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LOCAL GENERALIZATION OF TRANSVERSALITY CONDITIONS FOR OPTIMAL CONTROL PROBLEM☆

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Abstract. In this paper, we introduce the transversality conditions of optimal control problems formulated with the conformable derivative. Since the optimal control theory is based on variational calculus, the transversality conditions for variational calculus problems are first investigated and then supported by some illustrative examples. Utilizing from these formulations, the transversality conditions for optimal control problems are attained by using the Hamiltonian formalism and Lagrange multiplier technique. To illustrate the obtained results, the dynamical system on which optimal control problem constructed is taken as a diffusion process modeled in terms of the conformable derivative. The optimal control law is achieved by analytically solving the time dependent conformable differential equations occurring from the eigenfunction expansions of the state and the control functions. All figures are plotted using MATLAB.

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1. Introduction

Although the roots of fractional calculus are as old as classical calculus, it has been accepted as a powerful tool since the 1970s when its wide range applications have been realized such as viscoelasticity, diffusion phenomena, signal processing, bioengineering, control theory, etc. The main properties of fractional operators are to model memory and hereditary structures in the natural phenomena. There exists several fractional operators, the well known are Riemann–Liouville and Caputo [37, 46] which are nonlocal operators defined by convolution integrals with singular kernels. Due to the computational complexity of nonlocal fractional operators, solutions of the fractional order differential equations are generally obtained by numerical approximations [10, 25, 41, 51]. To remove the computational difficulties of the existing fractional operators, some new nonlocal operators with nonsingular kernels have also been defined such as Caputo–Fabrizio or Atangana–Baleanu operators [17, 27] whose real life applications can be found in [22, 23, 39, 50, 52]. On the other hand, all the nonlocal definitions do not obey some of the basic properties of classical derivatives such as Leibnitz or chain rules and are not suitable to investigate the local scaling or fractional differentiability [24]. Therefore, local derivatives with fractional order defined by references [28, 35, 36, 38] have attracted considerable attraction. The appropriate choice of local derivatives depends on the studied problem as similar to the nonlocal operators [14].

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In this paper, we consider the conformable derivative which is defined by expanding the usual limit definition of the classical derivative [36]. This local operator was generalized to the left and right derivatives and also to the sequential conformable derivative by Abdeljawad [1]. Additionally, he proposed chain rule, integration by parts, Taylor series expansion and Laplace transformation for the conformable derivative. Atangana *et al.* [18] introduced some useful properties and theorems for partial and sequential conformable derivatives. Since the conformable derivative provides the basic properties of the classical derivative, it has been shown that the conformable differential equations can be solved by analytical methods [15, 32, 54]. This advantage of the conformable derivative has quickly lead applications of the conformable differential equations to the real world problems both in the view of modeling [3, 20, 29, 30, 55] and control [33, 40, 56, 57].

Calculus of variations with fractional derivatives was born with the Riewe's work [48, 49]. Then, Agrawal proposed fractional Euler–Lagrange equations [4, 7]. The basic variational calculus problems contain two fixed endpoints which are sufficient to obtain the necessary optimality conditions. But some physical problems do not contain the appropriate number of endpoints, namely there are one or both endpoints are missing. This situation is known as free endpoint variational calculus problem and one or two auxiliary conditions known as transversality conditions (or natural boundary conditions) are needed to solve these types of problems. Transversality conditions for fractional variational problems were firstly considered by Agrawal [6, 8, 9] in the sense of both Caputo and Riemann–Liouville definitions. Later, these conditions in special cases including fractional derivatives and/or fractional integrals have been investigated in many studies [11–13, 42, 43]. Transversality conditions for fractional optimal control with different types of fractional operators have also been addressed in [5, 26, 31, 34, 47, 53].

Since the conformable calculus is a new tool, there are only a few studies on conformable calculus of variations and conformable optimal control. Variational calculus for conformable derivatives was firstly defined by Chung [29]. He proposed the conformable Euler–Lagrange equation and discussed the conformable version of the Newtonian mechanics in one-dimensional case. Then Lazo and Torres [40] studied the invariant conditions for both problems from conformable variational calculus and conformable optimal control with fixed endpoint conditions and also gave the conformable version of Noether's symmetry theorem for one- and multi-dimensional cases. They showed the possibleness and the convenience of formulations of action principle with conformable derivative for the frictional forces. Furthermore, İskender Eroğlu *et al.* [33] obtained the boundary optimal control law of a conformable heat equation. Motivated by the different types of endpoint conditions for conformable optimal control problems, we research the transversality conditions of conformable calculus of variations and conformable optimal control, respectively. Through the obtained transversality conditions, we examine the optimal control of a time-conformable diffusion process for free endpoint condition, whose analytical solutions in different coordinates were obtained in [2, 19, 21], as an application problem. It can be observed that the considered conformable optimal control law is achieved directly from analytical solutions without any need of numerical techniques. Also, it is worth to emphasize that the response of the conformable optimal control process has a similar manner with the fractional optimal control of the diffusion process [44].

The paper is organized as follows. In Section 2, the necessary definitions and the mathematical relations on conformable calculus needed in the subsequent formulations are given. In Section 3, conformable variational calculus problems are considered and their transversality conditions are obtained. In addition, some illustrative examples are also given. In Section 4, transversality conditions of conformable optimal control problems are acquired. Finally, conformable optimal control of a diffusion process is examined in Section 5.

2. Basic definitions and tools

Definition 2.1 (Conformable derivative [1, 36])**.** *The left conformable* derivative of a given function $f:[a,b]\to\mathbb{R}$ *starting from* $a\in\mathbb{R}$ of order $0<\alpha\leq 1$ is defined by

$$\frac{\mathrm{d}_a^\alpha}{\mathrm{d}t_a^\alpha} f(t) = f_a^{(\alpha)}(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\,(t-a)^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{2.1}$$

Furthermore, if the limit exists for $x \in (a,b)$ means f is left $\alpha-$differentiable then

$$f_a^{(\alpha)}(a) = \lim_{t\to a^+} f_a^{(\alpha)}(t)$$

and

$$f_a^{(\alpha)}(b) = \lim_{t\to b^-} f_a^{(\alpha)}(t).$$

The right conformable derivative of f terminating at $b \in \mathbb{R}$ of *order* $0 < \alpha \leq 1$ is defined by

$$\frac{{}_b\mathrm{d}^\alpha}{{}_b\mathrm{d}t^\alpha} f(t) = {}_b f^{(\alpha)}(t) = -\lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon(b-t)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{2.2}$$

Similarly, if the limit exists for $x\epsilon(a,b)$ means f is right $\alpha-$differentiable then

$${}_b f^{(\alpha)}(a) = \lim_{t\to a^+} {}_b f^{(\alpha)}(t)$$

and

$${}_b f^{(\alpha)}(b) = \lim_{t\to b^-} {}_b f^{(\alpha)}(t).$$

Note that if f is differentiable then $f_a^{(\alpha)}(t) = (t-a)^{1-\alpha} f'(t)$ and ${}_b f^{(\alpha)}(t) = -(b-t)^{1-\alpha} f'(t)$ where $f'(t)$ stands for first order derivative of $f(t)$.

In this paper, the left conformable derivative $f_a^{(\alpha)}(t)$ is usually used which satisfies all the basic properties given by the following theorem (see [1, 36, 40]):

Theorem 2.2. *Let* f *and* g *be* $\alpha-$*differentiable functions for* $0 < \alpha \leq 1$. *Then,*

(1) $(cf+dg)_a^{(\alpha)}(t) = cf_a^{(\alpha)}(t) + dg_a^{(\alpha)}(t)$ *for all* $c, d \in \mathbb{R}$.

