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TRIGONOMETRIC APPROXIMATION IN VARIABLE EXPONENT WEIGHTED LEBESGUE SPACES USING SUB-MATRIX METHOD

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Abstract. In this study, we obtain some results related to trigonometric approximation using matrix submethods of Fourier series of functions in the variable exponent weighted Lebesgue spaces. The estimations of our main results are sharper than those of the results obtained in [14]. Furthermore, we evaluate the approximation using a different matrix method.

1. Introduction

Variable exponent spaces are used for applications in harmonic analysis, partial differential equations, potential theory, image restoration, fluid mechanics and applied mathematics [5, 10, 13, 16, 22, 28, 29]. One of the most important of these spaces is variable exponent Lebesgue space. The theory of variable exponent Lebesgue spaces dates back to study given by W. Orlicz in [26]. Later, in weighted or non-weighted Lebesgue spaces with variable exponent, many authors obtain some results [1, 2, 3, 12, 14, 15, 19, 20, 21, 30, 31, 32, 33, 35].

Let $\mathbf{T} := [0, 2\pi]$ and $p(\cdot) : \mathbf{T} \to [0, \infty)$ be a Lebesgue measurable 2π-periodic function. The set of functions f Lebesgue measurable 2π-periodic satisfying the condition

$$\rho_{p(\cdot)}(f) := \int\limits_{\mathbf{T}} |f(x)|^{p(x)}\, dx < \infty$$

is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\xi > 0 : \rho_{p(\cdot)}\left(\frac{f}{\xi}\right) \le 1\right\}.$$

This space is called variable exponent Lebesgue space and denote $L^{p(\cdot)}(\mathbf{T})$. We suppose that the exponent functions $p(\cdot)$ satisfy the following conditions

$$1 < p_- := ess\inf_{x\in\mathbf{T}} p(x) \le ess\sup_{x\in\mathbf{T}} p(x) := p^+ < \infty, \tag{1.1}$$

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and

$$|p\left(x\right)-p\left(y\right)|\ln\left(\frac{1}{|x-y|}\right)\leq c,\ x,y\in\mathbf{T},\ 0<|x-y|\leq\frac{1}{2}. \tag{1.2}$$

By $\wp_0\left(\mathbf{T}\right)$, we denote the class of exponents $p\left(\cdot\right)$ satisfying conditions (1.1) and (1.2).

Detailed information about variable exponent Lebesgue spaces can be found in [9, 16].

A measurable 2π-periodic function $\omega:\mathbf{T}\to[0,\infty]$ is called a weight function if the set $\omega^{-1}(\{0,\infty\})$ has the Lebesgue measure zero.

The variable exponent weighted Lebesgue spaces $L_{\omega}^{p(\cdot)}(\mathbf{T})$ is defined as the set of all measurable 2π periodic functions f for which $f\omega\in L^{p(\cdot)}(\mathbf{T})$ and $\|f\|_{p(\cdot),\omega}:=\|\omega f\|_{p(\cdot)}<\infty$.

The weight functions ω used in this paper belong to the Muckenhoupt class $A_{p(\cdot)}(\mathbf{T})$ which is defined by

$$\underset{I}{sup}\,|I|^{-1}\left\|\omega\chi_I\right\|_{p(\cdot)}\left\|\omega^{-1}\chi_I\right\|_{q(\cdot)}<\infty,\ \frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}=1$$

where the supremum is taken with respect to all the Lebesgue measurable intervals $I\subset\mathbb{R}$ with χ_I is the characteristic function. $|I|$ denotes Lebesgue measure of I.

We can define the Fourier series of $f\in L_{\omega}^{p(\cdot)}(\mathbf{T})$

$$f\left(x\right)\backsim\frac{a_0\left(f\right)}{2}+\sum_{k=1}^{\infty}\left(a_k\left(f\right)\cos kx+b_k\left(f\right)\sin kx\right). \tag{1.3}$$

Here $a_0\left(f\right),a_k\left(f\right),b_k\left(f\right)$, $k=1,...,$ are Fourier coefficients of f.

Let $S_n(x,f)$, $(n=0,1,2,...)$ be the nth partial sums of the series (1.3) at the point x, that is,

$$S_n(f,x)=\sum_{k=0}^{n}U_k(f)(x),$$

where

$$U_0(f)(x):=\frac{a_0(f)}{2},$$

$$U_k(f)(x):=a_k(f)\cos kx+b_k(f)\sin kx,\quad k=1,2,....$$

By $E_n(f)_{L_{\omega}^{p(\cdot)}}$, we denote the best approximation of $f\in L_{\omega}^{p(\cdot)}(\mathbf{T})$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L_{\omega}^{p(\cdot)}}=\inf\left\|f-T_k\right\|_{p(\cdot),\omega},$$

where the infimum is taken with respect to all trigonometric polynomials of degree $k\leq n$.

The Hardy Littlewood maximal function of f is defined as

$$Mf\left(x\right):=\sup_{x\in I}\frac{1}{|I|}\int_{I}\left|f\left(t\right)\right|dt,\ x\in\mathbf{T},$$

where the supremum is taken over all open intervals $I\subset\mathbf{T}$, such that $x\in I$.

The boundedness of the maximal operator M in variable exponent weighted Lebesgue space $L_{\omega}^{p(\cdot)}$ was established by the authors in [19], see also [21].

Thus, we note that for a positive constant $c(p)$, the following inequality

$$\|M(f)\|_{p(\cdot),\omega} \leq c(p)\,\|f\|_{p(\cdot),\omega} \tag{1.4}$$

holds.

