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ANALYSIS OF ADVECTIVE–DIFFUSIVE TRANSPORT PHENOMENA MODELLED *VIA* NON-SINGULAR MITTAG-LEFFLER KERNEL

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Abstract. In this study, a linear advection–diffusion equation described by Atangana–Baleanu derivative with non-singular Mittag-Leffler kernel is considered. The Cauchy, Dirichlet and source problems are formulated on the half-line. The main motivation of this work is to find the fundamental solutions of prescribed problems. For this purpose, Laplace transform method with respect to time t and sine/cosine-Fourier transform methods with respect to spatial coordinate x are applied. It is remarkable that the obtained results are quite similar to the existing fundamental solutions of advection–diffusion equation with time-Caputo fractional derivative. Although the results are mathematically similar in both formulations, the AB derivative is a non-singular operator and provides a significant advantage in the computational processes. Therefore, it is preferable to replace the Caputo derivative in modelling such diffusive transports.

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1. Introduction

In the nature, transport of a solute is modelled by a parabolic type partial differential equation called as advection–diffusion equation (ADE). Pollution of groundwater flow, spreading of chemical pollution in sea water and atmospheric pollution caused by harmful gases are some real world problems modelled by ADEs. Novel applications of ADEs in different areas, such as hydrology, thermal engineering, environmental engineering, petroleum engineering, bio-science, chemical engineering, etc., have been an increasing interest among the researchers.

In the recent years, fractional calculus has been intensively used to describe the hereditary properties, dissipative effects, memory and damage structures in the real world problems. Riemann–Liouville (RL) and Caputo fractional operators are outstanding definitions of fractional calculus defined by the convolution of a given function/its derivative and a power decay function as a kernel [9]. By this definition, these operators are nonlocal. On the other hand, the singularity arising from power decay kernel function reveals many significant computational difficulties and therefore requires introducing numerical solutions [20, 35, 36].

Fractional differentiation has also been a powerful tool to model the anomalous diffusion and other transport processes in heterogeneous media. The analytical solutions of one-dimensional fractional ADE have been

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obtained in terms of H-function [18, 21]. The fundamental solutions to Cauchy, Dirichlet and source problems based on time-fractional ADE have been analyzed in different regions by using integral transform techniques [25, 27–29].

In the recent years, new fractional operators with non-singular kernels have been introduced. In this sense, Caputo and Fabrizio firstly have proposed a derivation operator in terms of exponential decay kernel function called by Caputo–Fabrizio (CF) derivative [12]. They have also emphasized that this new operator can succeed to model material heterogeneities, fatigue, damage and the structures with different scales [13]. Losada and Nieto [22] investigated some basic properties of CF derivative. A significant difference between CF and RL/Caputo derivatives appears in the solutions of differential equations. Although the solutions of fractional differential equations with RL/Caputo fractional derivative are usually obtained in terms of generalized functions such as Mittag-Leffler, Robotnov, H-Fox, Mainardi, Wright, Lorenzo-Hartley functions, etc., the solutions of fractional differential equations with CF derivative are found by elementary functions [10, 19, 32]. In different areas of engineering sciences, finance, health science, CF derivative has been successfully used to describe the complex dynamics of the problems [1–4, 6, 14–17, 23, 33, 34]. Remark that some authorities have accepted that CF operator acts like a filter regulator for engineering applications because of its regular kernel.

Atangana and Baleanu have also proposed two non-singular fractional order derivatives in sense of RL and Caputo which are called as Atangana–Baleanu (AB) derivatives and also based on the Mittag-Leffler kernel function [5]. AB operator has also been considered as filter with fractional regulator and also it has the fundamental properties of fractional derivatives [7]. The Mittag-Leffler kernel is non-local, non-singular and also has all the benefits of RL, Caputo and CF derivatives. In addition, the nonlocality of the kernel gives opportunity to have better description of memory properties in the structures with different scales. Similar to Caputo and CF derivatives, the Laplace transform of the AB derivative requires the physically interpretable initial conditions with integer-order derivatives and so it is rather preferred for modeling of different physical processes. Because of its considerable properties, it has an increasing interest in different fields [8, 11, 37].

In the present paper, we consider an ADE in terms of AB derivative on the half-line. The Cauchy, Dirichlet and source problems are formulated, respectively. We investigate the fundamental solutions for the prescribed problems by applying the Laplace and the Fourier transforms. The results are illustrated by the graphics. In addition, we give some remarkable differences and relations between the current fundamental solutions and the conventional solutions for the Caputo model. Therefore, we underline the advantage of AB derivative for a diffusive transport with an advection effect.

