Fixed Point Theory and Algorithms for Sciences and Engineering

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New perturbed parametric metric spaces for distance modeling: fixed-point analysis with applications to the fixed-circle problem

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Abstract

In this study, we introduce a new generalization of metric spaces, called Perturbed Parametric Metric Spaces (PPMS). This framework extends the classical metric space by incorporating perturbation functions and the presence of a non-negative parameter τ in its distance function, rather than a standard two-variable metric $d(\mu, \nu)$, providing a more flexible and robust treatment of distance measurement that reflects underlying variations and imperfections in complex systems. Specifically, the measurement of distance between two points is subject to errors, often arising from factors such as instrumental inaccuracies or environmental influences. We develop the foundational properties and initiate some topological notions of PPMS, provide illustrative examples, and establish several fixed-point theorems within this setting with application to fixed-circle problem. These results demonstrate the applicability of PPMS in nonlinear analysis and show how it unifies and generalizes various existing metric-type spaces. Our approach opens new perspectives for future research in functional analysis.

Keywords: Perturbed parametric metric; Perturbed function; Fixed point theory; Contraction; Fixed circle

Mathematics Subject Classification: 47H10; 54H25

1 Introduction and preliminaries

Fixed point theory forms a central pillar of nonlinear analysis and underlies numerous branches of pure and applied mathematics. It serves as a foundational tool for addressing diverse problems in differential equations, optimization, game theory, fuzzy systems, and various forms of mathematical modeling. Among its cornerstone results stands Banach's contraction principle, which guarantees that any self-map on a complete metric space admits a unique fixed point. Over successive decades, this principle has been systematically broadened and refined to expand its range of applicability and enrich its theoretical framework.

One major stream of generalization involves redefining the nature of the underlying space itself. The classical metric space (X, d) has been replaced by a variety of abstract and

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flexible structures such as fuzzy metric spaces [7, 18, 27], intuitionistic fuzzy metric spaces [32], b-metric spaces [1, 5], partial metric spaces [21], G-metric spaces [29], θ-parametric metric spaces [28], modular metric spaces [4], probabilistic metric spaces [22], and several other generalized metric-type configurations.

A second major research direction centers on modifying or relaxing the traditional contraction conditions. To this end, various analytical devices have been proposed, including control functions [2, 20], altering distance functions [15], simulation functions [16, 24, 25, 33], and numerous forms of generalized contractive mappings such as weak, integral-type, cyclic, and coupled contractions [8, 17, 35]. In parallel, relation-theoretic methodologies that utilize binary relations or graph-based structures have proven highly effective in extending classical theorems [6, 19, 26, 30, 34, 40]. Collectively, these advancements accommodate richer notions of contractiveness and capture dynamic behaviors beyond the reach of classical contraction mappings. Further developments have explored non-self mappings, which act between different subsets or regions of a given space [36]. This extension significantly increases the diversity of problems that can be modeled and analyzed within the fixed point framework. In recent years, the introduction of perturbation and parameterization techniques into metric-type structures has opened promising avenues for studying systems characterized by uncertainty, imprecision, and multi-valued interactions. Such approaches enhance modeling flexibility, enabling the description of more complex and adaptive phenomena within abstract, fuzzy, and analytical settings under perturbation.

Motivated by these developments, the present paper introduces a new construct, termed the perturbed parametric metric space, which unifies parameter-dependent and perturbative components within a flexible analytical framework. Within this setting, several novel fixed point results are established for self-mappings satisfying generalized contractive conditions. As an illustrative application, the fixed-circle problem is examined, demonstrating that the proposed perturbed parametric framework provides deeper insight and more refined analytical tools for the study of fixed-circle phenomena. The results not only extend numerous classical fixed point theorems, but also offer a coherent and unified basis for analyzing nonlinear systems in perturbed and generalized metric-type contexts. Consequently, this study enhances both the theoretical and practical scope of fixed point theory, particularly in settings characterized by relational complexity and perturbative effects.

In 2014, Hussain et al. [9] developed the concept of a parametric metric, extending the traditional notion of a metric, and derived multiple fixed-point results within this generalized setting.

Definition 1 [9] Let Y be a nonempty set, and consider a function $\kappa : Y \times Y \times (0, +\infty) \to [0, +\infty)$. The mapping κ is called a parametric metric on Y if it satisfies the following conditions:

1. $\kappa(\mu, \nu, \tau) = 0$ for all $\tau > 0$ if and only if $\mu = \nu$,
2. $\kappa(\mu, \nu, \tau) = \kappa(\nu, \mu, \tau)$ for all $\tau > 0$,
3. $\kappa(\mu, \nu, \tau) \leq \kappa(\mu, \omega, \tau) + \kappa(\omega, \nu, \tau)$ for all $\mu, \nu, \omega \in Y$ and all $\tau > 0$.

