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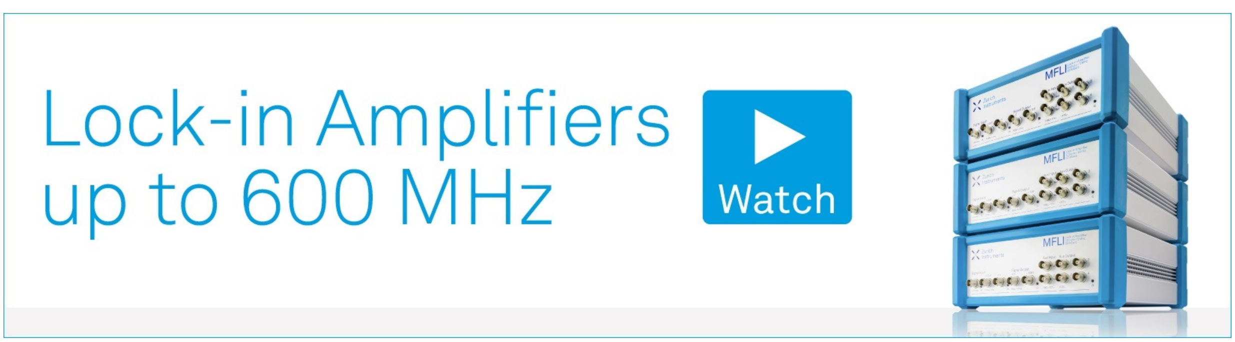

Sequential Definitions of Fuzzy Continuity in Fuzzy Spaces

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Abstract. Çakalli extended the concept of G-sequential compactness to a fuzzy topological group and introduced the notion of G-fuzzy sequential compactness, where G is a function from a suitable subset of the set of all sequences of fuzzy points in a fuzzy first countable topological space X. The aim of this paper is to investigate whether an idea like the G-fuzzy continuity can be introduced and consequently can be extended to a more general approach to fuzzy continuity in fuzzy topological spaces. In this article, we introduce the concepts of G-fuzzy sequential continuity and G-fuzzy sequential closedness in a fuzzy topological space and give some characterization theorems.
Keywords: Fuzzy points, G-fuzzy convergence, G-fuzzy sequential closedness, G-fuzzy sequential continuity.
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INTRODUCTION

Sometimes, we can not use traditional classical methods to handle some problems in some parts of real life such as medical sciences, social sciences, economics, engineering etc. Because, these problems involve various types of uncertainities. To cope with these problems, some new theories were given by scholars.

Let f be a mapping on a topological space X. It is well known that the mapping f on X is sequentially continuous if convergence of a sequence of points $\mathbf{x} = (x_n)$ in X implies the convergence of the sequence $\mathbf{y} = (f(x_n))$. This notion of sequential continuity has been modified by Connor and Grosse-Erdmann [8] for real functions by using an arbitrary linear function G defined on a linear subspace of the vector space of all real sequences. Cakalli extended this concept to the topological group case and introduced the concept of G-sequential continuity [3] in the sense that a function $f : X \to X$ is G-sequentially continuous at a point u if, given a sequence $\mathbf{x} = (x_n)$ of points in X, $G(\mathbf{x}) = u$ implies that $G(f(\mathbf{x})) = f(u)$, where G is an additive function from a subgroup of the group of all sequences of points in X. G is called sequential method on X (see also [7, 10, 12, 18, 19]).

Fuzzy topology and fuzzy continuity were first studied in [9]. Since then, many investigations have been done in this field as in [16, 20]. Different types of fuzzy continuity have been defined by many authors taking different approaches, some of which can be found in [2, 9, 14, 15, 17, 21, 22, 23, 24].

Cakalli and Das [11] introduced the concept of fuzzy sequential method G and define G-fuzzy sequential compactness in a fuzzy topological space and extended the notion of G-sequential compactness in the fuzzy settings.