(2) $(fg)_a^{(\alpha)}(t) = f_a^{(\alpha)}(t)\, g(t) + f(x)\, g_a^{(\alpha)}(t)$.

(3) $(f/g)_a^{(\alpha)}(t) = \left(f_a^{(\alpha)}(t)\, g(t) - f(t)\, g_a^{(\alpha)}(t)\right)/g^2(t)$.

(4) $(\lambda)_a^{(\alpha)} = 0$, *for all constant functions* $f(t) = \lambda$.

(5) $((t-a)^p)_a^{(\alpha)} = p(t-a)^{p-\alpha}$ *for all* $p \in \mathbb{R}$.

(6) $(f\circ g)_a^{(\alpha)}(t) = f_a^{(\alpha)}(g(t))\, g_a^{(\alpha)}(t)\, g^{\alpha-1}(t)$.

In the following, we provide all of the necessary theorem and definitions that we use in our formulations.

Definition 2.3 (Sequential conformable derivative [1])**.** Let $f : [a,b] \to \mathbb{R}$ such that $f^{(n)}(t)$ exists and continuous, $0 < \alpha \leq 1$ and $n \in \mathbb{N}^+$, then the left sequential conformable derivative of order n is defined by

$${}^n f_a^{(\alpha)}(t) = \underbrace{\frac{\mathrm{d}_a^\alpha}{\mathrm{d}t_a^\alpha} \frac{\mathrm{d}_a^\alpha}{\mathrm{d}t_a^\alpha} \cdots \frac{\mathrm{d}_a^\alpha}{\mathrm{d}t_a^\alpha}}_{n-\text{times}} f(t). \tag{2.3}$$

Definition 2.4 (Conformable Taylor series expansion [1]). Let f is an infinitely $\alpha-$differentiable function for some $0<\alpha\leq 1$ at a neighborhood of a point t_0. Then f has the conformable Taylor series expansion:

$$f(t)=f(t_0)+f_a^{(\alpha)}(t_0)\frac{(t-t_0)^\alpha}{\alpha}+\ {}^2f_a^{(\alpha)}(t_0)\frac{(t-t_0)^{2\alpha}}{2!\alpha^2}+\ldots+\ {}^nf_a^{(\alpha)}(t_0)\frac{(t-t_0)^{n\alpha}}{n!\alpha^n}+\ldots \tag{2.4}$$

where $t\in\left(t_0,t_0+R^{1/\alpha}\right)$ for $R>0$.

Definition 2.5 (Conformable partial derivative [18]). Let f be a function with m variables $x_1,x_2,\ldots,x_m$ then the conformable partial derivative of f with respect to x_i of order $0<\alpha\leq 1$ is defined by

$$\frac{\partial^\alpha}{\partial x_i^\alpha}f(x_1,x_2,\ldots,x_m)=\lim_{\varepsilon\to 0}\frac{f\left(x_1,\ldots,x_{i-1},x_i+\varepsilon x_i^{1-\alpha},\ldots,x_m\right)-f(x_1,\ldots,x_m)}{\varepsilon}. \tag{2.5}$$

Definition 2.6 (Conformable integral [1]). The left conformable integral of order $0<\alpha\leq 1$ starting from $a\in\mathbb{R}$ of a function f is defined by

$$I_a^\alpha f(t)=\int_a^t f(x)\,\mathrm{d}_a^\alpha x=\int_a^t f(x)(x-a)^{\alpha-1}\,\mathrm{d}x \tag{2.6}$$

and the right conformable integral of order $0<\alpha\leq 1$ terminating at $b\in\mathbb{R}$ of function f is defined

$${}_bI^\alpha f(t)=\int_t^b f(x)_b\,\mathrm{d}^\alpha x=\int_t^b f(x)(b-x)^{\alpha-1}\,\mathrm{d}x. \tag{2.7}$$

Theorem 2.7 (Integration by parts [1]). *Let $f,g:[a,b]\to\mathbb{R}$ be two functions such that fg differentiable. Then,*

$$\int_a^b f(t)\,g^{(\alpha)}(t)\,\mathrm{d}_a^\alpha t=f(t)\,g(t)\Big|_a^b-\int_a^b g(t)\,f_a^{(\alpha)}(t)\,\mathrm{d}_a^\alpha t. \tag{2.8}$$

3. Conformable variational calculus with transversality condition

The conformable variational calculus introduced by [29] can be defined as to find the minimizing (or maximizing) curve of a conformable or classical variational integral contains at least one conformable derivative term. We consider the following conformable variational calculus problem defined as

$$J(x)=\int_a^b F\left(t,x(t),x_a^{(\alpha)}(t)\right)\mathrm{d}_a^\alpha t \tag{3.1}$$

where $F:[a,b]\to\mathbb{R}$ is the Lagrangian from the class of $C^\alpha-$function in each of its argument, $x=x(t)$ is an unknown $C^\alpha-$function on the interval $[a,b]$ and

$$x(a)=x_a\quad\text{and}\quad x(b)=x_b \tag{3.2}$$

are endpoints for $x_a, x_b \in \mathbb{R}$. Note that, if $\alpha = 1$, then the conformable variational calculus problem coincides to the classical one. The necessary concepts to solve the conformable variational calculus problem are depicted below.

Definition 3.1. Functions x that are C^α and satisfy the endpoint conditions equation (3.2) are called *admissible functions.*

Definition 3.2. Let $0 < \alpha \leq 1$, $x^*(t)$ is a minimizing curve and $x(t)$ is an admissible function. If there exist small numbers ε_1 and ε_2 such that

$$|x^*(t) - x(t)| < \varepsilon_1 \text{ and } |x^{*(\alpha)}(t) - x^{(\alpha)}(t)| < \varepsilon_2, \text{ for all } t \epsilon [a, b] \tag{3.3}$$

then $x(t)$ is said to be a *weak variation* of $x^*(t)$. For the calculation purpose, the weak variation of $x(t)$ can be alternatively written in the form of $x(t) = x^*(t) + \varepsilon^\alpha \eta(t)$, where $\eta \epsilon C^\alpha [a, b]$ is a perturbation function satisfies $\eta(a) = \eta(b) = 0$.

Lemma 3.3 (Fundamental lemma for conformable calculus of variation [40])**.** *Let $0 < \alpha \leq 1$ and, μ and η be continuous functions on $[a, b]$. If for any $\eta \epsilon C^\alpha [a, b]$ with $\eta(a) = \eta(b) = 0$,*

$$\int_a^b \mu(t)\, \eta(t)\, \mathrm{d}_a^\alpha t = 0 \tag{3.4}$$

then for all $t \epsilon [a, b]$

$$\mu(t) = 0. \tag{3.5}$$

The following theorem proved by [40] presents the conformable Euler–Lagrange equation for fixed endpoints.

Theorem 3.4 (The conformable fractional Euler–Lagrange Eq. [40])**.** *Let J be a functional in the form of equations (3.1)–(3.2) for $0 < \alpha \leq 1$, $F \epsilon C^\alpha \left([a, b] \times \mathbb{R}^2\right)$ and $x : [a, b] \to \mathbb{R}$ be an $\alpha-$differentiable function. If $x(t)$ is a minimizer (or maximizer) of J, then $x(t)$ satisfies the following conformable Euler–Lagrange equation:*

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}_a^\alpha}{\mathrm{d}t_a^\alpha} \left(\frac{\partial F}{\partial x_a^{(\alpha)}} \right) = 0. \tag{3.6}$$

When one or two endpoints are missing, we need some additional conditions known as transversality conditions to solve the conformable Euler–Lagrange equation. By the following theorem, we will propose the transversality conditions of the conformable variational calculus.