Let $p(\cdot) \in \wp_0(\mathbf{T})$ and $\omega \in A_{p(\cdot)}(\mathbf{T})$. The modulus of smoothness of the function $f \in L_\omega^{p(\cdot)}(\mathbf{T})$ is defined as

$$\Omega(f,\delta)_{L_\omega^{p(\cdot)}} = \sup_{|h|\leq\delta} \|A_h f\|_{p(\cdot),\omega}, \qquad \delta > 0,$$

where

$$(A_h f)(x) := \frac{1}{h}\int_0^h |f(x+t) - f(x)|\,dt.$$

$\Omega(f,\delta)_{L_\omega^{p(\cdot)}}$ is well defined because of the inequality $\Omega(f,\delta)_{L_\omega^{p(\cdot)}} \leq c(p)\,\|f\|_{p(\cdot),\omega}$, by the estimation (1.4) . Furthermore, one can see that for $f_1, f_2 \in L_\omega^{p(\cdot)}(\mathbf{T})$

$$\lim_{\delta\to 0}\Omega(f,\delta)_{L_\omega^{p(\cdot)}} = 0, \quad \Omega(f_1+f_2,\delta)_{L_\omega^{p(\cdot)}} \leq \Omega(f_1,\delta)_{L_\omega^{p(\cdot)}} + \Omega(f_2,\delta)_{L_\omega^{p(\cdot)}}.$$

For $0 < \alpha \leq 1$, the variable exponent Lipschitz class $Lip(\alpha, p(\cdot), \omega)$ is defined as follow

$$Lip(\alpha,p(\cdot),\omega) = \left\{f \in L_\omega^{p(\cdot)}(\mathbf{T}) : \Omega(f,\delta)_{L_\omega^{p(\cdot)}} = O(\delta^\alpha),\ \delta > 0\right\}.$$

Let $(\lambda_n)_{n=1}^\infty$ be a strictly increasing sequence of positive integers. For a sequence (x_k) of the real or complex numbers, the sub-Cesàro method C_λ is defined by

$$(C_\lambda x)_n := \frac{1}{\lambda_n}\sum_{k=1}^{\lambda_n} x_k,\ (\lambda_n = 1, 2, ...).$$

Particularly, when $\lambda_n = n$ we note that $(C_\lambda x)_n$ is the classical Cesàro method C_1. Therefore, the sub-Cesàro method C_λ yields a subsequence of the Cesàro method C_1. The basic properties of the method C_λ were investigated firstly by Armitage and Maddox in [4] and Osikiewicz [27]. In these works, the relations between the classical Cesàro method and sub-Cesàro method were obtained. Further information about the method C_λ can be found in [4, 27].

We denote by $A \equiv (a_{n,k})$ a lower triangular regular matrix with nonnegative entries and let $S_n^{(A)}$ $(n = 0, 1, ...)$ be the row sums of this matrix, that is

$$S_n^{(A)} = \sum_{k=0}^n a_{n,k}.$$

The sub-matrix means of Fourier series are defined as

$$\tau_n^\lambda(f,x) := \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} S_k(f,x),\ \lambda_n = 0, 1, 2, ... \tag{1.5}$$

and

$$T_n^\lambda(f;x) = \sum_{k=0}^{\lambda_n} a_{\lambda_n,\lambda_n-k} S_k(f;x),\ \lambda_n = 0, 1, 2, ... \tag{1.6}$$

When $\lambda_n = n$, the method (1.5) turns into matrix method $\tau_n(f,x)$ defined by

$$\tau_n(f,x) := \sum_{k=0}^{n} a_{n,k} S_k(f,x), \ n = 0,1,2,... \tag{1.7}$$

and the method (1.6) turns into matrix method $T_n(f;x)$ defined by

$$T_n(f;x) = \sum_{k=0}^{n} a_{n,n-k} S_k(f;x), \text{ for all } n = 0,1,2,...$$

respectively.

When $a_{\lambda_n,\lambda_n-m} = \frac{p_{\lambda_n-m}}{P_{\lambda_n}}$, the method T_n^{λ} turns into sub-Nörlund method given as

$$N_n^{\lambda}(f;x) := \frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n} p_{\lambda_n-m} S_m(f;x)$$

and when $a_{\lambda_n,m} = \frac{p_m}{P_{\lambda_n}}$, the method τ_n^{λ} turns into sub-Riesz method given as

$$R_n^{\lambda}(f;x) := \frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n} p_m S_m(f;x)$$

where $P_{\lambda_n} = p_0+p_1+p_2+...+p_{\lambda_n} \neq 0 \quad (\lambda_n \geqslant 0)$ and by convention $p_{-1} = P_{-1} = 0$.

Also, in the case $p_{\lambda_n} = 1$, $\lambda_n \geq 0$, $\lambda_n = n$, both of $N_n^{\lambda}(f)(x)$ and $R_n^{\lambda}(f)(x)$ are equal to the Cesàro method

$$\sigma_n(f)(x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f;x).$$

A nonnegative sequence $u := (u_n)$ is called almost monotone decreasing (increasing), if there exists a constant $K := K(u)$, depending on the sequence u only, such that $u_n \leq K u_m \ \ (K u_n \geq u_m)$ for all $n \geq m$. Such sequences will be denoted by $u \in AMDS \ (u \in AMIS)$.

Let

$$A_{\lambda_n,k} := \frac{1}{k+1} \sum_{i=1}^{k} a_{\lambda_n,i}.$$

If $\{A_{\lambda_n,k}\} \in AMDS \ (\{A_{\lambda_n,k}\} \in AMIS)$, then we will say that $\{a_{\lambda_n,k}\}$ is an almost monotone decreasing (increasing) mean sequence with respect to $k = 1,2,...,\lambda_n$ for all λ_n. Briefly, we will write $\{a_{\lambda_n,k}\} \in AMDMS \ (\{a_{\lambda_n,k}\} \in AMIMS)$. For detailed information (in the case of $\lambda_n = n$) can be found in [25].

