a  $\mathcal{Z}_S$ -contraction. Therefore, by Theorem 1,  $T^*$  has a unique fixed point in  $X$  say  $u$ , that is  $T^*u = u$ . Hence,

$$Tu = T(T^*u) = u.$$

Therefore,  $T$  has a fixed point in  $X$ . Now, assume that there exists  $w \in X$  such that  $u \neq w$  and  $Tw = w$ . Hence, we get:

$$T^*w = T^*(Tw) = w.$$

Thus,  $u$  and  $w$  are two fixed points of  $T^*$  in  $X$ , but  $T^*$  has a unique fixed point in  $X$ . Therefore,  $u = w$ , and that is  $T$  has a unique fixed point in  $X$ , as desired.  $\square$

**Corollary 1.** *If a self-mapping  $T$  is a surjective  $\mathcal{Z}_S$ -expanding map with respect to  $\zeta$ , then the inverse mapping  $T^*$  of  $T$  is a  $\mathcal{Z}_S$ -contraction with respect to the same simulation function  $\zeta$ .*

**Example 4.** Let  $X = [0, \frac{1}{4}]$  and  $(X, \mathcal{S})$  be the  $S$ -metric space defined in Example 1. Then,  $(X, \mathcal{S})$  is a complete  $S$ -metric space. Let us define the self-mapping  $T : X \rightarrow X$  as:

$$Tx = \frac{x}{-x + 1},$$

for all  $x \in X$ . It is easy to verify that  $T$  is a surjective  $\mathcal{Z}_S$ -expanding map with respect to  $\zeta$  defined as:

$$\zeta(t, s) = \frac{s}{s + \frac{1}{4}} - t,$$

for all  $t, s \in [0, \infty)$ . By Theorem 2, the self-mapping  $T$  has a unique fixed point  $u = 0$ . Notice that the inverse mapping  $T^* : X \rightarrow X$  of  $T$  is:

$$T^*x = \frac{x}{x + 1}.$$

From Example 3, we know that the self-mapping  $T^*$  is a  $\mathcal{Z}_S$ -contraction with respect to the same simulation function  $\zeta$  and  $T^*$  has a unique fixed point  $u = 0$ .

#### 4. An Application to the Fixed-Circle Problem

In this section, we investigate new solutions to the fixed-circle problem raised by Özgür and Taş in [23]. Some fixed-circle or fixed-disc results, as the direct solutions of this problem, have been studied using various methods on a metric space or some generalized metric spaces (see [24]).

Let  $(X, \mathcal{S})$  be an  $S$ -metric space and  $T : X \rightarrow X$  be a self-mapping. Now, we recall the notion of a disc on  $S$ -metric spaces ([8]) as follows:

$$D_{x_0,r}^S = \{x \in X : \mathcal{S}(x, x, x_0) \leq r\}.$$

If  $Tx = x$  for all  $x \in D_{x_0,r}^S$ , then the disc  $D_{x_0,r}^S$  is called the fixed disc of  $T$ .

**Definition 7.** A self-mapping  $T$  is called a  $\mathcal{Z}_S^C$ -contraction with respect to  $\zeta$  if there exists  $x_0 \in X$  such that:

$$\mathcal{S}(Tx, Tx, x) > 0 \implies \zeta(\mathcal{S}(Tx, Tx, x), \mathcal{S}(Tx, Tx, x_0)) \geq 0,$$

for all  $x \in X$ .

**Theorem 3.** If  $T$  is a  $\mathcal{Z}_S^C$ -contraction with respect to  $\zeta$  for  $x_0 \in X$  and the condition  $0 < \mathcal{S}(Tx, Tx, x_0) \leq r$  holds for all  $x \in D_{x_0,r}^S - \{x_0\}$ , then  $D_{x_0,r}^S$  is a fixed disc of  $T$ , where:

$$r = \inf_{x \in X} \{\mathcal{S}(Tx, Tx, x) : Tx \neq x\}. \tag{8}$$

**Proof.** Case 1: Let  $r = 0$ . Then, we get  $D_{x_0,r}^S = \{x_0\}$ . If  $Tx_0 \neq x_0$ , then we have  $\mathcal{S}(Tx_0, Tx_0, x_0) > 0$ . By the  $\mathcal{Z}_S^C$ -contractive property and the condition  $(\zeta_2)$ , we get

$$\begin{aligned} 0 &\leq \zeta(\mathcal{S}(Tx_0, Tx_0, x_0), \mathcal{S}(Tx_0, Tx_0, x_0)) \\ &< \mathcal{S}(Tx_0, Tx_0, x_0) - \mathcal{S}(Tx_0, Tx_0, x_0) = 0, \end{aligned}$$

a contradiction. Therefore, it should be  $Tx_0 = x_0$ . Case 2: Let  $r > 0$  and  $x \in D_{x_0,r}^S - \{x_0\}$  be any point such that  $Tx \neq x$ . By the definition of  $r$ , we know:

$$0 < r \leq \mathcal{S}(Tx, Tx, x). \tag{9}$$

Using the inequality (9), the  $\mathcal{Z}_S^C$ -contractive property, the hypothesis  $0 < \mathcal{S}(Tx, Tx, x_0) \leq r$ , and the condition  $(\zeta_2)$ , we obtain:

$$\begin{aligned} 0 &\leq \zeta(\mathcal{S}(Tx, Tx, x), \mathcal{S}(Tx, Tx, x_0)) < \mathcal{S}(Tx, Tx, x_0) - \mathcal{S}(Tx, Tx, x) \\ &\leq r - \mathcal{S}(Tx, Tx, x) \leq r - r = 0, \end{aligned}$$

a contradiction. Hence, it should be  $Tx = x$ . Consequently, under the above cases,  $D_{x_0,r}^S$  is a fixed disc of  $T$ .  $\square$

From the above theorem, we give the following corollary modifying the Banach-type contractive condition (3) and Rhoades-type contractive condition (7) on an  $S$ -metric space.