Theorem 3.5 (Transversality conditions for conformable variational calculus)**.** *Let $x : [t_0, t_f] \to \mathbb{R}$ be an $\alpha-$differentiable function and F is a function in the class of $C^\alpha \left([t_0, t_f] \times \mathbb{R}^2\right)$ for $0 < \alpha \leq 1$. If $x(t)$ is a minimizer of*

$$J(x) = \int_{t_0}^{t_f} F\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right) \mathrm{d}_{t_0}^\alpha t, \tag{3.7}$$

when $x(t_0)=x_0$ $(x_0\epsilon\mathbb{R})$ is fixed but $x(t_f)$ lies on a some given curve $x=\gamma(t)$, then the general transversality condition is

$$F\left(t_f, x(t_f), x_{t_0}^{(\alpha)}(t_f)\right)\Delta\tau^{\alpha}+\frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f}\eta(t_f)=0, \tag{3.8}$$

where $\eta\in C^{\alpha}\left([t_0,t_f]\times\mathbb{R}^2\right)$ and $\Delta\tau$ are perturbations for the weak variation of x and t_f, respectively.

Proof. Let $x^*(t)$ is a minimizing curve which intersects with the target curve $\gamma(t)$ at $t=t_f^*$. To find the optimal solution, first of all we assume the following weak variations for $|\varepsilon|\ll 1$

$$\begin{aligned} x(t) &= x^*(t)+\varepsilon^{\alpha}\eta(t), \\ x_{t_0}^{(\alpha)}(t) &= x_{t_0}^{*(\alpha)}(t)+\varepsilon^{\alpha}\eta_{t_0}^{(\alpha)}(t), \\ t_f &= t_f^*+\varepsilon\Delta\tau, \end{aligned}$$

where $\eta(t_0)=0$ and $\eta(t_f)\neq 0$ since t_f is free. Then the variation of J is calculated as

$$\Delta J=\int_{t_0}^{t_f^*+\varepsilon\Delta\tau} F\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right)\mathrm{d}_{t_0}^{\alpha}t-\int_{t_0}^{t_f^*} F\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right)\mathrm{d}_{t_0}^{\alpha}t \tag{3.9}$$

which can be arranged in the following form

$$\begin{aligned} \Delta J = & \int_{t_f^*}^{t_f^*+\varepsilon\Delta\tau} F\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right)\mathrm{d}_{t_f^*}^{\alpha}t \\ & +\int_{t_0}^{t_f^*}\left(F\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right)-F\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right)\right)\mathrm{d}_{t_0}^{\alpha}t. \end{aligned}$$

When the function F is expanded to the Taylor series according to the pair of variables $\left(\varepsilon^{\alpha}\eta, \varepsilon^{\alpha}\eta_{t_0}^{(\alpha)}\right)$ near the point $\left(x^*, x_{t_0}^{*(\alpha)}\right)$ for $t\in[t_0,t_f]$ and then the expansion is substituted in ΔJ leads to

$$\begin{aligned} \Delta J = & \int_{t_f^*}^{t_f^*+\varepsilon\Delta\tau}\left(F\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right)+\frac{\partial F}{\partial x}\varepsilon^{\alpha}\eta(t)+\frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\varepsilon^{\alpha}\eta_{t_0}^{(\alpha)}(t)\right)\mathrm{d}_{t_f^*}^{\alpha}t \\ & +\int_{t_0}^{t_f^*}\left(F\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right)+\frac{\partial F}{\partial x}\varepsilon^{\alpha}\eta(t)+\frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\varepsilon^{\alpha}\eta_{t_0}^{(\alpha)}(t)-F\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right)\right)\mathrm{d}_{t_0}^{\alpha}t+O\left(\varepsilon^{2\alpha}\right). \end{aligned}$$

By calculating the first integral, we obtain the following equation

$$\Delta J = \left(F\left(t_f^*, x^*\left(t_f^*\right), x_{t_0}^{*(\alpha)}\left(t_f^*\right)\right) + \frac{\partial F}{\partial x}\Big|_{t_f^*} \varepsilon^\alpha \eta\left(t_f^*\right) + \frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f^*} \varepsilon^\alpha \eta_{t_0}^{(\alpha)}\left(t_f^*\right) \right) \varepsilon^\alpha \Delta\tau^\alpha$$
$$+ \int_{t_0}^{t_f^*} \left(\frac{\partial F}{\partial x} \varepsilon^\alpha \eta(t) + \frac{\partial F}{\partial x_{t_0}^{(\alpha)}} \varepsilon^\alpha \eta_{t_0}^{(\alpha)}(t) \right) \mathrm{d}_{t_0}^\alpha + O\left(\varepsilon^{2\alpha}\right).$$

Using the integration by parts formula equation (2.8), we get the first variation denoted by δ as

$$\delta J = F\left(t_f^*, x^*\left(t_f^*\right), x_{t_0}^{*(\alpha)}\left(t_f^*\right)\right) \Delta\tau^\alpha + \frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f^*} \eta\left(t_f^*\right) + \int_{t_0}^{t_f^*} \eta(t) \left(\frac{\partial F}{\partial x} - \frac{\mathrm{d}_a^\alpha}{\mathrm{d}t_a^\alpha} \left(\frac{\partial F}{\partial x_{t_0}^{(\alpha)}} \right) \right) \mathrm{d}_{t_0}^\alpha t = 0. \tag{3.10}$$

The last integral vanishes because of the conformable Euler–Lagrange equation and the transversality condition of conformable variational calculus is achieved as

$$F\left(t_f^*, x^*\left(t_f^*\right), x_{t_0}^{*(\alpha)}\left(t_f^*\right)\right) \Delta\tau^\alpha + \frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f^*} \eta\left(t_f^*\right) = 0. \tag{3.11}$$

□

In order to find the values of the unknown arbitrary functions of $\eta\left(t_f^*\right)$ and $\Delta\tau^\alpha$ in equation (3.8), the transversality conditions in particular cases will be investigated.

Corollary 3.6 (Conformable variational calculus for specialized transversality conditions). *First of all, consider the weak variation of $x(t) = x^*(t) + \varepsilon^\alpha \eta(t)$ at $t = t_f = t_f^* + \varepsilon\Delta\tau$ and expand the right hand side of this variation in a conformable Taylor series with respect to $\varepsilon\Delta\tau$ near the point t_f^*:*

$$x(t) = x^*\left(t_f^* + \varepsilon\Delta\tau\right) + \varepsilon^\alpha \eta\left(t_f^* + \varepsilon\Delta\tau\right) = x^*\left(t_f^*\right) + \frac{x_{t_f^*}^{*(\alpha)}\left(t_f^*\right)}{\alpha} \varepsilon^\alpha \Delta\tau^\alpha + \varepsilon^\alpha \eta\left(t_f^*\right) + O\left(\varepsilon^{2\alpha}\right). \tag{3.12}$$

Since it is assumed that $x(t)$ intersects with the target curve $\gamma(t)$ at $t = t_f$, we also need the expansion of $\gamma\left(t_f^* + \varepsilon\Delta\tau\right)$ in a Taylor series with respect to $\Delta\tau$ about the point t_f^*

$$\gamma\left(t_f^* + \varepsilon\Delta\tau\right) = \gamma\left(t_f^*\right) + \frac{\gamma_{t_f^*}^{(\alpha)}\left(t_f^*\right)}{\alpha} \varepsilon^\alpha \Delta\tau^\alpha + O\left(\varepsilon^{2\alpha}\right). \tag{3.13}$$