If we choose $\lambda_n = n^2$, then the sequence $(P_{\lambda_n}) = \left(\lambda_n e^{(-1)^{\lambda_n}}\right)$ belongs to $AMIMS$ and the sequence $(K_{\lambda_n}) = \left(\frac{1}{\lambda_n} e^{(-1)^{\lambda_n}}\right)$ belongs to $AMDMS$.

We will use sums up to λ_n in S_n and σ_n and write these sums S_n^{λ} and σ_n^{λ} (or S_{λ_n} and σ_{λ_n}), respectively.

The operator Δ_k is defined by $\Delta_k a_{\lambda_n,k} = a_{\lambda_n,k} - a_{\lambda_n,k+1}$.

The relation $\preceq$ is defined as "$A \preceq B \Leftrightarrow$there exists a positive constant C, independent of essential parameters, such that $A \leq CB$".

We set $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$.

2. Historical Background

The basic properties of the sub-Cesàro method C_λ were investigated firstly by Armitage and Maddox in [4] and Osikiewicz [27].

Later, some researchers obtained results about trigonometric approximation using different methods in wider spaces than Lebesgue spaces in [6, 7, 8, 11, 12, 17, 18, 23, 24, 34]

Lebesgue space may be generalized in different ways. One of the important generalizations of this space is variable exponent weighted Lebesgue space. In the variable exponent Lebesgue spaces (weighted or non-weighted), some researchers obtained results about trigonometric approximation using different methods [8, 11, 12, 14, 17, 18].

But in these papers degree of approximation using matrix methods obtained by sub-Cesàro method was not investigated in the variable exponent weighted Lebesgue spaces. In [14], the approximation properties of the matrix method T_n of partial sums of Fourier series of functions in the weighted variable exponent Lebesgue spaces were investigated.

In this study, we examine trigonometric approximation by the matrix methods $T_n^\lambda(f;x)$ and $\tau_n^\lambda(f;x)$ to the functions in variable exponent weighted Lebesgue spaces with the degree of $O\left(\lambda_n^{-\alpha}\right)$ $(0<\alpha\leq 1)$. Also, our main theorems give the error of estimation in terms of $\lambda_n^{-\alpha}$ $(0<\alpha\leq 1)$ which is sharper than the results given in [14], because $\lambda_n^{-\alpha}\leq n^{-\alpha}$ for $0<\alpha\leq 1$. We obtain this estimation using sub-Cesàro method instead of classical Cesàro method.

Note that, in [14] the approximation by $\tau_n^\lambda(f;x)$ method was not examined.

3. Auxiliary Results

To achieve the aim, which we mentioned above, we need some helpful lemmas given below.

Lemma 3.1. [14] *Let* $f\in L_\omega^{p(\cdot)}(\mathbf{T})$ *and* $p(\cdot)\in\wp_0(\mathbf{T})$. *If* $\omega(\cdot)\in A_{p(\cdot)}(\mathbf{T})$, *then there exists a positive constant* $c(p)$ *such that the following inequality holds:*

$$\|S_n(f)\|_{p(\cdot),\omega}\leq c(p)\|f\|_{p(\cdot),\omega},\ n=1,2,\dots .$$

Lemma 3.2. [14] *Let* $p(\cdot)\in\wp_0(\mathbf{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbf{T})$ *and* $0<\alpha\leq 1$. *If* $f\in Lip(\alpha,p(\cdot),\omega)$, *then*

$$\|f-S_n(f)\|_{p(\cdot),\omega}=O(n^{-\alpha}),\ n=1,2,\dots .$$

Lemma 3.3. [14] *Let* $p(\cdot)\in\wp_0(\mathbf{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbf{T})$. *If* $f\in Lip(1,p(\cdot),\omega)$, *then*

$$\|S_n(f)-\sigma_n(f)\|_{p(\cdot),\omega}=O(\frac{1}{n}),\ n=1,2,\dots .$$

Lemma 3.4. [23] *Let* $T\equiv(a_{\lambda_n,k})$ *be an infinite lower triangular regular matrix with nonnegative entries and row sums* 1. *If either*

$$(i)\quad (a_{\lambda_n,k})\in AMIMS$$

or

$$(ii)\quad (a_{\lambda_n,k})\in AMDMS \text{ and } (\lambda_n+1)a_{\lambda_n,0}=O(1),$$

then, for $0<\alpha<1$,

$$\sum_{k=0}^{\lambda_n}a_{\lambda_n,k}(k+1)^{-\alpha}=O\left(\lambda_n^{-\alpha}\right).$$

Lemma 3.5. *Let* $A = (a_{\lambda_n,k})$ *be an infinite lower triangular matrix and let* $0 < \alpha < 1$. *If one of the following conditions:*

$$(i)\ \ (a_{\lambda_n,k}) \in AMDS \text{ and } (\lambda_n + 1)a_{\lambda_n,0} = O(1),$$

$$(ii)\ \ (a_{\lambda_n,k}) \in AMIS \text{ and } (\lambda_n + 1)a_{\lambda_n,r} = O(1),$$

where r *integer part of* $\frac{\lambda_n}{2}$ *and* $|S^{(A)}_{\lambda_n} - 1| = O({\lambda_n}^{-\alpha})$ *holds, then*

$$\sum_{k=1}^{\lambda_n} k^{-\alpha} a_{\lambda_n,k} = O(\lambda_n^{-\alpha}).$$

Proof of Lemma 3.5

(i) Since $\sum\limits_{k=1}^{\lambda_n} k^{-\alpha} = O(\lambda_n^{1-\alpha})$ and $a_{\lambda_n,k} \le K a_{\lambda_n,0}$ for $k = 1, ..., \lambda_n$, we get

$$\begin{aligned} \sum_{k=1}^{\lambda_n} k^{-\alpha} a_{\lambda_n,k} &\le K a_{\lambda_n,0} \sum_{k=1}^{\lambda_n} k^{-\alpha} \\ &= O(\frac{1}{\lambda_n + 1}) O(\lambda_n^{1-\alpha}) \\ &= O(\lambda_n^{-\alpha}). \end{aligned}$$