Throughout $\mathbb{N}$ will denote the set of all positive integers. X will denote a fuzzy topological space, which allows countable local base at any point. We will use bold-face letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ for sequences $\mathbf{x} = (\lambda_{a_n}^{x_n})$, $\mathbf{y} = (\lambda_{b_n}^{y_n})$, $\mathbf{z} = (\lambda_{c_n}^{z_n})$ of fuzzy points of X. $\mathbf{s}(X)$ and $\mathbf{c}(X)$ will denote the set of all sequences of fuzzy points of X and the set of all fuzzy convergent sequences of fuzzy points of X respectively.

The aim of this paper is to introduce fuzzy G-sequential continuity and prove characterization theorems.

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G-fuzzy sequential continuity

Let us recall some definitions and results on fuzzy topological space which will be used in this paper and can be found in [2, 9, 11, 17, 24, 25].

Let X be a non-empty set and let $I = [0, 1]$. I^X will denote the set of all functions $\lambda : X \to I$. The member of I^X is called fuzzy subset of X.

For any two members λ and μ of I^X; $\lambda \subset \mu$ if and only if $\lambda(x) \leq \mu(x)$ for each $x \in X$, and in this case μ is said to contain λ or λ is said to be contained in μ.

0 and **1** denote constant mappings taking whole of X to 0 and 1, respectively.

λ_a^x will denote the fuzzy point of X which takes the value $a \in (0, 1]$ at the point $x \in X$ and 0 elsewhere.

If $\mu \in I^X$ and $\mu(x) \geq a$, then we write $\lambda_a^x \in \mu$.

Definition 1 *A collection τ of fuzzy subsets of X satisfying*
(i) $\mathbf{0}$ and $\mathbf{1} \in \tau$;
(ii) $\mu_i \in \tau$, $\forall i \in \Delta \Rightarrow \vee\{\mu_i : i \in \Delta\} \in \tau$;
(iii) $\mu, \lambda \in \tau \Rightarrow \mu \wedge \lambda \in \tau$,
is called a fuzzy topology on X. The pair (X, τ) is called a fuzzy topological space. Members of τ are called the fuzzy open sets and the fuzzy sets $1 - \mu; \mu \in \tau$ defined by $(1 - \mu)(x) = 1 - \mu(x)$ for each $x \in X$, are called the fuzzy closed sets of X. The fuzzy set $1 - \mu$ is called the complement of the fuzzy set μ.

The readers may refer to [11] for the definitions of fuzzy limit point (Definition 2, p. 1666), fuzzy sequential convergence (Definition 6, p. 1666) and fuzzy sequential closed (p.1666).

The method of fuzzy sequential convergence, or briefly a fuzzy method, is a function G defined on a subset $\mathbf{c}_G(X)$ of $\mathbf{s}(X)$ into the set of all fuzzy points of X, where $\mathbf{c}_G(X)$ contains all sequences of the form $(\lambda_{a_n}^{x_n})$ where $x_n = x\ \forall n$ and $G(\lambda_{a_n}^{x_n}) = \lambda_a^x$ where $a \leq \sup a_n$.

Definition 2 *A method G is called fuzzy regular if every fuzzy convergent sequence $\boldsymbol{x} = (\lambda_{a_n}^{x_n})$ is G-fuzzy convergent with $G(\boldsymbol{x}) = f - \lim \boldsymbol{x}$.*

Definition 3 *A fuzzy point λ_a^x is called a G-fuzzy sequential accumulation point of α if there is a sequence $\mathbf{x} = (\lambda_{a_n}^{x_n})$ of fuzzy points in α with $x_n \neq x$ for any $n \in \mathbb{N}$ such that $G(\mathbf{x}) = \lambda_a^x$. In other words, a fuzzy point λ_a^x is in the G-fuzzy sequential closure of α if there is a sequence $\mathbf{x} = (\lambda_{a_n}^{x_n})$ of fuzzy points in α such that $G(\mathbf{x}) = \lambda_a^x$.*

We denote the G-fuzzy sequential closure of a set α by $\overline{\alpha}^G$. We say that a fuzzy set α is G-fuzzy sequentially closed if it contains all fuzzy points in its G-fuzzy closure.

Theorem 1 *Let G be a fuzzy regular method and $\{\alpha_i\}$ be a collection of fuzzy subsets of X, $i \in \Delta$ where Δ is an index set. Then the following holds:*

(i) $\bigvee_{i\in\Delta} \overline{\alpha_i}^G \subset \overline{\bigvee_{i\in\Delta} \alpha_i}^G$.