Ignoring the remainder terms of $O\left(\varepsilon^{2\alpha}\right)$ and equating these two expansions give the subsequent relation:

$$x^*\left(t_f^*\right) + \frac{x_{t_f^*}^{*(\alpha)}\left(t_f^*\right)}{\alpha} \varepsilon^\alpha \Delta\tau^\alpha + \varepsilon^\alpha \eta\left(t_f^*\right) = \gamma\left(t_f^*\right) + \frac{\gamma_{t_f^*}^{(\alpha)}\left(t_f^*\right)}{\alpha} \varepsilon^\alpha \Delta\tau^\alpha. \tag{3.14}$$

Therefore, the perturbation function is acquired from equation (3.14) as

$$\eta\left(t_f^*\right) = \frac{\gamma_{t_f^*}^{(\alpha)}\left(t_f^*\right) - x_{t_f^*}^{*(\alpha)}\left(t_f^*\right)}{\alpha} \Delta\tau^\alpha. \tag{3.15}$$

Substituting equation (3.15) into equation (3.11) gives the following transversality condition:

$$\left(F\left(t_f^*, x^*\left(t_f^*\right), x_{t_0}^{*(\alpha)}\left(t_f^*\right)\right) + \frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f^*}\left(\frac{\gamma_{t_f^*}^{(\alpha)}\left(t_f^*\right) - x_{t_f^*}^{*(\alpha)}\left(t_f^*\right)}{\alpha}\right)\right)\Delta\tau^\alpha = 0.$$

Because $\Delta\tau^\alpha$ is an arbitrary function, the transversality condition is finally achieved as

$$F\left(t_f^*, x^*\left(t_f^*\right), x_{t_0}^{*(\alpha)}\left(t_f^*\right)\right) + \frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f^*}\left(\frac{\gamma_{t_f^*}^{(\alpha)}\left(t_f^*\right) - x_{t_f^*}^{*(\alpha)}\left(t_f^*\right)}{\alpha}\right) = 0. \tag{3.16}$$

According to equation (3.16), three types of transversality conditions will be examined in below.

A. Terminal curve:

If the terminal point $x(t_f)$ belongs to an $\alpha-$differentiable target curve γ, means $x(t_f) = \gamma(t_f)$, then the transversality condition is obtained as:

$$F\left(t_f, x(t_f), x_{t_0}^{(\alpha)}(t_f)\right) + \frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f}\left(\frac{\gamma_{t_f}^{(\alpha)}(t_f) - x_{t_f}^{(\alpha)}(t_f)}{\alpha}\right) = 0. \tag{3.17}$$

This condition is the transversality condition in the most general sense for conformable variational calculus.

B. Vertical terminal line (fixed-time horizon problem):

If t_f is fixed and $x(t_f)$ is free means the target curve is a straight line perpendicular to the x axis whose slope $\gamma(t_f)$ is infinite, then the transversality condition for infinite $\gamma_{t_f}^{(\alpha)}(t_f)$ is deduced as

$$\frac{1}{\alpha}\frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f} = 0 \tag{3.18}$$

which is referred to as the "natural boundary condition" because of their natural arising in variational formulation.

C. Horizontal terminal line (fixed-endpoint problem):

If $x(t_f)$ is fixed and t_f is free, means $x(t_f) = \gamma(t_f) = c$ and $\gamma_{t_f}^{(\alpha)}(t_f) = 0$, then the transversality condition is obtained as

$$F\left(t_f, x(t_f), x_{t_0}^{(\alpha)}(t_f)\right) - \frac{x_{t_f}^{(\alpha)}(t_f)}{\alpha}\frac{\partial F}{\partial x_{t_0}^{(\alpha)}}\Big|_{t_f} = 0. \tag{3.19}$$

Now, we give a conformable variational problem whose integer order versions can be found in [45].

Example 3.7. Find the minimum of

$$J(x) = \int_0^{t_f}\left(x_0^{(\alpha)}(t)\right)^2 \mathrm{d}_0^\alpha t \tag{3.20}$$

for each of the following cases: (i) $x(0) = 1$, $x(t_f) = 2$; (ii) $x(0) = 1$, $t_f = 2$; (iii) $x(0) = 1$ and $x(t_f)$ lies on the curve $\gamma(t) = 2 + (t^\alpha - 1)^2$.

Solution. The conformable Euler–Lagrange equation in the sequential form ${}^2x_0^{(\alpha)}(t) = 0$ has two-fold roots of $r_{1,2} = 0$. Therefore, the solution is as $x(t) = At^\alpha e^0 + Be^0$, see [16]. The unknown coefficient B is determined from the initial condition $x(0) = 1$ as $B = 1$ which leads to

$$x(t) = At^\alpha + 1. \tag{3.21}$$

Case (i) Since t_f is unspecified and $x(t_f) = 2$, the transversality condition equation (3.19) gives $A = \pm\sqrt{\frac{2}{\alpha}}$. Therefore, the extremals should be either $\sqrt{\frac{2}{\alpha}}t^\alpha$ or $-\sqrt{\frac{2}{\alpha}}t^\alpha$ with $\sqrt{\frac{2}{\alpha}}t_f^\alpha = 1$ or $-\sqrt{\frac{2}{\alpha}}t_f^\alpha = 1$ *via* $x(t_f) = 2$. The second one has no positive solution for t_f. Thus, the extremal is

$$x(t) = \sqrt{\frac{2}{\alpha}}t^\alpha + 1$$

with

$$t_f = \left(\sqrt{\frac{\alpha}{2}}\right)^{\frac{1}{\alpha}}.$$

Case (ii) Since $x(t_f)$ is unspecified and $t_f = 2$ the appropriate transversality condition is equation (3.18) which gives $x_0^{(\alpha)}(2) = 0$, so $A = 0$. The extremal is

$$x(t_f) = 1$$

and

$$x(2) = 1.$$

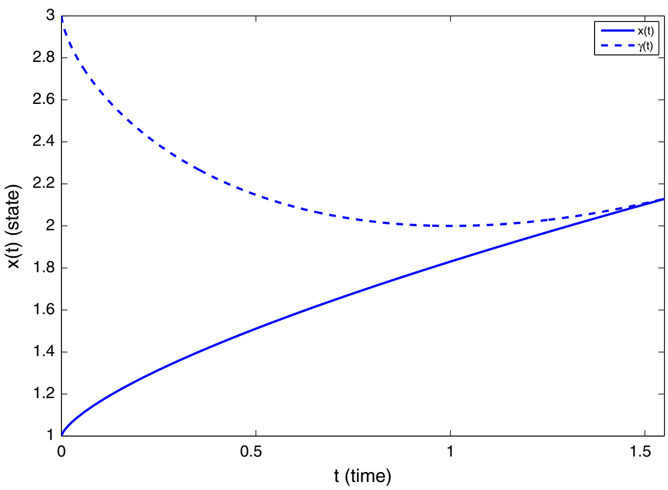

FIGURE 1. Minimizing curve which reaches target curve at $t_f = 1.5598$.

Case (iii) Since $x(t)$ function is not known at the unspecified endpoint t_f, the values of t_f and A are found from the equality of $x(t_f) = \gamma(t_f)$ and the transversality condition equation (3.17) as

$$\begin{aligned} At_f^{\alpha} - \left(t_f^{\alpha} - 1\right)^2 - 1 &= 0, \\ (A\alpha)^2 + 2A\alpha\left(2\left(t_f^{\alpha} - 1\right)^{2-\alpha} - A\right) &= 0. \end{aligned} \tag{3.22}$$

These equations are solved for the chosen value of $\alpha = 0.7$ by using symbolic toolbox of MATLAB. The solutions are then obtained as $t_f = 1.5598$ and $A = 0.8302$. Figure 1 is also plotted by MATLAB.