(ii) Since $a_{\lambda_n,k} \le K a_{\lambda_n,r}$ for $k = 1, ..., r$ and $|S^{(A)}_{\lambda_n} - 1| = O(\lambda_n^{-\alpha})$,

$$\begin{aligned} \sum_{k=1}^{\lambda_n} k^{-\alpha} a_{\lambda_n,k} &= \sum_{k=1}^{r} k^{-\alpha} a_{\lambda_n,k} + \sum_{k=r+1}^{\lambda_n} k^{-\alpha} a_{\lambda_n,k} \\ &\le K a_{\lambda_n,r} \sum_{k=1}^{r} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=r+1}^{\lambda_n} a_{\lambda_n,k} \\ &\le K a_{\lambda_n,r} \sum_{k=1}^{\lambda_n} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} \\ &= O\left(\frac{1}{\lambda_n + 1}\right) O(\lambda_n^{1-\alpha}) + O(\lambda_n^{-\alpha}) S^{(A)}_{\lambda_n} \\ &= O\left(\lambda_n^{-\alpha}\right). \end{aligned}$$

Lemma 3.6. *For every* $k = 1, 2, ..., \lambda_n$ *the relation*

$$\left| \sum_{m=0}^{k} a_{\lambda_n,m} - (k+1) a_{\lambda_n,k} \right| \le \sum_{m=1}^{k} m \left| a_{\lambda_n,m-1} - a_{\lambda_n,m} \right|$$

holds.

Proof of Lemma 3.6 If $k = 1$, then

$$\left| \sum_{m=0}^{1} a_{\lambda_n,m} - 2a_{\lambda_n,1} \right| = \left| a_{\lambda_n,0} - a_{\lambda_n,1} \right|.$$

We assume that the inequality is true for $k=v$. then for $k=v+1$ we get

$$\begin{aligned}\left|\sum_{m=0}^{v+1} a_{\lambda_n,m}-(v+2)a_{\lambda_n,v+1}\right| &\leq \left|\sum_{m=0}^{v} a_{\lambda_n,m}-(v+1)a_{\lambda_n,v}\right|+\left|(v+1)a_{\lambda_n,v}-(v+1)a_{\lambda_n,v+1}\right| \\ &\leq \sum_{m=1}^{v} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|+(v+1)\left|a_{\lambda_n,v}-a_{\lambda_n,v+1}\right| \\ &= \sum_{m=1}^{v+1} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|.\end{aligned}$$

4. Main Results

Theorem 4.1. *Let* $f\in Lip\left(\alpha,p\left(\cdot\right),\omega\right)$, $0<\alpha<1$, $p\left(\cdot\right)\in\wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right)\in A_{p(\cdot)}(\mathbf{T})$ *and* $A=(a_{\lambda_n,k})$ *be a lower triangular matrix with* $|S_{\lambda_n}^{(A)}-1|=O({\lambda_n}^{-\alpha})$. *If one of the following conditions*

$$(i)\ (a_{\lambda_n,k})\in AMDS \text{ and } (\lambda_n+1)(a_{\lambda_n,0})=O(1)$$

$$(ii)\ (a_{\lambda_n,k})\in AMIS \text{ and } (\lambda_n+1)(a_{\lambda_n,r})=O(1),$$

where r *is integer part of* $\lambda_n/2$, *holds, then*

$$\|f-T_n^{\lambda}(f)\|_{p(\cdot),\omega}=O(\lambda_n^{-\alpha}).$$

Corollary 4.2. *Let* $f\in Lip\left(\alpha,p\left(\cdot\right),\omega\right)$, $0<\alpha<1$, $p\left(\cdot\right)\in\wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right)\in A_{p(\cdot)}(\mathbf{T})$ *and* (p_{λ_n}) *be a positive sequence. If one of the following conditions*

$$(i)(p_{\lambda_n})\in AMDS,$$

$$(ii)\ (p_{\lambda_n})\in AMIS \text{ and } (\lambda_n+1)(p_{\lambda_n})=O(P_{\lambda_n}),$$

holds, then

$$\|f-N_n^{\lambda}(f)\|_{p(\cdot),\omega}=O(\lambda_n^{-\alpha}).$$

Theorem 4.3. *Let* $f\in Lip\left(1,p\left(\cdot\right),\omega\right)$, $p\left(\cdot\right)\in\wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right)\in A_{p(\cdot)}(\mathbf{T})$ *and* $A=(a_{\lambda_n,k})$ *be a lower triangular matrix with* $|S_{\lambda_n}^{(A)}-1|=O(\frac{1}{\lambda_n})$. *If the following condition*

$$\sum_{k=1}^{\lambda_n-1}\left(\lambda_n-k\right)\left|a_{\lambda_n,k-1}-a_{\lambda_n,k}\right|=O\left(1\right)$$

holds, then

$$\|f-T_n^{\lambda}\|_{p(\cdot),\omega}=O(\lambda_n^{-1}).$$

Corollary 4.4. *Let* $f\in Lip\left(1,p\left(\cdot\right),\omega\right)$, $p\left(\cdot\right)\in\wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right)\in A_{p(\cdot)}(\mathbf{T})$ *and* (p_{λ_n}) *be a positive sequence. If the following condition*

$$\sum_{k=1}^{\lambda_n-1}\left|p_k-p_{k+1}\right|=O(\frac{P_{\lambda_n}}{\lambda_n})$$

