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C-parallel and C-proper Slant Curves of S-manifolds

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Abstract. In the present paper, we define and study C-parallel and C-proper slant curves of S-manifolds. We prove that a slant curve γ in an S-manifold of order $r \geq 3$, under certain conditions, is C-parallel or C-parallel in the normal bundle if and only if it is a non-Legendre slant helix or Legendre helix, respectively. Moreover, under certain conditions, we show that γ is C-proper or C-proper in the normal bundle if and only if it is a non-Legendre slant curve or Legendre curve, respectively. We also give two examples of such curves in $\mathbb{R}^{2m+s}(-3s)$.

1. Introduction

Let M^m be an integral submanifold of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$. Then M is called *integral C-parallel* if $\nabla^{\perp}B$ is parallel to the characteristic vector field ξ, where B is the second fundamental form of M and $\nabla^{\perp}B$ is given by

$$(\nabla^{\perp}B)(X, Y, Z) = \nabla^{\perp}_{X}B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z),$$

where X, Y, Z are vector fields on M, $\nabla^{\perp}$ and ∇ are the normal connection and the Levi-Civita connection on M, respectively [8]. Now, let γ be a curve in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$. Lee, Suh and Lee introduced the notions of C-parallel and C-proper curves along slant curves of Sasakian 3-manifolds in the tangent and normal bundles [12]. A curve γ in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be *C-parallel* if $\nabla_T H = \lambda\xi$, *C-proper* if $\Delta H = \lambda\xi$, *C-parallel in the normal bundle* if $\nabla^{\perp}_{T}H = \lambda\xi$, *C-proper in the normal bundle* if $\Delta^{\perp}H = \lambda\xi$, where T is the unit tangent vector field of γ, H is the mean curvature vector field, Δ is the Laplacian, λ is a non-zero differentiable function along the curve γ, $\nabla^{\perp}$ and $\Delta^{\perp}$ denote the normal connection and Laplacian in the normal bundle, respectively [12]. For a submanifold M of an arbitrary Riemannian manifold $\widetilde{M}$, if $\Delta H = \lambda H$, then M is called *submanifold with a proper mean curvature vector field H* [6]. If $\Delta^{\perp}H = \lambda H$, then M is said to be *submanifold with a proper mean curvature vector field H in the normal bundle* [1].

Let $\gamma(s)$ be a Frenet curve parametrized by the arc-length parameter s in an almost contact metric manifold M. The function $\theta(s)$ defined by $cos[\theta(s)] = g(T(s), \xi)$ is called *the contact angle function*. A curve γ is called a *slant curve* if its contact angle is a constant [7]. If a slant curve is with contact angle $\frac{\pi}{2}$, then it is called a *Legendre curve* [4].

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Lee, Suh and Lee studied C-parallel and C-proper slant curves of Sasakian 3-manifolds in [12]. As a generalization of this paper, in [9], the present authors studied C-parallel and C-proper slant curves in trans-Sasakian manifolds. In [14], the second author investigated C-parallel Legendre curves of non-Sasakian contact metric manifolds. In the present paper, our aim is to consider C-parallel and C-proper slant curves of S-manifolds.

The paper is organized as follows: In Section 2, we give a brief introduction about S-manifolds. Furthermore, we define the notions of C-parallel and C-proper curves in S-manifolds both in tangent and normal bundles. In Section 3, we consider C-parallel slant curves in S-manifolds in tangent and normal bundles, respectively. In Section 4, we study C-proper slant curves in S-manifolds in tangent and normal bundles, respectively. In the last section, we present two examples of these kinds of curves in $\mathbb{R}^{2m+s}(-3s)$.