2. Preliminaries

In this section, we recall some basic definitions and properties related to AB derivative. To obtain the fundamental solutions, we apply the Laplace transform with respect to t variable and sine/cosine Fourier transforms with respect to spatial coordinate x. Let us remind these mathematical concepts as follows.

Let $(a,b) \subset \mathbb{R}$ and let u be a function of the Hilbert space $L^2(a,b)$. u' denotes the derivative of u as distribution on (a,b).

Definition 2.1. The Sobolev space of order 1 in (a,b) is defined as

$$H^1(a,b) = \left\{ u \in L^2(a,b) \mid u' \in L^2(a,b) \right\}.$$

Definition 2.2. Let $\alpha \in (0,1)$ and a function $u \in H^1(a,b)$, $b > a$. The AB derivative in Caputo sense of order α of u with a based point a is defined as [5]

$$^{ABC}D_t^{\alpha} u(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t u'(s)\, E_\alpha \left[-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right] \mathrm{d}s, \tag{2.1}$$

where $B(\alpha)$ is the normalization function which has the following form

$$B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)},$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function, defined in terms of a series as the following entire function [24]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} (\alpha, \beta > 0).$$

Definition 2.3. Let $\alpha \in (0,1)$ and a function $u \in H^1(a,b), b > a$. The AB derivative in the RL sense of order α of u is defined as [5]

$$^{ABR}D_t^{\alpha} u(t) = \frac{B(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t u(s) E_{\alpha} \left[-\frac{\alpha}{1-\alpha} (t-s)^{\alpha} \right] \mathrm{d}s. \tag{2.2}$$

The Laplace transform of AB derivative in RL sense is given as [5]

$$\mathcal{L}\left\{ {}_0^{ABR}D_t^{\alpha} \{f(t)\} \right\}(s) = \frac{B(\alpha)}{1-\alpha} \frac{s^{\alpha} \mathcal{L}\{f(t)\}(s)}{s^{\alpha} + \frac{\alpha}{1-\alpha}}. \tag{2.3}$$

Similarly, the Laplace transform of AB derivative in Caputo sense is as follows [5]

$$\mathcal{L}\left\{ {}_0^{ABC}D_t^{\alpha} \{f(t)\} \right\}(s) = \frac{B(\alpha)}{1-\alpha} \frac{s^{\alpha} \mathcal{L}\{f(t)\}(s) - s^{\alpha-1} f(0)}{s^{\alpha} + \frac{\alpha}{1-\alpha}}. \tag{2.4}$$

The sine-Fourier transform is defined as [26]:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \int_0^{\infty} f(x) \sin(x\xi) \mathrm{d}x \tag{2.5}$$

with the inverse transform [26]:

$$\mathcal{F}^{-1}\left\{ \widetilde{f}(\xi) \right\} = f(x) = \frac{2}{\pi} \int_0^{\infty} \widetilde{f}(\xi) \sin(x\xi) \mathrm{d}\xi. \tag{2.6}$$

The sine-Fourier transform of the second order derivative of a given function is calculated according to the following relation [26]:

$$\mathcal{F}\left\{ \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} \right\} = -\xi^2 \widetilde{f}(\xi) + \xi f(x) \mid_{x=0}. \tag{2.7}$$

Similarly, the cosine-Fourier transform can be given as [26]:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \int_0^{\infty} f(x) \cos(x\xi) \mathrm{d}x, \tag{2.8}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\xi)\right\} = f(x) = \frac{2}{\pi}\int\limits_0^\infty \widetilde{f}(\xi)\cos(x\xi)\mathrm{d}\xi. \tag{2.9}$$

and the cosine-Fourier transform of the second order derivative of a given function can be given as [26]:

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\widetilde{f}(\xi) - \frac{\mathrm{d}f(x)}{\mathrm{d}x}\mid_{x=0}. \tag{2.10}$$

3. Formulation of the main problem

3.1. Fundamental solutions to the Cauchy problems

3.1.1. Problem 1

Let us consider the following ADE in terms of time-AB derivative

$$^{ABC}D_t^\alpha c(x,t) = a\frac{\partial^2 c(x,t)}{\partial x^2} - \nu\frac{\partial c(x,t)}{\partial x}, \tag{3.1}$$

$$0 < \alpha \leq 1, 0 < x < \infty, 0 < t < \infty, a > 0, \nu > 0$$

with the initial condition

$$t = 0 : c(x,0) = \delta(x), \tag{3.2}$$

and the zero Neumann boundary condition

$$x = 0 : c(x,t) = 0, \frac{\partial c(x,t)}{\partial x} = 0, \tag{3.3}$$

where the physical quantities a represents the diffusivity coefficient and ν is the velocity parameter.