4. Conformable optimal control problem with transversality condition

The conformable optimal control problem firstly examined by [40] can be defined as to find a pair of functions $(x(.), u(.))$ that minimizing (or maximizing) of a performance index defined by a conformable or classical integral subject to a conformable dynamic constraints. In this study, we consider the conformable optimal control problem defined as

$$J(x, u) = \int_a^b F(t, x(t), u(t))\, \mathrm{d}_a^{\alpha} t \tag{4.1}$$

which is subjected to the conformable dynamical system

$$x_a^{(\alpha)}(t) = g(t, x(t), u(t)), \tag{4.2}$$

where x and u are the state and control functions, respectively. We assume that the Lagrangian F and the function g are the functions at least from the C^{α} class in their domain $\left([a, b] \times \mathbb{R}^2\right)$ for $0 < \alpha \leq 1$. Also, the admissible state functions $x(t)$ are such that $x_a^{(\alpha)}(t)$ exists. The pair $(x(.), u(.))$ that minimizes the performance index equation (4.1) subjected to equation (4.2) is called as an optimal process. It is worth to note that for $\alpha = 1$, conformable optimal control problem coincides to classical optimal control problem.

The necessary optimality conditions were obtained by [40] from the conformable Hamiltonian formalism can be given as

$$\begin{aligned} x_a^{(\alpha)}(t) &= \frac{\partial H}{\partial \lambda}(t, x, u, \lambda) && \text{(state)} \\ \lambda_a^{(\alpha)}(t) &= -\frac{\partial H}{\partial x}(t, x, u, \lambda) && \text{(costate)} \\ \frac{\partial H}{\partial u}(t, x, u, \lambda) &= 0 && \text{(control)} \end{aligned} \tag{4.3}$$

where

$$H(t, x, u, \lambda) = -F(t, x, u) + \lambda(t)\, g(t, x, u) \tag{4.4}$$

is the Hamiltonian function and λ is an α-differentiable function known as Lagrange multiplier.

Now, we will give the transversality condition of the conformable optimal control problem by the following theorem.

Theorem 4.1 (Transversality conditions for conformable optimal control)**.** *Let $x:[t_0,t_f]\to\mathbb{R}$ is an $\alpha-$differentiable function, F and g are functions in the class of $C^\alpha\left([t_0,t_f]\times\mathbb{R}^2\right)$ for $0<\alpha\leq 1$. If $(x(t),u(t))$ is a minimizer of*

$$J(x,u)=\int_{t_0}^{t_f}F(t,x(t),u(t))\,\mathrm{d}_{t_0}^{\alpha}t \tag{4.5}$$

subject to the system dynamics constraint

$$x_{t_0}^{(\alpha)}(t)=g(t,x(t),u(t)) \tag{4.6}$$

when $x(t_0)=x_0$ is fixed, and $x(t_f)=x_f$ is free. Then the general transversality condition is

$$\left[-H(t_f,x(t_f),u(t_f))+\lambda(t_f)\,x_{t_0}^{(\alpha)}(t)\right]\Delta\tau^\alpha+\lambda(t_f)\,\eta(t_f)=0, \tag{4.7}$$

where $\eta\in C^\alpha\left([t_0,t_f]\times\mathbb{R}^2\right)$ and $\Delta\tau$ are perturbations for the weak variation of x and t_f, respectively.

Proof. Suppose that $x^*(t)$ is a minimizing curve which intersects with the target curve $\gamma(t)$ at $t=t_f^*$. To find the optimal solution for state, control and costate functions at $t\epsilon[t_0,t_f]$ assume the following weak variations for $|\varepsilon|\ll 1$

$$\begin{aligned}
x(t)&=x^*(t)+\varepsilon^\alpha\eta(t),\\
x_{t_0}^{(\alpha)}(t)&=x_{t_0}^{*(\alpha)}(t)+\varepsilon^\alpha\eta_{t_0}^{(\alpha)}(t),\\
u(t)&=u^*(t)+\varepsilon^\alpha\xi(t),\\
\lambda(t)&=\lambda^*(t)+\varepsilon^\alpha\Lambda(t),\\
t_f&=t_f^*+\varepsilon\Delta\tau,
\end{aligned} \tag{4.8}$$

where $\eta(t),\eta_{t_0}^{(\alpha)}(t),\xi(t),\Lambda(t)$ and $\Delta\tau$ are perturbations for the weak variation of $x(t)$, $x_{t_0}^{(\alpha)}(t),u(t),\lambda(t)$ and t_f, respectively. Note that $\eta(t_0)=0$ and $\eta(t_f)\neq 0$ since t_f is free. To use the method of Lagrange multipliers technique, the performance index can be defined as:

$$I(x,u,\lambda)=\int_{t_0}^{t_f}F(t,x(t),u(t))\,\mathrm{d}_{t_0}^{\alpha}t+\int_{t_0}^{t_f}\lambda(t)\left(x_{t_0}^{(\alpha)}(t)-g(t,x(t),u(t))\right)\mathrm{d}_{t_0}^{\alpha}t. \tag{4.9}$$

For the sake of easy computations, the conformable integrals in equation (4.9) will be examined separately, the first one is obviously J and the second one is denoted by Φ. Assume that $x^*(t)$ and $u^*(t)$ are the optimum functions of the problem equations (4.5)–(4.6), then the variation of J and Φ are respectively given as follows,

$$\Delta J=\int_{t_f^*}^{t_f^*+\varepsilon\Delta\tau}F(t,x(t),u(t))\,\mathrm{d}_{t_f^*}^{\alpha}t+\int_{t_0}^{t_f^*}\left(F(t,x(t),u(t))-F(t,x^*(t),u^*(t))\right)\mathrm{d}_{t_0}^{\alpha}t, \tag{4.10}$$

$$\Delta\Phi = \int_{t_f^*}^{t_f^*+\varepsilon\Delta\tau} \lambda(t)\left(x_{t_0}^{(\alpha)}(t) - g(t, x(t), u(t))\right) \mathrm{d}_{t_f^*}^{\alpha} t + \int_{t_0}^{t_f^*} \lambda(t)\left(x_{t_0}^{(\alpha)}(t) - g(t, x(t), u(t))\right) \mathrm{d}_{t_0}^{\alpha} t$$
$$- \int_{t_0}^{t_f^*} \lambda^*(t)\left(x_{t_0}^{*(\alpha)}(t) - g(t, x^*(t), u^*(t))\right) \mathrm{d}_{t_0}^{\alpha} t. \tag{4.11}$$

The functions F and g in equations (4.10)–(4.11) are expanded in a Taylor series according to the pair of variables $(\varepsilon^\alpha \eta, \varepsilon^\alpha \xi)$ near the point (x^*, u^*) for $t \epsilon [t_0, t_f]$ gives

$$\Delta J = \int_{t_f^*}^{t_f^*+\varepsilon\Delta\tau} \left(F(t, x^*(t), u^*(t)) + \frac{\partial F}{\partial x}\varepsilon^\alpha \eta(t) + \frac{\partial F}{\partial u}\varepsilon^\alpha \xi(t)\right) \mathrm{d}_{t_f^*}^{\alpha} t$$
$$+ \int_{t_0}^{t_f^*} \left(F(t, x^*(t), u^*(t)) + \frac{\partial F}{\partial x}\varepsilon^\alpha \eta(t) + \frac{\partial F}{\partial u}\varepsilon^\alpha \xi(t) - F(t, x^*(t), u^*(t))\right) \mathrm{d}_{t_0}^{\alpha} t + O\left(\varepsilon^{2\alpha}\right), \tag{4.12}$$