2. Preliminaries

Let (M, g) be a $(2m+s)$-dimensional Riemann manifold. M is called a *framed metric manifold* [17] with a *framed metric structure* $(\varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, if this structure satisfies the following equations:

$$\varphi^2 = -I + \sum_{\alpha=1}^{s} \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \varphi(\xi_\alpha) = 0, \quad \eta^\alpha \circ \varphi = 0 \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y), \tag{2}$$

$$d\eta^\alpha(X, Y) = g(X, \varphi Y) = -d\eta^\alpha(Y, X), \quad \eta^\alpha(X) = g(X, \xi), \tag{3}$$

where, φ is a $(1,1)$ tensor field of rank $2m$; $\xi_1, ..., \xi_s$ are vector fields; $\eta^1, ..., \eta^s$ are 1-forms and g is a Riemannian metric on M; $X, Y \in TM$ and $\alpha, \beta \in \{1, ..., s\}$. $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is also called a *framed φ-manifold* [13] or an *almost r-contact metric manifold* [16]. $(\varphi, \xi_\alpha, \eta^\alpha, g)$ is said to be an *S-structure*, if the Nijenhuis tensor of φ is equal to $-2d\eta^\alpha \otimes \xi_\alpha$, where $\alpha \in \{1, ..., s\}$ [3, 5].

When $s = 1$, a framed metric structure turns into an almost contact metric structure and an S-structure turns into a Sasakian structure. For an S-structure, the following equations are satisfied [3, 5]:

$$(\nabla_X \varphi)Y = \sum_{\alpha=1}^{s} \left\{ g(\varphi X, \varphi Y)\xi_\alpha + \eta^\alpha(Y)\varphi^2 X \right\}, \tag{4}$$

$$\nabla_X \xi_\alpha = -\varphi X, \ \alpha \in \{1, ..., s\}. \tag{5}$$

If M is Sasakian ($s = 1$), (5) can be directly calculated from (4).

Firstly, we give the following definition:

Definition 2.1. *Let $\gamma : I \to (M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a unit speed curve in an S-manifold. Then γ is called*

i) C-parallel (in the tangent bundle) if

$$\nabla_T H = \lambda \sum_{\alpha=1}^{s} \xi_\alpha,$$

ii) C-parallel in the normal bundle if

$$\nabla^\perp_T H = \lambda \sum_{\alpha=1}^{s} \xi_\alpha,$$

iii) C-proper (in the tangent bundle) if

$$\Delta H = \lambda \sum_{\alpha=1}^{s} \xi_\alpha,$$

iv) C-proper in the normal bundle if

$$\Delta^{\perp} H = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha},$$

where H is the mean curvature field of γ, λ is a real-valued non-zero differentiable function, ∇ is the Levi-Civita connection, $\nabla^{\perp}$ is the Levi-Civita connection in the normal bundle, Δ is the Laplacian and $\Delta^{\perp}$ is the Laplacian in the normal bundle.

Let $\gamma : I \to M$ be a curve parametrized by arc length in an n-dimensional Riemannian manifold (M, g). Denote by the Frenet frame and curvatures of γ by $\{E_1, E_2, ..., E_r\}$ and $\kappa_1, ..., \kappa_{r-1}$, respectively. We know that (see [1])

$$\nabla_T H = -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3,$$

$$\nabla_T^{\perp} H = \kappa_1' E_2 + \kappa_1 \kappa_2 E_3,$$

$$\begin{aligned} \Delta H &= -\nabla_T \nabla_T \nabla_T T \\ &= 3\kappa_1 \kappa_1' E_1 + \left(\kappa_1^3 + \kappa_1 \kappa_2^2 - \kappa_1''\right) E_2 \\ &\quad -(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 \end{aligned}$$

and

$$\begin{aligned} \Delta^{\perp} H &= -\nabla_T^{\perp} \nabla_T^{\perp} \nabla_T^{\perp} T \\ &= \left(\kappa_1 \kappa_2^2 - \kappa_1''\right) E_2 - \left(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\right) E_3 \\ &\quad -\kappa_1 \kappa_2 \kappa_3 E_4. \end{aligned}$$