For only convenience, the zero condition at infinity is also assumed

$$\lim_{x\to\infty} c(x,t) = 0. \tag{3.4}$$

In the present work, we give a comparative interpretation with [30] in sense of singularity structures of AB and Caputo derivatives. The main advantage of AB derivative stem from its non-singular kernel can remove the computational complexities as compared to Caputo operator.

We use an auxiliary function $u(x,t)$ and hence assume that

$$c(x,t) = \exp\left(\frac{\nu x}{2a}\right)u(x,t). \tag{3.5}$$

By considering a basic property of Dirac delta function $f(x)\delta(x) = f(0)\delta(x)$, the Cauchy problem given by equations (3.1)–(3.4) is reduced to the following form:

$$^{ABC}D_t^\alpha u(x,t) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\nu^2}{4a}u(x,t), \tag{3.6}$$

$$u(x,0) = \delta(x), \tag{3.7}$$

$$u(0,t)=0, \frac{\partial u(x,t)}{\partial x}=0, \tag{3.8}$$

$$\lim_{x\to\infty} u(x,t)=0. \tag{3.9}$$

By applying the Laplace transform with respect to the time t and cosine-Fourier transform with respect to the variable x to equation (3.6) with the initial condition equation (3.7) and boundary conditions equations (3.8), (3.9), we get

$$\overline{u}^*(\xi,s)=\frac{\gamma}{\gamma+a\xi^2+\frac{\nu^2}{4a}}\frac{s^{\alpha-1}}{s^\alpha+\frac{\alpha\gamma\left(a\xi^2+\frac{\nu^2}{4a}\right)}{\gamma+a\xi^2+\frac{\nu^2}{4a}}}, \gamma=\frac{1}{1-\alpha}. \tag{3.10}$$

Notice that s denotes the Laplace transform variable and ξ is the Fourier transform variable in equation (3.11). Next, by taking into account the Mittag-Leffler function formula [26]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha+b}\right\}=E_\alpha\left(-bt^\alpha\right), \tag{3.11}$$

and by applying inverse transforms of Laplace and cosine-Fourier, we arrive

$$u(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\gamma}{\gamma+a\xi^2+\frac{\nu^2}{4a}}E_\alpha\left(-\frac{\alpha\gamma\left(a\xi^2+\frac{\nu^2}{4a}\right)}{\gamma+a\xi^2+\frac{\nu^2}{4a}}t^\alpha\right)\cos(x\xi)\mathrm{d}\xi. \tag{3.12}$$

Returning to the $c(x,t)$ according to equation (3.5), we obtain the fundamental solution

$$c(x,t)=\frac{2}{\pi}\exp\left(\frac{\nu x}{2a}\right)\int_0^\infty \frac{\gamma}{\gamma+a\xi^2+\frac{\nu^2}{4a}}E_\alpha\left(-\frac{\alpha\gamma\left(a\xi^2+\frac{\nu^2}{4a}\right)}{\gamma+a\xi^2+\frac{\nu^2}{4a}}t^\alpha\right)\cos(x\xi)\mathrm{d}\xi. \tag{3.13}$$

Finally, we take limit case $\alpha=1$ and $\gamma\to\infty$ and then obtain the fundamental solution of classical ADE

$$c(x,t)=\frac{2}{\pi}\exp\left(\frac{\nu x}{2a}\right)\int_0^\infty \exp\left(-\left(a\xi^2+\frac{\nu^2}{4a}\right)t\right)\cos(x\xi)\mathrm{d}\xi. \tag{3.14}$$

In Figure 1a, we analyze the dependence of 1st type Cauchy solutions on the fractional order of the AB derivative. Therefore, we fix the time and velocity parameters and obtain the results with respect to the position of diffusive transport. We see that the diffusion curves obey the sub-diffusion phenomena while the order is changing from 1 to 0. This kind of behavior can be observed in the diffusion models with Caputo fractional derivative. It means that AB derivative can successfully characterize the sub/super structures in the diffusion process. It is a significant result to prefer this non-singular fractional operator to model diffusion phenomena. We research the effect of variation of velocity parameter on the diffusion curves by fixing the fractional order $\alpha=0.5$ in Figure 1b. However, it should be noted that we have taken these problem parameters only as mathematical quantities. But of course, according to the real data coming from the experimental studies, these results have more realistic physical meanings.