$$\Delta\Phi = \int_{t_f^*}^{t_f^*+\varepsilon\Delta\tau} (\lambda^*(t) + \varepsilon^\alpha \Lambda(t)) \left(x_{t_0}^{*(\alpha)}(t) + \varepsilon^\alpha \eta_{t_0}^{(\alpha)}(t) - \left(g(t, x^*(t), u^*(t)) + \frac{\partial g}{\partial x}\varepsilon^\alpha \eta(t) + \frac{\partial g}{\partial u}\varepsilon^\alpha \xi(t)\right)\right) \mathrm{d}_{t_f^*}^{\alpha} t$$
$$+ \int_{t_0}^{t_f^*} (\lambda^*(t) + \varepsilon^\alpha \Lambda(t)) \left(x_{t_0}^{*(\alpha)}(t) + \varepsilon^\alpha \eta_{t_0}^{(\alpha)}(t) - \left(g(t, x^*(t), u^*(t)) + \frac{\partial g}{\partial x}\varepsilon^\alpha \eta(t) + \frac{\partial g}{\partial u}\varepsilon^\alpha \xi(t)\right)\right) \mathrm{d}_{t_0}^{\alpha} t$$
$$- \int_{t_0}^{t_f^*} \lambda^*(t)\left(x_{t_0}^{*(\alpha)}(t) - g(t, x^*(t), u^*(t))\right) \mathrm{d}_{t_0}^{\alpha} t + O\left(\varepsilon^{2\alpha}\right). \tag{4.13}$$

The first variations of equations (4.12) and (4.13) are respectively obtained as

$$\delta J = F\left(t_f^*, x^*\left(t_f^*\right), u^*\left(t_f^*\right)\right) \Delta\tau^\alpha + \int_{t_0}^{t_f^*} \left(\frac{\partial F}{\partial x}\eta(t) + \frac{\partial F}{\partial u}\xi(t)\right) \mathrm{d}_{t_0}^{\alpha} t = 0, \tag{4.14}$$

$$\delta\Phi = \lambda^*\left(t_f^*\right)\left(x_{t_0}^{*(\alpha)}\left(t_f^*\right) - g\left(t_f^*, x^*\left(t_f^*\right), u^*\left(t_f^*\right)\right)\right) \Delta\tau^\alpha$$
$$+ \int_{t_0}^{t_f^*} \left(\left(\lambda^*(t)\left(\eta_{t_0}^{(\alpha)}(t) - \frac{\partial g}{\partial x}\eta(t) - \frac{\partial g}{\partial u}\xi(t)\right) + \Lambda(t)\left(x_{t_0}^{*(\alpha)}(t) - g(t, x^*(t), u^*(t))\right)\right)\right) \mathrm{d}_{t_0}^{\alpha} t = 0. \tag{4.15}$$

Using integration by parts formula equation (2.8) for equation (4.15) and then aggregating δJ and $\delta\Phi$ gives the following equation in the Hamiltonian form

$$\delta J+\delta\Phi=\left(-H\left(t_f^*,x^*\left(t_f^*\right),u^*\left(t_f^*\right)\right)+\lambda^*\left(t_f^*\right)x_{t_0}^{*(\alpha)}\left(t_f^*\right)\right)\Delta\tau^\alpha+\int_{t_0}^{t_f^*}\eta\left(t\right)\left(-\lambda_{t_0}^{*(\alpha)}\left(t\right)-\frac{\partial H}{\partial x}\right)\mathrm{d}_{t_0}^\alpha t$$
$$+\int_{t_0}^{t_f^*}\xi\left(t\right)\left(\frac{\partial H}{\partial u}\right)\mathrm{d}_{t_0}^\alpha t\left(t\right)+\int_{t_0}^{t_f^*}\Lambda\left(t\right)\left(x_{t_0}^{*(\alpha)}\left(t\right)-g\left(t,x^*\left(t\right),u^*\left(t\right)\right)\right)\mathrm{d}_{t_0}^\alpha t+\lambda^*\left(t_f^*\right)\eta\left(t_f^*\right)=0. \quad (4.16)$$

Since the necessary optimality conditions equation (4.3), the integral vanishes and then the transversality condition of the conformable optimal control problem is achieved as

$$\left(-H\left(t_f^*,x^*\left(t_f^*\right),u^*\left(t_f^*\right)\right)+\lambda^*\left(t_f^*\right)x_{t_0}^{*(\alpha)}\left(t_f^*\right)\right)\Delta\tau^\alpha+\lambda^*\left(t_f^*\right)\eta\left(t_f^*\right)=0. \quad (4.17)$$

□

The unknown arbitrary functions of $\eta\left(t_f^*\right)$ and $\Delta\tau^\alpha$ in formula can be specialized with the transversality conditions in particular cases given below.

Corollary 4.2 (**Conformable optimal control for specialized transversality conditions**). *As a result of the conformable Taylor series expansions of the functions $x\left(t\right)$ and $\gamma\left(t\right)$, the perturbation function is again obtained with the similar process in Section 3 (see Eqs. (3.12)–(3.14)) as*

$$\eta\left(t_f^*\right)=\frac{\gamma_{t_f^*}^{(\alpha)}\left(t_f^*\right)-x_{t_f^*}^{*(\alpha)}\left(t_f^*\right)}{\alpha}\Delta\tau^\alpha.$$

Therefore, by substituting $\eta\left(t_f^*\right)$ in equation (4.17), we get

$$\delta J+\delta\Phi=\left(-H\left(t_f^*,x^*\left(t_f^*\right),u^*\left(t_f^*\right)\right)+\lambda^*\left(t_f^*\right)x_{t_0}^{*(\alpha)}\left(t_f^*\right)+\lambda^*\left(t_f^*\right)\left(\frac{\gamma_{t_f^*}^{(\alpha)}\left(t_f^*\right)-x_{t_f^*}^{*(\alpha)}\left(t_f^*\right)}{\alpha}\right)\right)\Delta\tau^\alpha=0.$$

Because $\Delta\tau^\alpha$ is an arbitrary function, the transversality condition is finally achieved in the following form:

$$-H\left(t_f^*,x^*\left(t_f^*\right),u^*\left(t_f^*\right)\right)+\lambda^*\left(t_f^*\right)x_{t_0}^{*(\alpha)}\left(t_f^*\right)+\lambda^*\left(t_f^*\right)\left(\frac{\gamma_{t_f^*}^{(\alpha)}\left(t_f^*\right)-x_{t_f^*}^{*(\alpha)}\left(t_f^*\right)}{\alpha}\right)=0. \quad (4.18)$$

According to this equation three types of transversality conditions are examined as in below.

A. Terminal curve: *If the terminal point $x\left(t_f\right)$ belongs to an $\alpha-$differentiable target curve γ, means $x\left(t_f\right)=\gamma\left(t_f\right)$, then the transversality condition is obtained as:*

$$-H\left(t_f,x\left(t_f\right),u\left(t_f\right)\right)+\lambda\left(t_f\right)x_{t_0}^{(\alpha)}\left(t_f\right)+\lambda\left(t_f\right)\left(\frac{\gamma_{t_f}^{(\alpha)}\left(t_f\right)-x_{t_f}^{(\alpha)}\left(t_f\right)}{\alpha}\right)=0. \quad (4.19)$$

This condition is the transversality condition in the most general sense for conformable optimal control.