So we can directly state the following Proposition:

Proposition 2.2. *Let $\gamma : I \to (M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a unit speed curve in an S-manifold. Then*

i) γ is C-parallel (in the tangent bundle) if and only if

$$-\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}, \tag{6}$$

ii) γ is C-parallel in the normal bundle if and only if

$$\kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}, \tag{7}$$

iii) γ is C-proper (in the tangent bundle) if and only if

$$3\kappa_1 \kappa_1' E_1 + \left(\kappa_1^3 + \kappa_1 \kappa_2^2 - \kappa_1''\right) E_2 - (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}, \tag{8}$$

iv) γ is C-proper in the normal bundle if and only if

$$\left(\kappa_1 \kappa_2^2 - \kappa_1''\right) E_2 - \left(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\right) E_3 - \kappa_1 \kappa_2 \kappa_3 E_4 = \lambda \sum_{\alpha=1}^{s} \xi_{\alpha}. \tag{9}$$

Now, our aim is to apply Proposition 2.2 to slant curves in S-manifolds.
Let $\gamma : I \to (M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a slant curve. Then, if we differentiate

$$\eta^\alpha(T) = \cos\theta,$$

we get

$$\eta^\alpha(E_2) = 0,$$

where θ denotes the constant contact angle satisfying

$$\frac{-1}{\sqrt{s}} \leq \cos\theta \leq \frac{1}{\sqrt{s}}.$$

The equality case is only valid for geodesics corresponding to the integral curves of

$$T = \frac{\pm 1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha,$$

(see [10]).

3. *C*-parallel Slant Curves of *S*-manifolds

Our first Theorem below is a result of Proposition 2.2 i).

Theorem 3.1. *Let $\gamma : I \to M^{2m+s}$ be a unit-speed slant curve. Then γ is C-parallel (in the tangent bundle) if and only if it is a non-Legendre slant helix of order $r \geq 3$ satisfying*

$$\sum_{\alpha=1}^{s} \xi_\alpha \in sp\{T, E_3\},$$

$$\varphi T \in sp\{E_2, E_4\},$$

$$\kappa_2 = \frac{-\kappa_1 \sqrt{1 - s\cos^2\theta}}{\sqrt{s}\cos\theta},\ \kappa_2 \neq 0,$$

$$\lambda = \frac{-\kappa_1^2}{s\cos\theta} = constant,$$

and moreover if $\kappa_3 = 0$, then

$$\kappa_1 = -s\cos\theta\sqrt{1 - s\cos^2\theta}, \tag{10}$$

$$\kappa_2 = \sqrt{s}\left(1 - s\cos^2\theta\right). \tag{11}$$

Proof. Let us assume that γ is C-parallel (in the tangent bundle). Then, if we take the inner product of equation (6) with E_2, we find $\kappa_1' = 0$, that is, κ_1 =constant. Now, taking the inner product of equation (6) with T, we have

$$\lambda s\cos\theta = -\kappa_1^2.$$

Here, $\theta \neq \frac{\pi}{2}$ since $\kappa_1 \neq 0$. Hence, γ is non-Legendre slant. So, we get

$$\lambda = \frac{-\kappa_1^2}{s\cos\theta} = \text{constant}.$$

Equation (6) can be rewritten as

$$\sum_{\alpha=1}^{s}\xi_\alpha = \frac{-\kappa_1^2}{\lambda}T + \frac{\kappa_1\kappa_2}{\lambda}E_3,$$

which is equivalent to

$$\sum_{\alpha=1}^{s}\xi_\alpha = s\cos\theta T - \frac{\kappa_2 s\cos\theta}{\kappa_1}E_3. \tag{12}$$

If we calculate the norm of both sides, we obtain

$$\kappa_2 = \frac{-\kappa_1\sqrt{1-s\cos^2\theta}}{\sqrt{s}\cos\theta}. \tag{13}$$