(ii) $\overline{\bigwedge_{i\in\Delta} \alpha_i}^G \subset \bigwedge_{i\in\Delta} \overline{\alpha_i}^G$.

Proof 1 *The proof is straightforward and hence omitted.*

The following result is immediate from Theorem 1.

Theorem 2 *Let G be a fuzzy method. The intersection of the collection $\{\alpha_i : i \in I\}$ of G-fuzzy sequential closed sets is G-fuzzy sequential closed.*

Proof 2 *The result follows directly from Theorem 1.*

Theorem 3 *If G is fuzzy regular, then $\alpha \subseteq \overline{\alpha} \subseteq \overline{\alpha}^G$, for any fuzzy subset α of X.*

Proof 3 *The result is obvious and hence omitted.*

Theorem 4 *Let G be a fuzzy regular method. Then $\overline{\alpha}^G = \overline{\alpha}$, for every fuzzy set α of X if and only if G is a fuzzy subsequential method.*

Proof 4 *Let G be a fuzzy subsequential method and $\lambda_a^x \in \overline{\alpha}^G$. Then there is a sequence of fuzzy points $(\lambda_{a_n}^{x_n})$ in α such that $G(\lambda_{a_n}^{x_n}) = \lambda_a^x$. Since G is a fuzzy subsequential method, there is a fuzzy subsequence $(\lambda_{b_k}^{y_k})$ of the fuzzy sequence $(\lambda_{a_n}^{x_n})$ such that $f - \lim_k \lambda_{b_k}^{y_k} = \lambda_a^x$. Hence $\lambda_a^x \in \overline{\alpha}$. Since G is regular, $\overline{\alpha}^G = \overline{\alpha}$.*

Conversely, suppose that $\overline{\alpha}^G = \overline{\alpha}$, for every fuzzy subset α of X. Let $(\lambda_{a_n}^{x_n})$ be a G-fuzzy convergent sequence with $G(\lambda_{a_n}^{x_n}) = \lambda_a^x$. Now, since G is fuzzy regular, $\lambda_a^x \in \overline{\left\{\lambda_{a_n}^{x_n} : n \geq n_0\right\}}^G$ for $n_0 \in \mathbb{N}$.

As given, $\overline{\left\{\lambda_{a_n}^{x_n} : n \geq n_0\right\}}^G = \overline{\left\{\lambda_{a_n}^{x_n} : n \geq n_0\right\}}$ and we obtain $\lambda_a^x \in \inf_n \left\{\lambda_{a_n}^{x_n} : n \geq n_0\right\}$.

Thus there is a fuzzy subsequence $(\lambda_{b_k}^{y_k})$ of fuzzy sequence $(\lambda_{a_n}^{x_n})$ such that $f - \lim_k \lambda_{b_k}^{y_k} = \lambda_a^x$. Thus G is a fuzzy subsequential method.

Now we give the definition of G-fuzzy continuity:

Definition 4 *A mapping f on X is said to be G-fuzzy continuous at a fuzzy point $u = \lambda_a^x$ if there is a sequence of fuzzy points $\boldsymbol{x} = (\lambda_{a_n}^{x_n})$ such that $G(\boldsymbol{x}) = u$ implies $G(f(\boldsymbol{x})) = f(u)$.*

Theorem 5 *Let $f : X \to X$ and $g : X \to X$ be G-fuzzy continuous functions, then the composition function gof is G-fuzzy continuous.*

Proof 5 *The proof can be obtained easily, so is omitted.*

Conclusion

In this paper, we have introduced the concepts of G-fuzzy sequential closedness and G-fuzzy sequential continuity in a fuzzy topological space, new results are given, and some characterization theorems are obtained. We expect that our introduced notion and investigation might be a reference for further studies, so that one may expect it to be a more useful tool in the field of fuzzy theory in modeling various problems occurring in many areas of science, computer science, smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, economics, game theory, operations research, and in many kinds of real life problems. For a further study, we suggest to investigate soft G-sequential continuity, soft G-sequential closedness, and soft G-sequential compactness in a soft topological space (see [1, 13]).

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