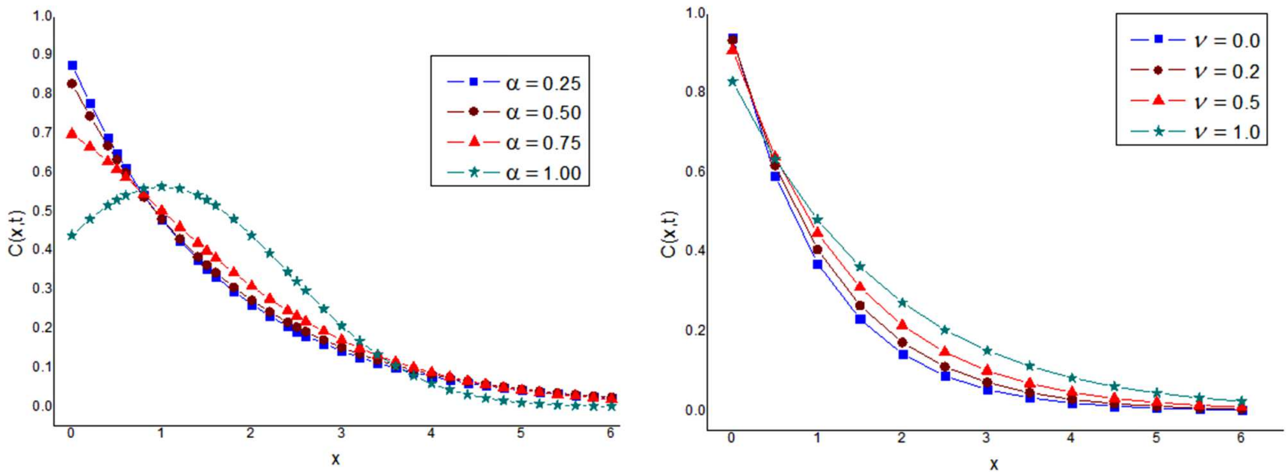

FIGURE 1. (a) The diffusion profiles corresponding to Problem 3.1.1 for different values of α: $a = t = \nu = 1$. (b) The influence of velocity parameter ν on the fundamental solution to the problem 1: $\alpha = 0.5$, $a = t = 1$.

3.1.2. Problem 2

In this case, we modify the Problem 3.1.1 as a similar manner:

$$^{ABC}D_t^{\alpha}c(x,t) = a\frac{\partial^2 c(x,t)}{\partial x^2} - \nu\frac{\partial c(x,t)}{\partial x}, \tag{3.15}$$

$$0 < \alpha \leq 1, 0 < x < \infty, 0 < t < \infty, a > 0, \nu > 0,$$

with the following modified initial condition

$$t = 0 : c(x,t) = \delta(x - \zeta),\ 0 < \zeta < \infty, \tag{3.16}$$

and the zero boundary conditions

$$x = 0 : c(x,t) = 0, \tag{3.17}$$

$$\lim_{x\to\infty} c(x,t) = 0. \tag{3.18}$$

We assume that

$$c(x,t) = \exp\left(\frac{\nu x}{2a}\right)u(x,t). \tag{3.19}$$

and so the Problem 3.1.2 reduces to

$$^{ABC}D_t^{\alpha}u(x,t) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\nu^2}{4a}u(x,t), \tag{3.20}$$

$$u(x,0) = \exp\left(-\frac{\upsilon\zeta}{2a}\right)\delta(x-\zeta), \tag{3.21}$$

$$u(0,t) = 0, \tag{3.22}$$

$$\lim_{x\to\infty} u(x,t) = 0. \tag{3.23}$$

Applying the Laplace and sine-Fourier transforms to the modified problem leads to

$$\overline{u}^*(\xi,s) = \exp\left(-\frac{v\zeta}{2a}\right)\sin(\zeta\xi)\,\frac{\gamma}{\gamma + a\xi^2 + \frac{\nu^2}{4a}}\,\frac{s^{\alpha-1}}{s^\alpha + \frac{\alpha\gamma\left(a\xi^2+\frac{\nu^2}{4a}\right)}{\gamma + a\xi^2+\frac{\nu^2}{4a}}}, \qquad \gamma = \frac{1}{1-\alpha}. \tag{3.24}$$