The pair (Y, κ) is then called a parametric metric space.

Example 1 [9] Let Y be the set of all real-valued functions defined on $(0,+\infty)$. Define $\kappa : Y \times Y \times (0,+\infty) \to [0,+\infty)$ by

$$\kappa(\psi,\varphi,\tau) = |\psi(\tau) - \varphi(\tau)|,$$

for all $\psi,\varphi \in Y$ and $\tau > 0$. It can be verified that (Y,κ) forms a parametric metric space.

2 Perturbed parametric metrics

Following the analytical framework introduced by Jleli and Samet [11] and further developed by Moussaoui in [23], we introduce a generalized form of a metric space, referred to as a perturbed parametric metric space.

Definition 2 Let $\mathcal{P}$ and $\hbar$ be two mappings defined as $\mathcal{P},\hbar : Y \times Y \times (0,+\infty) \to [0,+\infty)$. We say that $\mathcal{P}$ is a parametric perturbed metric on Y with respect to $\hbar$ if the following holds: The function $\mathcal{P} - \hbar : Y \times Y \times (0,+\infty) \to \mathbb{R}$, defined by:

$$(\mu,\nu,\tau) \mapsto \mathcal{P}(\mu,\nu,\tau) - \hbar(\mu,\nu,\tau),$$

is a metric on Y. That is, for all $\mu,\nu,\omega \in Y$ and all $\tau > 0$, the following conditions must be satisfied:

($\mathcal{P}1$) $(\mathcal{P}-\hbar)(\mu,\nu,\tau) \geq 0$. (Non-negativity);
($\mathcal{P}2$) $(\mathcal{P}-\hbar)(\mu,\nu,\tau) = 0 \iff \mu = \nu$. (Identity of indiscernibles);
($\mathcal{P}3$) $(\mathcal{P}-\hbar)(\mu,\nu,\tau) = (\mathcal{P}-\hbar)(\nu,\mu,\tau)$. (Symmetry);
($\mathcal{P}4$) $(\mathcal{P}-\hbar)(\mu,\nu,\tau) \leq (\mathcal{P}-\hbar)(\mu,\omega,\tau) + (\mathcal{P}-\hbar)(\omega,\nu,\tau)$. (Triangle inequality).

We refer to $\hbar$ as a *parametric perturbed mapping*, $p = \mathcal{P} - \hbar$ as the *exact parametric metric*, and the triple $(Y,\mathcal{P},\hbar)$ as a *perturbed parametric metric space*.

Remark 1 It is crucial to emphasize that a perturbed parametric metric on Y is not necessarily a parametric metric on Y. This distinction will be clarified in the following example.

3 Examples of perturbed parametric metrics

Example 2 Consider the mapping $\mathcal{P} : \mathbb{R} \times \mathbb{R} \times (0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{P}(\mu,\nu,\tau) = \frac{|\mu-\nu|}{\tau} + e^{\tau}\mu\nu^3, \quad \text{for all } \mu,\nu \in \mathbb{R} \text{ and } \tau > 0.$$

Then $\mathcal{P}$ constitutes a perturbed parametric metric on $\mathbb{R}$ with respect to the perturbation $\hbar : \mathbb{R} \times \mathbb{R} \times (0,+\infty) \to [0,+\infty)$ given by

$$\hbar(\mu,\nu,\tau) = e^{\tau}\mu\nu^3, \quad \text{for all } \mu,\nu \in \mathbb{R} \text{ and } \tau > 0.$$

Accordingly, the corresponding exact parametric metric $p : \mathbb{R} \times \mathbb{R} \times (0,+\infty) \to [0,+\infty)$ is expressed as

$$p(\mu,\nu,\tau) = \frac{|\mu-\nu|}{\tau}, \quad \text{for all } \mu,\nu \in \mathbb{R} \text{ and } \tau > 0.$$

It is worth noting that $\mathcal{P}$ is not a parametric metric on $\mathbb{R}$, since

$$\mathcal{P}(1,1,\tau) = e^{\tau} \neq 0, \quad \text{for all } \tau > 0.$$

Example 3 Consider the mapping $\mathcal{P} : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ defined by

$$\mathcal{P}(\mu, \nu, \tau) = g(\tau)|\mu - \nu| + \tau^{\alpha}\mu^{\beta}\nu^{\gamma}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0,$$

where $g : (0, +\infty) \to (0, +\infty)$ is a continuous function and α, β, γ are positive constants. In this setting, $\mathcal{P}$ defines a perturbed parametric metric on $\mathbb{R}$ with respect to the perturbation $\hbar : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ given by

$$\hbar(\mu, \nu, \tau) = \tau^{\alpha}\mu^{\beta}\nu^{\gamma}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Accordingly, the associated exact parametric metric $p : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ is expressed as

$$p(\mu, \nu, \tau) = g(\tau)|\mu - \nu|, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