If we assume $\kappa_2 = 0$, then $\sum_{\alpha=1}^{s}\xi_\alpha$ is parallel to T. Hence $\kappa_1 = 0$ or $\theta = \frac{\pi}{2}$, both of which is a contradiction. So, we have $\kappa_2 \neq 0$ and $r \geq 3$. If we write equation (13) in (12), we get

$$\sum_{\alpha=1}^{s}\xi_\alpha = s\cos\theta T + \sqrt{s}\sqrt{1-s\cos^2\theta}E_3.$$

If we differentiate this last equation along the curve γ, we find

$$\varphi T = \frac{-\kappa_1}{s\cos\theta}E_2 - \frac{\kappa_3\sqrt{1-s\cos^2\theta}}{\sqrt{s}}E_4. \tag{14}$$

If we calculate $g(\varphi T, \varphi T)$, we have

$$s\cos\theta\left(1-s\cos^2\theta\right)\left(s\cos\theta-\kappa_3^2\right) = \kappa_1^2,$$

which gives us κ_3 =constant. In particular, if $\kappa_3 = 0$, then we find equations (10) and (11). If $\kappa_3 \neq 0$, we differentiate equation (14) along the curve γ and find that κ_4 =constant. If we continue differentiating and calculating the norm of both sides, we easily obtain κ_i =constant for all $i = \overline{1,r}$, that is, γ is a slant helix of order r. Thus, we have just proved the necessity.

To prove sufficiency, if γ satisfies the equations given in the Theorem, then it is easy to show that equation (6) is satisfied. So, γ is C-parallel (in the tangent bundle). □

For C-parallel slant curves in the normal bundle, we have the following Theorem:

Theorem 3.2. *Let $\gamma : I \to M^{2m+s}$ be a unit-speed slant curve. Then γ is C-parallel in the normal bundle if and only if it is a Legendre helix of order $r \geq 3$ satisfying*

$$\sum_{\alpha=1}^{s}\xi_\alpha = \sqrt{s}E_3,$$

$$\varphi T = \frac{\kappa_2}{\sqrt{s}}E_2 - \frac{\kappa_3}{\sqrt{s}}E_4,$$

$$\kappa_2 \neq 0,\ \lambda = \frac{\kappa_1\kappa_2}{\sqrt{s}}$$

and moreover if $\kappa_3 = 0$, then

$$\kappa_2 = \sqrt{s},\ \varphi T = E_2.$$

Proof. Let us assume that γ is C-parallel in the normal bundle. Then, if we take the inner product of equation (7) with T, we have $\eta^{\alpha}(T) = 0$, so γ is Legendre. Next, we take the inner product with E_2 and find κ_1 =constant. Thus, equation (7) becomes

$$\kappa_1\kappa_2 E_3 = \lambda \sum_{\alpha=1}^{s} \xi_\alpha,$$

which gives us

$$E_3 = \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha, \tag{15}$$

$$\kappa_2 \neq 0,\ \lambda = \frac{\kappa_1\kappa_2}{\sqrt{s}}.$$

If we differentiate equation (15), we get

$$\varphi T = \frac{\kappa_2}{\sqrt{s}} E_2 - \frac{\kappa_3}{\sqrt{s}} E_4. \tag{16}$$

If we differentiate this last equation, we obtain

$$\begin{aligned} \nabla_T \varphi T &= \sum_{\alpha=1}^{s} \xi_\alpha + \kappa_1 \varphi E_2 \\ &= \frac{\kappa_2'}{\sqrt{s}} E_2 + \frac{\kappa_2}{\sqrt{s}}(-\kappa_1 T + \kappa_2 E_3) - \frac{\kappa_3'}{\sqrt{s}} E_4 - \frac{\kappa_3}{\sqrt{s}}(-\kappa_3 E_3 + \kappa_4 E_5). \end{aligned} \tag{17}$$

If we take the inner product of both sides with E_2, we find κ_2 =constant. Then, the norm of equation (16) gives us κ_3 =constant. In particular, if $\kappa_3 = 0$, from equation (16), we have

$$\kappa_2 = \sqrt{s},\ \varphi T = E_2.$$

Otherwise, from the norm of both sides in (17), we also have κ_4 =constant. If we continue differentiating equation (17), we find that γ is a helix of order r.