After taking the inverse transforms, we get

$$u(x,t) = \frac{2}{\pi}\exp\left(-\frac{v\zeta}{2a}\right)\int_0^\infty \frac{\gamma}{\gamma + a\xi^2 + \frac{\nu^2}{4a}} E_\alpha\left(-\frac{\alpha\gamma\left(a\xi^2 + \frac{\nu^2}{4a}\right)}{\gamma + a\xi^2 + \frac{\nu^2}{4a}} t^\alpha\right)\sin(\zeta\xi)\sin(x\xi)\mathrm{d}\xi. \tag{3.25}$$

Substituting equation (3.25) into equation (3.19), we have the fundamental solution

$$c(x,t) = \frac{2}{\pi}\exp\left(\frac{\nu}{2a}(x-\zeta)\right)\int_0^\infty \frac{\gamma}{\gamma + a\xi^2 + \frac{\nu^2}{4a}} E_\alpha\left(-\frac{\alpha\gamma\left(a\xi^2 + \frac{\nu^2}{4a}\right)}{\gamma + a\xi^2 + \frac{\nu^2}{4a}} t^\alpha\right)\sin(\zeta\xi)\sin(x\xi)\mathrm{d}\xi. \tag{3.26}$$

We obtain the classical solution in the limit case for $\alpha = 1$ and $\gamma \to \infty$

$$c(x,t) = \frac{2}{\pi}\exp\left(\frac{\nu}{2a}(x-\zeta)\right)\int_0^\infty \exp\left(-\left(a\xi^2 + \frac{\nu^2}{4a}\right)t\right)\sin(\zeta\xi)\sin(x\xi)\mathrm{d}\xi. \tag{3.27}$$

Similar to 1st type Cauchy problem, we plot the diffusion curves for changing values of α in Figure 2a. We observe that the maximum values of the concentration values have been smaller than the previous problem. In addition, depending on the type of the problem there is sharpness around the position $x = 1$ which is also different from the 1st problem. On the other hand, the dependency of solutions on velocity parameter acts like to the 1st problem as seen in Figure 2b.

3.2. Fundamental solution to the Dirichlet problem

In this case, we research the Dirichlet problem corresponding to ADE with AB derivative:

$${}^{ABC}D_t^\alpha c(x,t) = a\frac{\partial^2 c(x,t)}{\partial x^2} - \nu\frac{\partial c(x,t)}{\partial x}, \tag{3.28}$$

$$0 < x < \infty, 0 < t < \infty, 0 < \alpha \le 1, a > 0, \nu > 0$$

under the zero initial condition

$$t = 0: \quad c(x,0) = 0, \tag{3.29}$$

the Dirichlet boundary condition

$$x = 0: c(x,t) = \delta(t) \tag{3.30}$$

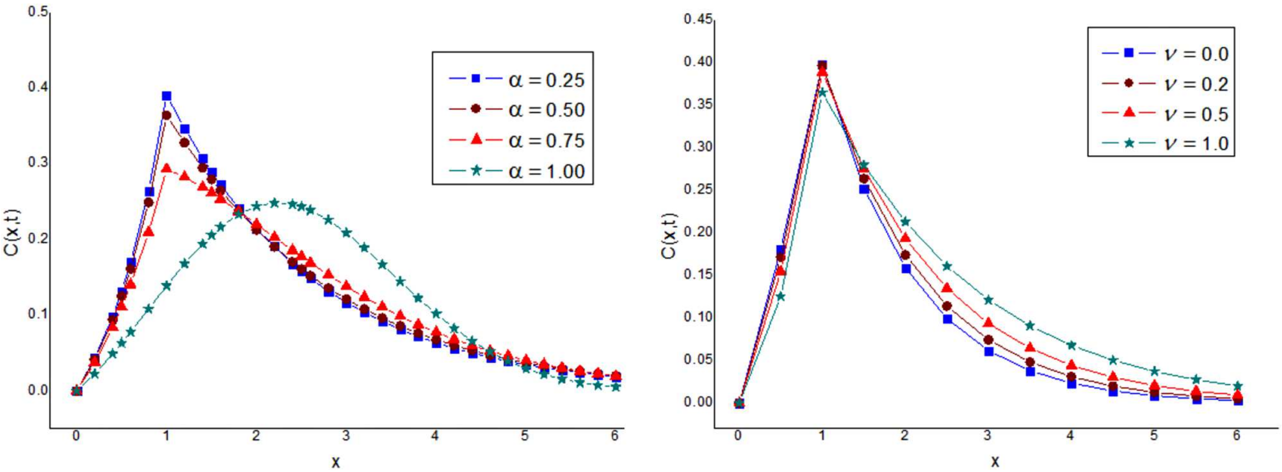

FIGURE 2. (a) The diffusion profiles corresponding to Problem 3.1.1 for different values of α: $a = t = \nu = 1$. (b) The influence of velocity parameter ν on the fundamental solution to the problem 2: $\alpha = 0.5$, $a = t = 1$.