B. Vertical terminal line (fixed-time horizon problem):

If t_f is fixed and $x(t_f)$ is free, then the target curve is a straight line perpendicular to the x axis whose slope $\gamma(t_f)$ is infinite. Therefore, the transversality condition for infinite $\gamma_{t_f}^{(\alpha)}$ is achieved as

$$\lambda(t_f) = 0. \tag{4.20}$$

C. Horizontal terminal line (fixed-endpoint problem):

If $x(t_f)$ is fixed and t_f is free, means $x(t_f) = \gamma(t_f) = c$ and $\gamma_{t_f}^{(\alpha)} = 0$, then the transversality condition is obtained as

$$-H(t_f, x(t_f), u(t_f)) + \lambda(t_f)\left(x_{t_0}^{(\alpha)}(t_f) - \frac{x_{t_f}^{(\alpha)}(t_f)}{\alpha}\right) = 0. \tag{4.21}$$

5. Conformable optimal control of a two-dimensional diffusion system

To depict conformable transversality condition, we present an optimal control problem for a conformable diffusion system which previously taken into account by Özdemir *et al.* [44] in Riemann–Liouville sense. By using eigenfunction expansion method, the optimal control law is here achieved analytically while achieved numerically in [44].

We aim to find the optimal process of the following optimal control problem defined by the performance index

$$J(x,u) = \frac{1}{2}\int_0^1\int_0^L\int_0^L \left(x^2(\zeta,\rho,t) + u^2(\zeta,\rho,t)\right) \mathrm{d}\zeta \mathrm{d}\rho \mathrm{d}_0^\alpha t \tag{5.1}$$

subjected to the conformable dynamical system

$$x_0^{(\alpha)}(\zeta,\rho,t) = \left(\frac{\partial^2 x(\zeta,\rho,t)}{\partial\zeta^2} + \frac{\partial^2 x(\zeta,\rho,t)}{\partial\rho^2}\right) + u(\zeta,\rho,t) \tag{5.2}$$

with the initial

$$x(\zeta,\rho,0) = 1 + \zeta + \rho \tag{5.3}$$

and the boundary conditions

$$\frac{\partial x(0,\rho,t)}{\partial\zeta} = \frac{\partial x(L,\rho,t)}{\partial\zeta} = \frac{\partial u(\zeta,0,t)}{\partial\rho} = \frac{\partial u(\zeta,L,t)}{\partial\rho} = 0, \tag{5.4}$$

where $x(\zeta,\rho,t)$ is the state and $u(\zeta,\rho,t)$ is the control functions which depend on time t and the space parameters $(\zeta,\rho)\,\epsilon\,[0,L]\times[0,L]$ and $x_0^{(\alpha)}(\zeta,\rho,t)$ represent the conformable derivative of state function of order $0<\alpha\le 1$ with respect to t.

Using the eigenfunctions $\phi_{mn}(\zeta,\rho)$, $m,n=0,1,2,...,\infty$, the state and the control functions can be written as

$$x(\zeta,\rho,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}x_{mn}(t)\,\phi_{mn}(\zeta,\rho) \tag{5.5}$$

$$u(\zeta,\rho,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{mn}(t)\,\phi_{mn}(\zeta,\rho), \tag{5.6}$$

where $x_{mn}(t)$ is the state and $u_{mn}(t)$ is the control eigencoordinates. Applying the method of separation of variables, it could be demonstrated that the eigenfunctions for the problem are obtained as

$$\phi_{mn}(\zeta,\rho)=\cos\left(\frac{m\pi\zeta}{L}\right)\cos\left(\frac{n\pi\rho}{L}\right). \tag{5.7}$$

For the simplicity, the upper limit of the indexes m and n is taken as a finite value demonstrated by k. If the state and the control functions in equations (5.5) and (5.6) are substituting in the performance index, the following form is achieved

$$\begin{aligned}J=&-\frac{L^2}{2}\int_0^1\left(x_{00}^2(t)+\sum_{m=1}^{k}\frac{1}{2}x_{m0}^2(t)+\sum_{n=1}^{k}\frac{1}{2}x_{0n}^2(t)+\sum_{m=1}^{k}\sum_{n=1}^{k}\frac{1}{4}x_{mn}^2(t)\right)\mathrm{d}_0^{\alpha}t\\&-\frac{L^2}{2}\int_0^1\left(u_{00}^2(t)+\sum_{m=1}^{k}\frac{1}{2}u_{m0}^2(t)+\sum_{n=1}^{k}\frac{1}{2}u_{0n}^2(t)+\sum_{m=1}^{k}\sum_{n=1}^{k}\frac{1}{4}u_{mn}^2(t)\right)\mathrm{d}_0^{\alpha}t.\end{aligned} \tag{5.8}$$

Finally, writing equation (5.5) into equation (5.3), multiplying both sides by $\cos\left(\frac{m\pi\zeta}{L}\right)\cos\left(\frac{n\pi\rho}{L}\right)$ and integrating *via* ζ,ρ from 0 to L, we get

$$x_{mn}(0)=\frac{1}{L^2}\begin{cases}L^2+L^3 & m=0,n=0\\ \frac{2L^3}{n^2\pi^2}\left(\cos(n\pi)-1\right) & m=0,n>0\\ \frac{2L^3}{m^2\pi^2}\left(\cos(m\pi)-1\right) & m>0,n=0\\ 0 & m>0,n>0.\end{cases} \tag{5.9}$$

Using the above approximations, the Hamiltonian for the system can be defined as

$$\begin{aligned}H=&-\frac{L^2}{2}\left(x_{00}^2(t)+\sum_{m=1}^{k}\frac{1}{2}x_{m0}^2(t)+\sum_{n=1}^{k}\frac{1}{2}x_{0n}^2(t)+\sum_{m=1}^{k}\sum_{n=1}^{k}\frac{1}{4}x_{mn}^2(t)\right)\\&-\frac{L^2}{2}\left(u_{00}^2(t)+\sum_{m=1}^{k}\frac{1}{2}u_{m0}^2(t)+\sum_{n=1}^{k}\frac{1}{2}u_{0n}^2(t)+\sum_{m=1}^{k}\sum_{n=1}^{k}\frac{1}{4}u_{mn}^2(t)\right)\\&-\frac{L^2}{2}\left(\lambda_{00}(t)\,u_{00}(t)+\sum_{m=1}^{k}\sum_{n=1}^{k}\lambda_{mn}(t)\left(-\left\{\left(\frac{m\pi}{L}\right)^2+\left(\frac{n\pi}{L}\right)^2\right\}x_{mn}(t)+u_{mn}(t)\right)\right).\end{aligned} \tag{5.10}$$

The necessary optimality conditions of the system are given as

$$\text{State equation: } x_{0_{mn}}^{(\alpha)}(t)+\left\{\left(\frac{m\pi}{L}\right)^2+\left(\frac{n\pi}{L}\right)^2\right\}x_{mn}(t)+u_{mn}(t)=0 \tag{5.11}$$

$$\text{Costate equation: } \frac{L^2}{4} x_{mn}(t) + \left\{ \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \right\} \lambda_{mn}(t) - \lambda_{0_{mn}}^{(\alpha)}(t) = 0 \tag{5.12}$$

$$\text{Control equation: } \lambda_{mn}(t) - \frac{L^2}{4} u_{mn}(t) = 0. \tag{5.13}$$

When equations (5.11)–(5.13) are solved using the analytical method of conformable differential equations (see, [54]), the state and the control functions are found as

$$u_{mn}(t) = c_1^{mn} e^{-r\frac{t^\alpha}{\alpha}} + c_2^{mn} e^{r\frac{t^\alpha}{\alpha}} \tag{5.14}$$

$$x_{mn}(t) = \left(\left\{ \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \right\} - r \right) c_1^{mn} e^{-r\frac{t^\alpha}{\alpha}} + \left(\left\{ \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \right\} + r \right) c_2^{mn} e^{r\frac{t^\alpha}{\alpha}}, \tag{5.15}$$

where $\pm r$, $\left(r = \sqrt{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2 + 1} \right)$ are the roots of characteristic equation of conformable differential equation obtained from equations (5.11)–(5.12) and, c_1^{mn} and c_2^{mn} are the coefficients determined from the initial and the transversality conditions. Because of fixed $t_f = 1$ and free $x(t_f)$, the transversality condition of the problem is

$$\lambda_{mn}(1) = 0.$$

Also it is found from equation (5.13) that $u(1) = 0$. Therefore, the unknown coefficients of c_1^{mn} and c_2^{mn} are calculated from the equations

$$u_{mn}(1) = c_1^{mn} e^{-\frac{r}{\alpha}} + c_2^{mn} e^{\frac{r}{\alpha}} = 0 \tag{5.16}$$