Conversely, let γ be a Legendre helix of order $r \geq 3$ satisfying the stated equations. Then, it is easy to show that equation (7) is verified. Thus, γ is C-parallel in the normal bundle. □

4. *C*-proper Slant Curves of *S*-manifolds

For C-proper slant curves in the tangent bundle, we can state the following Theorem:

Theorem 4.1. *Let $\gamma : I \to M^{2m+s}$ be a unit-speed slant curve. Then γ is C-proper (in the tangent bundle) if and only if it is a non-Legendre slant curve satisfying*

$$\sum_{\alpha=1}^{s} \xi_\alpha \in sp\{T, E_3, E_4\},$$

$$\varphi T \in sp\{E_2, E_3, E_4, E_5\},$$

$$\kappa_1 \neq constant,\ \kappa_2 \neq 0,$$

$$\lambda = \frac{3\kappa_1\kappa_1'}{s\cos\theta}, \tag{18}$$

$$\kappa_1^2 + \kappa_2^2 = \frac{\kappa_1''}{\kappa_1}, \tag{19}$$

$$\lambda s \eta^\alpha(E_3) = -(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'), \tag{20}$$

$$\lambda s \eta^\alpha(E_4) = -\kappa_1\kappa_2\kappa_3, \tag{21}$$

$$\eta^\alpha(E_3)^2 + \eta^\alpha(E_4)^2 = \frac{1 - s\cos^2\theta}{s} \tag{22}$$

and moreover if $\kappa_3 = 0$, *then*

$$\varphi T = \sqrt{1 - s\cos^2\theta}\, E_2, \tag{23}$$

$$E_3 = \frac{1}{\sqrt{s}\sqrt{1 - s\cos^2\theta}}\left(-s\cos\theta T + \sum_{\alpha=1}^{s}\xi_\alpha\right), \tag{24}$$

$$\kappa_2 = \sqrt{s}\left(1 + \frac{\kappa_1\cos\theta}{\sqrt{1 - s\cos^2\theta}}\right). \tag{25}$$

Proof. Let γ be C-proper (in the tangent bundle). If we take the inner product of equation (8) with T, we find

$$\lambda s\cos\theta = 3\kappa_1\kappa_1'.$$

Let us assume that γ is Legendre. Then we have $\kappa_1' = 0$, that is, κ_1 =constant. If we take the inner product of equation (8) with E_2, we get

$$0 = \kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1'' = \kappa_1\left(\kappa_1^2 + \kappa_2^2\right),$$

which gives us $\kappa_1 = 0$. Then equation (8) becomes

$$\lambda\sum_{\alpha=1}^{s}\xi_\alpha = 0,$$

which is a contradiction. Thus, γ is non-Legendre slant and $\kappa_1 \neq$constant. We find equations (18), (19), (20) and (21) taking the inner product with T, E_2, E_3 and E_4, respectively. Then, we write these equations in (8) and calculate the norm of both sides to obtain equation (22). Now, let us assume $\kappa_2 = 0$. Then, from equation (8), we have

$$\lambda\sum_{\alpha=1}^{s}\xi_\alpha = 3\kappa_1\kappa_1' T,$$

which is only possible when

$$T = \frac{1}{\sqrt{s}}\sum_{\alpha=1}^{s}\xi_\alpha.$$

If we calculate $\nabla_T T$, we find $\kappa_1 = 0$, which is a contradiction. Hence, $\kappa_2 \neq 0$. Differentiating equation (8), we can easily see that

$$\varphi T \in sp\{E_2, E_3, E_4, E_5\}.$$

In particular, if $\kappa_3 = 0$, we obtain equations (23), (24) and (25). See our paper [10], Case III, equation (4.9), which is also valid when κ_1 and κ_2 are not constants.