and the zero condition at infinity

$$\lim_{x\to\infty} c(x,t) = 0. \tag{3.31}$$

Using an auxiliary function, we assume

$$c(x,t) = \exp\left(\frac{vx}{2a}\right) u(x,t). \tag{3.32}$$

Therefore, Dirichlet problem is simplified as follows

$${}^{ABC}D_t^{\alpha} u(x,t) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\nu^2}{4a}u(x,t), \tag{3.33}$$

with the initial condition

$$u(x,0) = 0, \tag{3.34}$$

and the boundary conditions

$$u(0,t) = \delta(t), \tag{3.35}$$

$$\lim_{x\to\infty} u(x,t) = 0. \tag{3.36}$$

Taking the relevant integral transforms

$$\overline{u}^*(\xi,s) = a\xi\left(\frac{1}{\gamma + a\xi^2 + \frac{\nu^2}{4a}} + \frac{\alpha\gamma^2}{\left(\gamma + a\xi^2 + \frac{\nu^2}{4a}\right)^2}\frac{1}{s^\alpha + \frac{\alpha\gamma\left(a\xi^2+\frac{\nu^2}{4a}\right)}{\gamma+a\xi^2+\frac{\nu^2}{4a}}}\right) \tag{3.37}$$

and inverting the calculations, we get

$$u(x,t)=\frac{2}{\pi}a\int\limits_0^\infty \xi\left(\frac{\delta(t)}{\gamma+a\xi^2+\frac{\nu^2}{4a}}+\frac{\alpha\gamma^2}{\left(\gamma+a\xi^2+\frac{\nu^2}{4a}\right)^2}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\alpha\gamma\left(a\xi^2+\frac{\nu^2}{4a}\right)}{\gamma+a\xi^2+\frac{\nu^2}{4a}}t^\alpha\right)\right)\sin(x\xi)\mathrm{d}\xi. \tag{3.38}$$

Let us return to $c(x,t)$, by using equations (3.38) and (3.32), we arrive the following fundamental solution

$$c(x,t)=\frac{2}{\pi}a\exp\left(\frac{vx}{2a}\right)\int\limits_0^\infty \xi\left(\frac{\delta(t)}{\gamma+a\xi^2+\frac{v^2}{4a}}+\frac{\alpha\gamma^2}{\left(\gamma+a\xi^2+\frac{\nu^2}{4a}\right)^2}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\alpha\gamma\left(a\xi^2+\frac{\nu^2}{4a}\right)}{\gamma+a\xi^2+\frac{\nu^2}{4a}}t^\alpha\right)\right)\sin(x\xi)\mathrm{d}\xi \tag{3.39}$$

by taking into account the Mittag-Leffler function formula [26]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha+b}\right\}=t^{\beta-1}E_{\alpha,\beta}\left(-bt^\alpha\right). \tag{3.40}$$

We can remark that the first term in equation (3.39) equals to 0 for $t>0$. Such a similar mathematical result has been reached for the fundamental solutions of ADE with CF derivative in [23]. The results obtained for the present ADE model with AB derivative in here and also with the CF derivative in [23] differ from existing results arising from the ADE with Caputo derivative with the first term in equation (3.39). From mathematical point of view, this is noteworthy.

In the classical sense for $\alpha=1$, *i.e.* $\gamma\to\infty$ for $t>0$, we obtain

$$c(x,t)=\frac{2}{\pi}a\exp\left(\frac{vx}{2a}\right)\int\limits_0^\infty \xi\sin\left(x\xi\right)\exp\left(-\left(a\xi^2+\frac{\nu^2}{4a}\right)t\right)\mathrm{d}\xi. \tag{3.41}$$

The obtained solutions corresponding to the Dirichlet problem are illustrated in Figure 3. It is remarkable that the results are meaningfully different from the Cauchy solutions. For example, the maximum value of concentration arises for the fractional value $\alpha=0.75$ instead of $\alpha=0.25$. In here, we only illustrate the dependency of Dirichlet solutions on the variation of fractional order. But, of course, the effect of velocity parameter can be visualized similar to the Cauchy problems.