holds, then

$$\|f-N_n^{\lambda}(f)\|_{p(\cdot),\omega}=O(\lambda_n^{-1}).$$

Theorem 4.5. *Let* $f \in Lip\left(\alpha, p\left(\cdot\right), \omega\right)$, $0<\alpha<1$, $p\left(\cdot\right) \in \wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right) \in A_{p(\cdot)}(\mathbf{T})$ *and* $A=(a_{\lambda_n,k})$ *be a lower triangular matrix with nonnegative entries and row sums* 1*. If one of the conditions*

$$(i)\ 0<\alpha<1 \text{ and } (a_{\lambda_n,k}) \in AMIMS,$$

$$(ii)\ 0<\alpha<1,\ (a_{\lambda_n,k}) \in AMDMS \text{ and } (\lambda_n+1)a_{\lambda_n,0}=O(1),$$

is valid, then

$$\left\|f-\tau_n^{\lambda}(f)\right\|_{p(\cdot),\omega}=O(\lambda_n^{-\alpha}). \tag{4.1}$$

Corollary 4.6. *Let* $f \in Lip\left(\alpha, p\left(\cdot\right), \omega\right)$, $p\left(\cdot\right) \in \wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right) \in A_{p(\cdot)}(T)$ *and* (p_{λ_n}) *be a positive sequence. If one of the conditions*

$$(i)\ 0<\alpha<1 \text{ and } (p_{\lambda_n}) \in AMIMS,$$

$$(ii)\ 0<\alpha<1,\ (p_{\lambda_n}) \in AMDMS \text{ and } (\lambda_n+1)p_{\lambda_n}=O\left(P_{\lambda_n}\right),$$

is valid, then

$$\left\|f-R_n^{\lambda}(f)\right\|_{p(\cdot),\omega}=O(\lambda_n^{-\alpha}).$$

Theorem 4.7. *Let* $f \in Lip\left(1, p\left(\cdot\right), \omega\right)$, $0<\alpha<1$, $p\left(\cdot\right) \in \wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right) \in A_{p(\cdot)}(T)$ *and* $A=(a_{\lambda_n,k})$ *be a lower triangular matrix with nonnegative entries and row sums* 1*. If the following condition*

$$\sum_{k=0}^{\lambda_n-2}\left|\Delta_k A_{\lambda_n,k}\right|=O\left(\lambda_n^{-1}\right)$$

holds, then

$$\left\|f-\tau_n^{\lambda}(f)\right\|_{p(\cdot),\omega}=O(\lambda_n^{-1}).$$

Corollary 4.8. *Let* $f \in Lip\left(1, p\left(\cdot\right), \omega\right)$, $p\left(\cdot\right) \in \wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right) \in A_{p(\cdot)}(T)$ *and* (p_{λ_n}) *be a positive sequence. If the following condition*

$$\sum_{k=0}^{\lambda_n-1}\left|\Delta_k \frac{P_k}{k+1}\right|=O\left(\frac{P_{\lambda_n}}{\lambda_n}\right)$$

holds, then

$$\left\|f-R_n^{\lambda}(f)\right\|_{p(\cdot),\omega}=O(\lambda_n^{-1}).$$

5. Proofs of Main Results

Proof of Theorem 4.1 Let $f \in Lip\left(\alpha, p\left(\cdot\right), \omega\right)$, $0<\alpha<1$, $p\left(\cdot\right) \in \wp_0\left(\mathbf{T}\right)$, $\omega\left(\cdot\right) \in A_{p(\cdot)}(\mathbf{T})$ and $A=(a_{\lambda_n,k})$ be a lower triangular matrix with $|S_{\lambda_n}^{(A)}-1|=O({\lambda_n}^{-\alpha})$. We suppose that one of the conditions (i) and (ii) of the theorem holds. Then by definitions of $T_n^{\lambda}(f)$ and $S_{\lambda_n}^{(A)}$ we have

$$\begin{aligned}
T_n^{\lambda}(f)\left(x\right)-f\left(x\right) &= \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} S_k f(x)-f(x) \\
&= \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} S_k f(x)-f(x)+S_{\lambda_n}^{(A)} f(x)-S_{\lambda_n}^{(A)} f(x) \\
&= \sum_{k=0}^{\lambda_n} a_{\lambda_n,k}\left[S_k f(x)-f(x)\right]+\left(S_{\lambda_n}^{(A)}-1\right) f(x).
\end{aligned}$$

Since $|S_{\lambda_n}^{(A)}-1| = O({\lambda_n}^{-\alpha})$, using Lemma 3.2 and Lemma 3.5 we get that

$$\begin{aligned}
\|f-T_n^{\lambda}(f)\|_{p(\cdot),\omega} &\le a_{\lambda_n,0}\|S_0f-f\|_{p(\cdot),\omega}+\sum_{k=1}^{\lambda_n} a_{\lambda_n,k}\|S_kf-f\|_{p(\cdot),\omega}\\
+|S_{\lambda_n}^{(A)}-1|\|f\|_{p(\cdot),\omega} &= O\left(\lambda_n^{-1}\right)+O(1)\sum_{k=1}^{\lambda_n} a_{\lambda_n,k}k^{-\alpha}+O\left(\lambda_n^{-\alpha}\right)\\
&= O\left(\lambda_n^{-\alpha}\right).
\end{aligned}$$