Conversely, if γ is a non-Legendre slant curve satisfying the stated equations, then Proposition 2.2 iii) is valid. So, γ is C-proper (in the tangent bundle). □

Finally, we give the following Theorem for C-proper slant curves in the normal bundle:

Theorem 4.2. *Let $\gamma : I \to M^{2m+s}$ be a unit-speed slant curve. Then γ is C-proper in the normal bundle if and only if it is a Legendre curve satisfying*

$$\sum_{\alpha=1}^{s}\xi_\alpha \in sp\left\{E_3, E_4\right\},$$

$$\varphi T \in sp\left\{E_2, E_3, E_4, E_5\right\}$$

$$\kappa_1 \neq constant,\ \kappa_2 \neq 0,$$

$$\kappa_1\kappa_2^2 - \kappa_1'' = 0,$$

$$\lambda s\eta^\alpha(E_3) = -(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'),$$

$$\lambda s\eta^\alpha(E_4) = -\kappa_1\kappa_2\kappa_3,$$

$$\eta^\alpha(E_3)^2 + \eta^\alpha(E_4)^2 = \frac{1}{s}$$

and moreover if $\kappa_3 = 0$, then

$$\sum_{\alpha=1}^{s}\xi_\alpha = \sqrt{s}E_3,$$

$$\kappa_2 = \sqrt{s},\ \varphi T = E_2.$$

Proof. The proof is similar to the proof of Theorem 4.1. For the case $\kappa_3 = 0$, we refer to [15]. □

5. Examples

In this section, we give the following two examples in the well-known S-manifold $\mathbb{R}^{2m+s}(-3s)$. For more information on $\mathbb{R}^{2m+s}(-3s)$, see [11].

Example 5.1. *Let us consider $\mathbb{R}^{2m+s}(-3s)$ with $m = 2$ and $s = 2$. The curve $\gamma : I \to \mathbb{R}^6(-6)$ given by*

$$\gamma(t) = (\sin t, 2 + \sin t, -\cos t, 3 - \cos t, -2t - \sin t\cos t, 1 - 2t - \sin t\cos t)$$

is a unit-speed non-Legendre slant helix with

$$\kappa_1 = \kappa_2 = \frac{1}{\sqrt{2}},\ \theta = \frac{2\pi}{3}.$$

It has the Frenet frame field

$$\left\{T,\ \sqrt{2}\varphi T, \left(T + \sum_{\alpha=1}^{2}\xi_\alpha\right)\right\}$$

and it is C-parallel (in the tangent bundle) with $\lambda = \frac{1}{2}$.

Example 5.2. *Let us consider* $\mathbb{R}^{2m+s}(-3s)$ *with* $m = 1$ *and* $s = 4$. *We define real valued functions on an open interval* I *as*

$$\gamma_1(t) = 2\int_0^t \cos(e^{2u})du,\ \gamma_2(t) = -2\int_0^t \sin(e^{2u})du,$$

$$\gamma_3(t) = ... = \gamma_6(t) = -4\int_0^t \cos(e^{2u})\left(\int_0^u \sin(e^{2v})dv\right)du.$$

The curve $\gamma : I \to \mathbb{R}^6(-12)$, $\gamma(t) = (\gamma_1(t), ..., \gamma_6(t))$ *is a unit-speed Legendre curve with*

$$\kappa_1 = 2e^{2t},\ \kappa_2 = 2,\ r = 3,$$

$$\varphi T = E_2,\ E_3 = \frac{1}{2}\sum_{\alpha=1}^{4}\xi_\alpha$$

and it is C-proper in the normal bundle with $\lambda = -8e^{2t}$.

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