3.3. Fundamental solution to the source problem

Finally, let us consider ADE with AB derivative with source term as follows

$$^{ABC}D_t^\alpha c(x,t)=a\frac{\partial^2 c(x,t)}{\partial x^2}-\nu\frac{\partial c(x,t)}{\partial x}+c_0\delta(x)\delta(t), \tag{3.42}$$

$$0<x<\infty,\ 0<t<\infty,\ 0<\alpha\le 1,$$

under the zero initial condition, Neumann boundary condition and also the zero condition at infinity:

$$t=0: c(x,0)=0 \tag{3.43}$$

$$x=0: c(x,t)=0,\frac{\partial c(x,t)}{\partial x}=0, \tag{3.44}$$

$$\lim_{x\to\infty}c(x,t)=0. \tag{3.45}$$

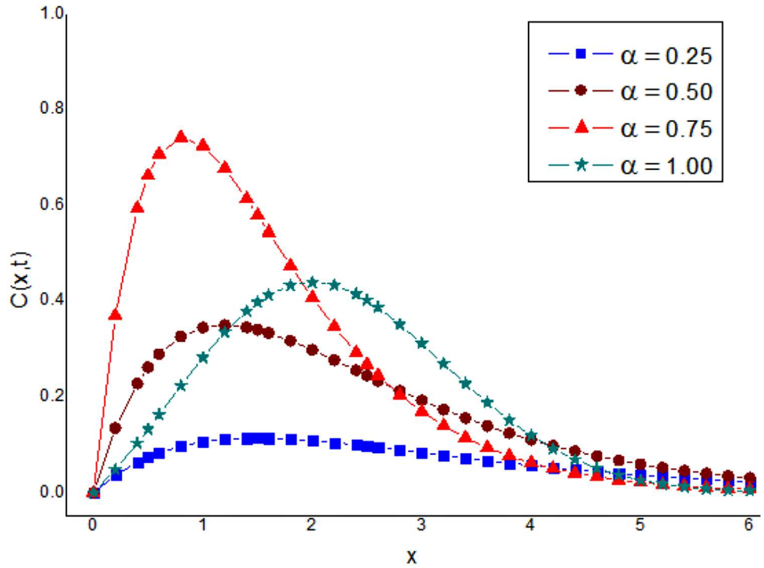

FIGURE 3. The diffusion profiles corresponding to the Dirichlet problem for different values of α: $a = t = \nu = 1$.

By the assumption

$$c(x,t) = \exp\left(\frac{vx}{2a}\right) u(x,t), \tag{3.46}$$

the source problem defined by equations (3.42)–(3.45) reduces to

$$^{ABC}D_t^{\alpha} u(x,t) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{v^2}{4a}u(x,t) + c_0\delta(x)\delta(t), \tag{3.47}$$

$$t = 0 : u(x,0) = 0, \tag{3.48}$$

$$x = 0 : u(x,t) = \frac{\partial u(x,t)}{\partial x} = 0, \tag{3.49}$$

$$\lim_{x\to\infty} u(x,t) = 0. \tag{3.50}$$

As similar to the Cauchy and Dirichlet formulations, we apply the integrals transforms and obtain

$$\widetilde{u}^*(\xi,s) = c_0\left(\frac{1}{\gamma + a\xi^2 + \frac{v^2}{4a}} + \frac{\alpha\gamma^2}{\left(\gamma + a\xi^2 + \frac{v^2}{4a}\right)^2}\frac{1}{s^{\alpha} + \frac{\alpha\gamma\left(a\xi^2 + \frac{v^2}{4a}\right)}{\gamma + a\xi^2 + \frac{v^2}{4a}}}\right). \tag{3.51}$$

Then by inverting the transforms, we get

$$u(x,t) = c_0\int_0^{\infty}\left(\frac{\delta(t)}{\gamma + a\xi^2 + \frac{v^2}{4a}} + \frac{\alpha\gamma^2}{\left(\gamma + a\xi^2 + \frac{v^2}{4a}\right)^2}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\alpha\gamma\left(a\xi^2 + \frac{v^2}{4a}\right)}{\gamma + a\xi^2 + \frac{v^2}{4a}}t^{\alpha}\right)\right)\cos(x\xi)\mathrm{d}\xi. \tag{3.52}$$