Proof of Theorem 4.3 We suppose that the condition

$$\sum_{k=1}^{\lambda_n-1}\left(\lambda_n-k\right)\left|a_{\lambda_n,k-1}-a_{\lambda_n,k}\right| = O(1)$$

holds. Using Lemma 3.6 we get

$$\begin{aligned}
\sum_{k=1}^{\lambda_n-1}\left|b_{\lambda_n,k}-b_{\lambda_n,k+1}\right| &\le \sum_{k=1}^{\lambda_n-1}\frac{1}{k\left(k+1\right)}\sum_{m=1}^{k} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|\\
&\le \sum_{k=1}^{r}\frac{1}{k\left(k+1\right)}\sum_{m=1}^{k} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|\\
&+\sum_{k=r}^{\lambda_n-1}\frac{1}{k\left(k+1\right)}\sum_{m=1}^{k} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right| \qquad (5.1)
\end{aligned}$$

where $r := \left[\frac{\lambda_n}{2}\right]$. For the first term standing on the right side of (5.1), using Abel's Transformation and the condition (i) we have

$$\begin{aligned}
\sum_{k=1}^{r}\frac{1}{k\left(k+1\right)}\sum_{m=1}^{k} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right| &\le \sum_{k=1}^{r}\left|a_{\lambda_n,k-1}-a_{\lambda_n,k}\right|\\
&= \sum_{k=1}^{r}\frac{1}{\left(\lambda_n-k\right)}\left(\lambda_n-k\right)\left|a_{\lambda_n,k-1}-a_{\lambda_n,k}\right|\\
&\le \frac{1}{\left(\lambda_n-r\right)}\sum_{k=1}^{r}\left(\lambda_n-k\right)\left|a_{\lambda_n,k-1}-a_{\lambda_n,k}\right|\\
&\le \frac{1}{\left(\lambda_n-r\right)}O(1) = O\left(\lambda_n^{-1}\right). \qquad (5.2)
\end{aligned}$$

For the second term, standing on the right side of (5.1), we get

$$\begin{aligned}
&\sum_{k=r}^{\lambda_n-1}\frac{1}{k\left(k+1\right)}\sum_{m=1}^{k} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|\\
&\le \sum_{k=r}^{\lambda_n-1}\frac{1}{k\left(k+1\right)}\left\{\sum_{m=1}^{r} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|+\sum_{m=r}^{k} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|\right\}\\
&= \sum_{k=r}^{\lambda_n-1}\frac{1}{k\left(k+1\right)}\sum_{m=1}^{r} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|+\sum_{k=r}^{\lambda_n-1}\frac{1}{k\left(k+1\right)}\sum_{m=r}^{k} m\left|a_{\lambda_n,m-1}-a_{\lambda_n,m}\right|\\
&:= I_1+I_2.
\end{aligned}$$

For I_1, using the inequality

$$\sum_{k=1}^{r} |a_{\lambda_n,k-1} - a_{\lambda_n,k}| = O\left(\lambda_n^{-1}\right),$$

obtained in the course of the proof of (5.2), we have the estimation

$$I_1 \le \sum_{k=r}^{\lambda_n-1} \frac{1}{(k+1)} \sum_{m=1}^{r} |a_{\lambda_n,m-1} - a_{\lambda_n,m}|$$

$$\le \frac{1}{(r+1)} \sum_{k=r}^{\lambda_n-1} \left(\sum_{m=r}^{k} |a_{\lambda_n,m-1} - a_{\lambda_n,m}| \right)$$

$$\le \frac{2}{\lambda_n} \sum_{k=r}^{\lambda_n-1} \left(\sum_{m=r}^{k} |a_{\lambda_n,m-1} - a_{\lambda_n,m}| \right)$$

$$\le \frac{2}{\lambda_n} \sum_{k=1}^{\lambda_n-1} (\lambda_n - k) |a_{\lambda_n,k-1} - a_{\lambda_n,k}| = \frac{2}{\lambda_n} O(1) = O\left(\lambda_n^{-1}\right).$$

Hence

$$\sum_{k=r}^{\lambda_n-1} \frac{1}{(k+1)} \sum_{m=1}^{k} m\, |a_{\lambda_n,m-1} - a_{\lambda_n,m}| \le I_{n_1} + I_{n_2} = O\left(\lambda_n^{-1}\right). \tag{5.3}$$

Using Lemma 3.2, we have

$$\|f - T_n^{\lambda}(f)\|_{p(\cdot),\omega} \le \|S_{\lambda_n}(f) - T_n^{\lambda}(f)\|_{p(\cdot),\omega} + \|f - S_{\lambda_n}(f)\|_{p(\cdot),\omega}$$

$$= \|S_{\lambda_n}(f) - T_n^{\lambda}(f)\|_{p(\cdot),\omega} + O\left(\lambda_n^{-1}\right). \tag{5.4}$$

In addition to this, the following inequality is valid.

$$\left\| S_n^{\lambda}(f) - T_n^{\lambda}(f) \right\|_{p(\cdot),\omega} = O(\lambda_n^{-1}\Big) O(1) + O(1) \sum_{k=1}^{\lambda_n-1} |b_{\lambda_n,k} - b_{\lambda_n,k+1}|. \tag{5.5}$$

Now combining $(5.1), (5.2)$ and (5.3) we get

$$\sum_{k=1}^{\lambda_n-1} |b_{\lambda_n,k} - a_{\lambda_n,k+1}| = O\left(\lambda_n^{-1}\right),$$

which together with the relations (5.4) and (5.5) imply that

$$\|f - T_n^{\lambda}(f)\|_{p(\cdot),\omega} = O(\lambda_n^{-1}).$$

Proof of Theorem 4.5 Since

$$\tau_n^{\lambda}(f)(x) - f(x) = \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} S_k(f)(x) - f(x).$$

therefore using Lemma 3.2 and Lemma 3.4 we have

$$\begin{aligned}\left\|\tau_n^\lambda(f)-f\right\|_{p(\cdot),\omega} &\leq \sum_{k=0}^{\lambda_n} a_{\lambda_n,k}\left\|S_k(f)-f\right\|_{p(\cdot),\omega} \\ &= O\left(\sum_{k=0}^{\lambda_n} a_{\lambda_n,k}(k+1)^{-\alpha}\right) \\ &= O(\lambda_n^{-\alpha}).\end{aligned}$$

This completes the proof of (i) and (ii).