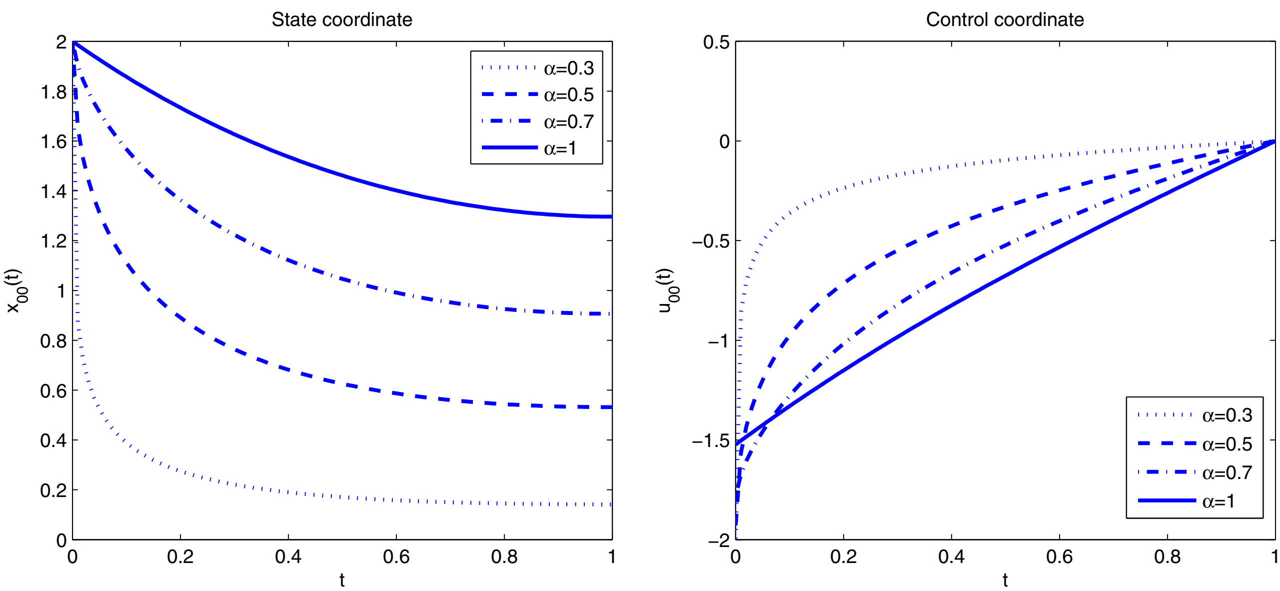

FIGURE 2. State and control coordinates for different values of α.

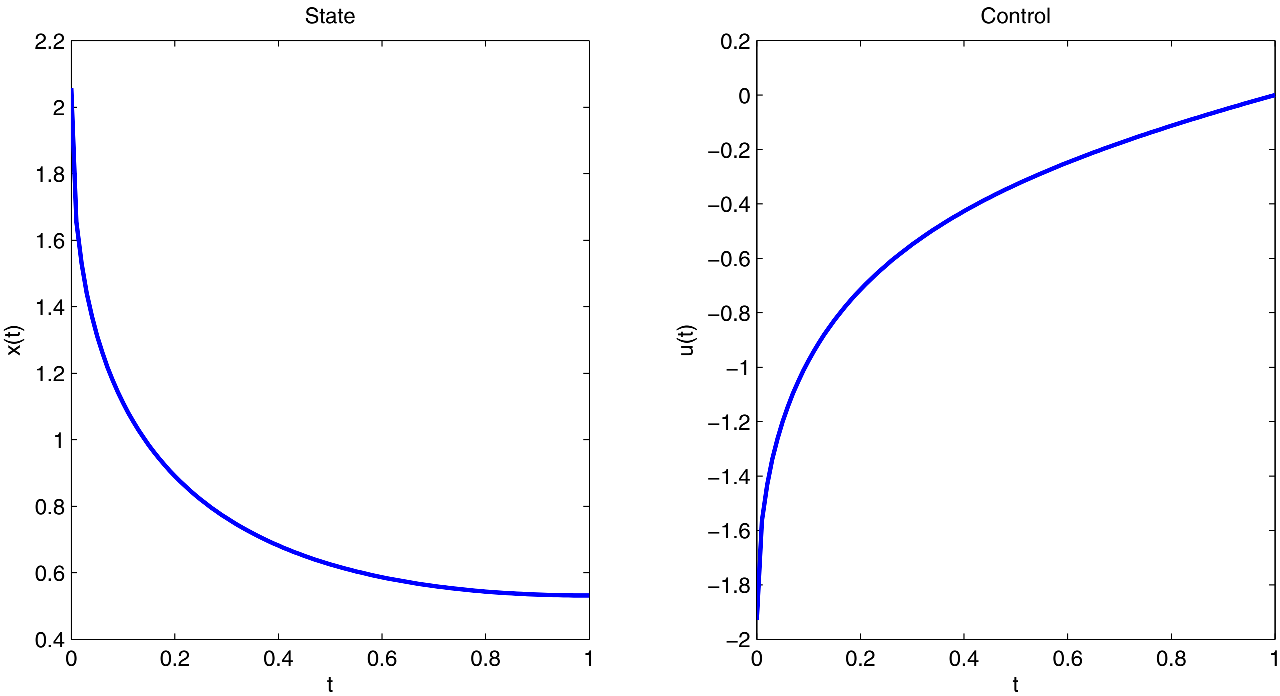

FIGURE 3. State and control functions for $\alpha = 0.5, \zeta = 0.5, \rho = 0.3$.

and

$$x_{mn}(0) = \left(\left\{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right\} - r\right) c_1^{mn} + \left(\left\{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right\} + r\right) c_2^{mn}, \tag{5.17}$$

by using symbolic toolbox of MATLAB.

To show the effect of order α on the optimal process, we choose the indexes $m = n = 0$ and $L = 1$ which gives $r = 1$. Then, we plot the state x_{00} and the control u_{00} eigencoordinates for different values of α in Figure 2. It can be seen from the left side of Figure 2 that the contribution of state eigencoordinates decrease while the order of α is reduced form 1 to 0. It can be read as the behaviors of state eigencoordinates changes from normal diffusion to subdiffusion. Also the effects of control eigencoordinates observed from the right side of Figure 2 increase parallel to state coordinates as expected. Finally, the state and the control functions are plotted for $\alpha = 0.5$ as a function of time by choosing $\zeta = 0.5$ and $\rho = 0.3$ in Figure 3. Note that, we cut the series in equations (5.5)–(5.6) after 5 terms to illustrate the last figure.

6. SUMMARY

the transversality conditions of the problems both from conformable variational calculus and conformable optimal control have been investigated and specialized for particular cases. To show the applications of the formulations the optimal control problem of conformable diffusion process with free final time has been considered. The optimal control law is obtained by using Hamiltonian formalism and Lagrange multiplier technique. Comparing the obtained results for conformable optimal control law *via* fractional optimal control law shows that the conformable derivative gives the opportunity of analytical solutions while both types supply a similar manner for the optimal control process. The transversality conditions for the generalized type that a performance index defined with classical integral whose integrand is containing conformable derivative term will be discussed in the next study.

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