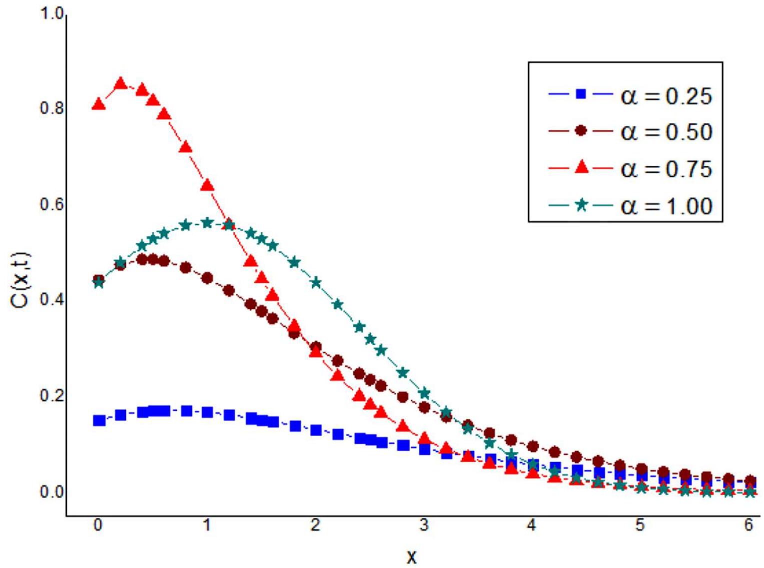

FIGURE 4. The diffusion profiles corresponding to the source problem for different values of α: $a = t = \nu = c_0 = 1$.

By substituting equation (3.52) into the equation (3.46), we obtain the fundamental solution $c(x,t)$ as

$$c(x,t) = \frac{2}{\pi} c_0 \exp\left(\frac{vx}{2a}\right) \int\limits_0^\infty \left(\frac{\delta(t)}{\gamma + a\xi^2 + \frac{v^2}{4a}} + \frac{\alpha\gamma^2}{\left(\gamma + a\xi^2 + \frac{v^2}{4a}\right)^2} t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\alpha\gamma\left(a\xi^2 + \frac{v^2}{4a}\right)}{\gamma + a\xi^2 + \frac{v^2}{4a}} t^\alpha \right) \right) \cos(x\xi) \mathrm{d}\xi. \tag{3.53}$$

Let us remind that the first term in the equation (3.53) equals to 0 for $t > 0$. For only computational purposes, we rewrite the result by removing this term as follows

$$c(x,t) = \frac{2}{\pi} c_0 \exp\left(\frac{vx}{2a}\right) t^{\alpha-1} \int\limits_0^\infty \frac{\alpha\gamma^2}{\left(\gamma + a\xi^2 + \frac{v^2}{4a}\right)^2} E_{\alpha,\alpha}\left(-\frac{\alpha\gamma\left(a\xi^2 + \frac{v^2}{4a}\right)}{\gamma + a\xi^2 + \frac{v^2}{4a}} t^\alpha \right) \cos(x\xi) \mathrm{d}\xi, \tag{3.54}$$

and the result for the classical ADE can be obtained in the limit case $\alpha = 1$ ($\gamma \to \infty$)

$$c(x,t) = \frac{2}{\pi} c_0 \exp\left(\frac{vx}{2a}\right) \int\limits_0^\infty \exp\left(-\left(a\xi^2 + \frac{v^2}{4a}\right) t \right) \cos(x\xi) \mathrm{d}\xi. \tag{3.55}$$

Finally, we give a comparison of solutions with respect to the source problem in Figure 4 for changing values of fractional parameter. The results behave like Dirichlet case. As similar to the previous problems, we assume the other problem parameters equal to 1 only for convenience.

4. CONCLUDING REMARKS

A linear ADE in terms of AB derivative with non-singular Mittag-Leffler kernel has been studied. The Cauchy, Dirichlet and source problems are formulated on the half real line. It is well known that the boundary properties of the domain, on which initial-boundary value problems are constructed, are significative for applying integral transform techniques to find the analytical solutions. Under this reality, the Laplace and sine/cosine Fourier transforms have been applied to obtain the fundamental solutions of prescribed problems. The results have been represented by the linear combinations of trigonometric and Mittag-Leffler functions. The effect of the fractional order α on the diffusion profiles has been illustrated by the graphics. For this purpose, Maple Software program has been performed. It has also been shown that the velocity coefficient ν leads to a drift on the diffusion along the x-axis. This study shows that AB derivative is a preferable alternative because of its non-singular kernel function to the Caputo fractional derivative for the modeling of diffusive transports represented by the present formulation.

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