Proof of Theorem 4.7 In the case of $\alpha=1$, we will use S_{λ_n} as S_n^λ. Applying Abel's transformation two times,

$$\begin{aligned}\tau_n^\lambda(f)(x)-f(x) &= \sum_{k=0}^{\lambda_n-1}\left[S_k(f;x)-S_{k+1}(f;x)\right]\sum_{i=0}^{k} a_{\lambda_n,i} \\ &\quad+\left[S_n^\lambda f(x)-f(x)\right] \\ &= S_n^\lambda(f)(x)-f(x)-\sum_{k=0}^{\lambda_n-1}(k+1)\,U_{k+1}f(x)\,A_{\lambda_n,k} \\ &= S_n^\lambda(f)(x)-f(x)-\sum_{k=0}^{\lambda_n-2}\left(A_{\lambda_n,k}-A_{\lambda_n,k+1}\right)\times \\ &\quad \sum_{i=0}^{k}(i+1)\,U_{i+1}f(x)-A_{\lambda_n,\lambda_{n-1}}\sum_{i=0}^{\lambda_n-1}(i+1)\,U_{i+1}f(x) \\ &= S_n^\lambda(f)(x)-f(x)-\sum_{k=0}^{\lambda_n-2}\left(A_{\lambda_n,k}-A_{\lambda_n,k+1}\right)\sum_{i=1}^{k+1} iU_if(x) \\ &\quad -\frac{1}{\lambda_n}\sum_{k=0}^{\lambda_n} a_{\lambda_n,k}\sum_{i=1}^{\lambda_n} iU_if(x).\end{aligned}$$

Therefore by Minkowski inequality, we get

$$\begin{aligned}\left\|\tau_n^\lambda(f)-f\right\|_{p(\cdot),\omega} \leq \left\|S_n^\lambda(f)-f\right\|_{p(\cdot)},\omega+\sum_{k=0}^{\lambda_n-2}\left\|\sum_{i=1}^{k+1} iU_if\right\|_{p(\cdot),\omega}\left|\Delta_k A^*_{\lambda_n,k}\right| \\ +\frac{1}{\lambda_n}\left\|\sum_{i=1}^{\lambda_n} iU_if\right\|_{p(\cdot),\omega}. \qquad (5.6)\end{aligned}$$

We get

$$S_n^\lambda(f)(x)-\sigma_n^\lambda(f)(x)=\frac{1}{\lambda_n+1}\sum_{i=1}^{\lambda_n} iU_if(x)$$

therefore by Lemma 3.3

$$\left\|\sum_{i=1}^{\lambda_n} iU_if\right\|_{p(\cdot),\omega}=(\lambda_n+1)\left\|S_n^\lambda(f)-\sigma_n^\lambda(f)\right\|_{p(\cdot),\omega}=O(1). \qquad (5.7)$$

If

$$\sum_{k=0}^{\lambda_n-2}|\Delta_k A_{\lambda_n,k}|=O\left(\lambda_n^{-1}\right),$$

then, from (5.6), (5.7) and Lemma 3.2, we write

$$\begin{aligned}\left\|\tau_n^{\lambda}(f)-f\right\|_{p(\cdot),\omega} &= O\left(\lambda_n^{-1}\right)+O\left(1\right)\cdot\sum_{k=0}^{\lambda_n-2}|\Delta_k A_{\lambda_n,k}| \\ &= O\left(\lambda_n^{-1}\right).\end{aligned}$$

Therefore, this yields (4.1).

6. Applications

In this section, we will give an application of some summability methods obtained by Fourier series and compare different summability methods of Fourier series using graphics, examples and value tables. The graphics show oscillation and overshoots of summability methods. The examples and the value tables give some algebraic results about approximation error.

The function f defined by

$$f\left(t\right)=\left\{\begin{array}{l}1,\ for\ -\pi\leq t<0\\ -1,\ for\ 0\leq t\leq\pi.\end{array}\right. \tag{6.1}$$

For all real values of t, we have $f(t+2\pi)=f(t)$.

Since it is an odd function, its Fourier series is given as

$$f(t)=\sum_{n=1}^{\infty}b_n\sin nt$$

where

$$b_n=\frac{2}{\pi}\int_0^{\pi}f\left(t\right)\sin nt dt=\frac{2}{\pi}\left(\frac{-1+(-1)^n}{n}\right).$$

Therefore, the Fourier series of $f(t)$ is given by

$$f(t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{-1+(-1)^n}{n}\right)\sin nt,\ t\in\left[-\pi,\pi\right]. \tag{6.2}$$

The partial sum $S_{\lambda_n}(t)$ of the series (6.2) is given as

$$S_{\lambda_n}(t)=\frac{4}{\pi}\left(\sin t+\frac{1}{3}\sin 3t+...+\frac{1}{\lambda_n}\sin\left(\lambda_n t\right)\right). \tag{6.3}$$

Sub-Cesàro mean for (6.2) is

$$C_{\lambda_n}(f)=\sum_{k=1}^{\lambda_n}\left(\frac{\lambda_n-k}{\lambda_n}\right)\left(\frac{-2}{\pi}\right)\left(\frac{-1+(-1)^k}{k}\right)\sin(kt).$$

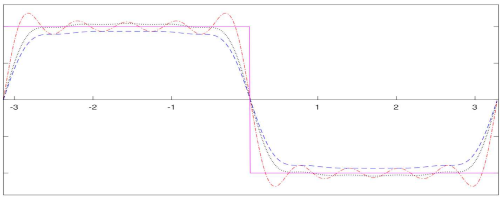
In (1.5), taking $(a_{\lambda_n,k})$ as below, we will calculate the Sub-Matrix mean for the series (6.2).