**Corollary 2.** Let  $x_0 \in X$  and  $r$  be defined as in (8). (1) (Banach-type fixed-disc result) If the condition  $0 < \mathcal{S}(Tx, Tx, x_0) \leq r$  holds for all  $x \in D_{x_0, r}^S - \{x_0\}$  and  $T$  satisfies the following inequality for all  $x \in X$ , then  $D_{x_0, r}^S$  is a fixed disc of  $T$  :

$$\mathcal{S}(Tx, Tx, x) \leq L\mathcal{S}(Tx, Tx, x_0), L \in [0, 1).$$

(2) (Rhoades-type fixed-disc result) If the condition  $0 < \mathcal{S}(Tx, Tx, x_0) \leq r$  holds for all  $x \in D_{x_0, r}^S - \{x_0\}$  and  $T$  satisfies the following inequality for all  $x \in X$ , then  $D_{x_0, r}^S$  is a fixed disc of  $T$  :

$$\mathcal{S}(Tx, Tx, x) \leq \mathcal{S}(Tx, Tx, x_0) - \varphi(\mathcal{S}(Tx, Tx, x_0)),$$

where the function  $\varphi$  is defined as in Remark 1 (3).

**Proof.** Let us consider the function  $\zeta_1$  defined as in (2) (resp. the function  $\zeta_3$  defined as in (6)). Using the hypothesis, we can easily see that  $T$  is a  $\mathcal{Z}_\zeta^C$ -contraction with respect to  $\zeta = \zeta_1$  (resp.  $\zeta = \zeta_3$ ) for  $x_0 \in X$ . From Theorem 3, the proof is completed.  $\square$

Now, we give two illustrative examples.

**Example 5.** Let  $X = \mathbb{R}$  and  $(X, \mathcal{S})$  be the  $S$ -metric space defined as in Example 1. Let us define the self-mapping  $T : X \rightarrow X$  as:

$$Tx = \begin{cases} x & ; \quad x \in [-2, 2] \\ 2x & ; \quad x \in (2, \infty) \\ -2x & ; \quad x \in (-\infty, -2) \end{cases},$$

for all  $x \in \mathbb{R}$ . Then,  $T$  is a  $\mathcal{Z}_\zeta^C$ -contraction with  $r = 4$ ,  $x_0 = 0$  and the simulation function  $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  defined by  $\zeta(t, s) = \frac{7}{4}s - t$  for all  $s, t \in [0, \infty)$ . By Theorem 3,  $D_{0, 4}^S = [-2, 2]$  is a fixed disc of  $T$ .

**Example 6.** Let  $X = \mathbb{R}$  and  $(X, \mathcal{S})$  be the  $S$ -metric space defined as in Example 1. Let us define the self-mapping  $T : X \rightarrow X$  as:

$$Tx = \begin{cases} x & ; \quad \mathcal{S}(x, x, x_0) \leq \mu \\ 2x_0 & ; \quad \mathcal{S}(Tx, Tx, x_0) > \mu \end{cases},$$

for all  $x \in \mathbb{R}$  with  $x_0 > 0$  and  $\mu \geq 4x_0$ . The self-mapping  $T$  is not a  $\mathcal{Z}_\zeta^C$ -contraction with respect to any  $\zeta \in \mathcal{Z}$  for  $x_0 \in \mathbb{R}$ . Indeed, using the condition  $(\zeta_2)$ , we get:

$$\begin{aligned} \zeta(\mathcal{S}(Tx, Tx, x), \mathcal{S}(Tx, Tx, x_0)) &= \zeta(2|2x_0 - x|, 2|x_0|) \\ &< 2(|x_0| - |2x_0 - x|) < 0, \end{aligned}$$

for all  $x \in (-\infty, x_0 - \frac{r}{2}) \cup (x_0 + \frac{r}{2}, \infty)$ . However,  $T$  fixes the disc  $D_{x_0, \mu}^S$ .

We note that Example 6 shows that the converse statement of Theorem 3 is not always true.

### 5. Conclusions

In closing, we would like to bring the reader’s attention to the following open questions:

- (1) Let  $(X, \mathcal{S})$  be a complete partial  $S$ -metric space and  $T : X \rightarrow X$  be a self-mapping. If  $T$  is a  $\mathcal{Z}_\zeta$ -contraction with respect to  $\zeta$ , does  $T$  have a unique fixed point  $a \in X$  and the Picard sequence  $\{x_n\}$  converges to the fixed point  $a$ ? If not, what is (are) the condition(s) that we need to add?
- (2) Let  $(X, \mathcal{S})$  be a complete partial  $S$ -metric space and  $T : X \rightarrow X$  be a surjective  $\mathcal{Z}_\zeta$ -expanding map. Does  $T$  have a unique fixed point in  $X$ ? If not, what is (are) the condition(s) that we need to add?

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