$$(a_{\lambda_n,k}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \\ \vdots & & & & & \ddots & 0 & 0 & \\ \frac{1}{\lambda_n+1} & \frac{1}{\lambda_n+1} & \frac{1}{\lambda_n+1} & \frac{1}{\lambda_n+1} & \frac{1}{\lambda_n+1} & \frac{1}{\lambda_n+1} & \frac{1}{\lambda_n+1} & 0 & \cdots \\ \vdots & & & & & & & \ddots & \end{bmatrix}$$

In $(a_{\lambda_n,k})$, $\frac{1}{\lambda_n+1}$ repeats (λ_n+1) times in the $(\lambda_n)^{th}$ *row.* For (6.2) and lower triangular matrix given above, the sub-Matrix mean is

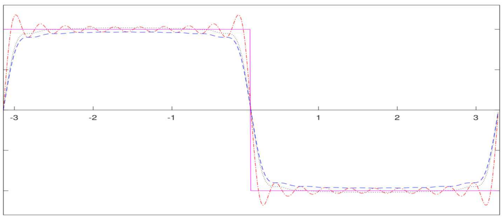
$$\tau_n^{\lambda}(f,t) := \sum_{k=0}^{\lambda_n} \frac{1}{\lambda_n+1} S_k(f,t),\ \lambda_n = 0,1,2,....$$

For $\lambda_n = 9$ and 19, using the definitions of $f(t)$ (solid), $S_{\lambda_n}(t)$ (dash dot), $C_{\lambda_n}(f)$ (dash) *and* $\tau_n^{\lambda}(f,t)$ (dot) we plot the following graphics. In these graphics, t is on the axis x. Axis y shows the value of given function or sum at the point t.



This graphic shows oscillation and overshoots for $\lambda_n = 9$.

Figure 1



This graphic shows oscillation and overshoots for $\lambda_n = 19$.

Figure 2

These graphics inform to us about oscillation and overshoots occurred when we approximate a periodic function having a jump discontinuity by a finite sum of Fourier series.

Note that, according to step function f, $S_{\lambda_n}(f,t)$ have peaks in the neighborhood of discontinuity and its oscillation moves towards discontinuity points $(-\pi, 0, -\pi)$.

If we take $\lambda_n = 19$ instead of $\lambda_n = 9$, then the errors of approximation decrease.

When using the sub-Cesàro mean $C_{\lambda_n}(f,t)$ and sub-matrix mean $\tau_n^{\lambda}(f,t)$ instead of partial sum or increasing the value λ_n, we will have better results in the sense of straightening effect. So, the peaks become flatter.

The oscillation amplitude on graphics tends to decrease when λ_n is increases.

We summarized a conclusion related to the approximation errors by following example. Note that, we calculated the approximation errors both weighted case (Muckenhoupt weight) and non-weighted case in the variable exponent Lebesgue spaces.

Let

$$\begin{aligned} p(t) &= \frac{9|t|+2}{2|t|+1},\ t\in \mathbf{T}, \\ \omega(t) &= |t|^{\alpha},\ -1<\alpha<p-1,\ p>1. \end{aligned}$$

Then, we write that

$$\left\|f-\tau_n^{\lambda}(f)\right\|_{p(\cdot),\omega} := \inf\left\{\xi>0 : \int\limits_{\mathbf{T}} \left|\frac{\omega(t)\left(f(t)-\tau_n^{\lambda}(f;t)\right)}{\xi}\right|^{p(t)} dt \le 1\right\}.$$

For some special cases, the following table gives the norm values of variable exponent Lebesgue spaces with Muckenhoupt weight.

	$\omega(t)=\lvert t\rvert^{\frac{1}{200}}$		$\omega(t)=\lvert t\rvert^{\frac{1}{2000}}$	
	$\lambda_n=9$	$\lambda_n=19$	$\lambda_n=9$	$\lambda_n=19$
$\left\|f-\tau_n^{\lambda}(f)\right\|_{p(\cdot),\omega}$	$0,6812$	$0,5855$	$0,6784$	$0,5827$
$\left\|f-C_{\lambda_n}f\right\|_{p(\cdot),\omega}$	$0,7166$	$0,6008$	$0,7137$	$0,5980$
$\left\|f-S_{\lambda_n}(f)\right\|_{p(\cdot),\omega}$	$0,5969$	$0,5054$	$0,5942$	$0,5028$

In the case of $\omega(t)=1$ (case of non-weighted space), we get the following values for the errors of deviations.

	$\lambda_n=9$		$\lambda_n=19$	
	$\omega(t)=\lvert t\rvert^{\frac{1}{200}}$	$\omega(t)=1$	$\omega(t)=\lvert t\rvert^{\frac{1}{200}}$	$\omega(t)=1$
$\left\|f-\tau_n^{\lambda}(f)\right\|_{p(\cdot),\omega}$	$0,6812$	$0,6819$	$0,5855$	$0,5868$
$\left\|f-C_{\lambda_n}(f)\right\|_{p(\cdot),\omega}$	$0,7166$	$0,7171$	$0,6008$	$0,6020$
$\left\|f-S_{\lambda_n}(f)\right\|_{p(\cdot),\omega}$	$0,5969$	$0,5982$	$0,5054$	$0,5075$

We say that the deviation of $\left\|f-\tau_n^{\lambda}(f)\right\|_{p(\cdot),\omega}$ is better than the deviation of $\left\|f-C_{\lambda_n}(f)\right\|_{p(\cdot),\omega}$ in the sense of approximation error in weighted case or non-weighted case.

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