If $g(\tau) = \tau$, then the expression of $\mathcal{P}$ reduces to

$$\mathcal{P}(\mu, \nu, \tau) = \tau|\mu - \nu| + \tau^{\alpha}\mu^{\beta}\nu^{\gamma}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

In this case, $\mathcal{P}$ still represents a perturbed parametric metric on $\mathbb{R}$ with respect to $\hbar$, and the exact parametric metric takes the form

$$p(\mu, \nu, \tau) = \tau|\mu - \nu|, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Furthermore, if one sets $g(\tau) = \tau$ and selects $\alpha = 3$, $\beta = 2$, and $\gamma = 4$, the mapping $\mathcal{P}$ becomes

$$\mathcal{P}(\mu, \nu, \tau) = \tau|\mu - \nu| + \tau^{3}\mu^{2}\nu^{4}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

In this situation, $\mathcal{P}$ remains a perturbed parametric metric on $\mathbb{R}$ associated with $\hbar$, and the corresponding exact parametric metric simplifies to

$$p(\mu, \nu, \tau) = \tau|\mu - \nu|, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Example 4 **(Exponential-Type Perturbation)** Consider the mapping $\mathcal{P} : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ defined by

$$\mathcal{P}(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau} + e^{-\tau(\mu^2+\nu^2)}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

The mapping $\mathcal{P}$ constitutes a perturbed parametric metric on $\mathbb{R}$ with respect to the perturbation $\hbar : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ defined by

$$\hbar(\mu, \nu, \tau) = e^{-\tau(\mu^2+\nu^2)}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Accordingly, the corresponding exact parametric metric $p : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ is given by

$$p(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Example 5 **(Logarithmic-Type Perturbation)** Consider the mapping $\mathcal{P} : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ defined by

$$\mathcal{P}(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau} + \ln(1 + \tau^2\mu^2\nu^2), \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

The mapping $\mathcal{P}$ represents a perturbed parametric metric on $\mathbb{R}$ with respect to the perturbation $\hbar : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ given by

$$\hbar(\mu, \nu, \tau) = \ln(1 + \tau^2\mu^2\nu^2), \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Consequently, the corresponding exact parametric metric $p : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ is given by

$$p(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Example 6 **(Gaussian-Type Perturbation)** Consider the mapping $\mathcal{P} : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ defined by

$$\mathcal{P}(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau} + e^{-\tau^2(\mu-\nu)^2}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

The mapping $\mathcal{P}$ defines a perturbed parametric metric on $\mathbb{R}$ with respect to the perturbation $\hbar : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ given by

$$\hbar(\mu, \nu, \tau) = e^{-\tau^2(\mu-\nu)^2}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Accordingly, the associated exact parametric metric $p : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ is expressed as

$$p(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Example 7 **(Rational-Type Perturbation)** Consider the mapping $\mathcal{P} : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ defined by

$$\mathcal{P}(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau} + \frac{|\mu\nu|}{1 + \tau^2}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

The mapping $\mathcal{P}$ defines a perturbed parametric metric on $\mathbb{R}$ with respect to the perturbation $\hbar : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ given by

$$\hbar(\mu, \nu, \tau) = \frac{|\mu\nu|}{1 + \tau^2}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Consequently, the associated exact parametric metric $p : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ is given by

$$p(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Example 8 **(Square Root-Type Perturbation)** Consider the mapping $\mathcal{P} : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ defined by

$$\mathcal{P}(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau} + \sqrt{1 + \tau^2 |\mu + \nu|}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

The mapping $\mathcal{P}$ defines a perturbed parametric metric on $\mathbb{R}$ with respect to the perturbation

$$\hbar(\mu, \nu, \tau) = \sqrt{1 + \tau^2 |\mu + \nu|}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Accordingly, the corresponding exact parametric metric $p : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to [0, +\infty)$ is given by

$$p(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau}, \quad \text{for all } \mu, \nu \in \mathbb{R} \text{ and } \tau > 0.$$

Example 9 Let $(Y, \mathcal{M}, N, *, \diamond)$ be a triangular intuitionistic fuzzy metric space (for further details, see [10]). Define the mapping $\mathcal{P} : Y \times Y \times (0, +\infty) \to [0, +\infty)$ by

$$\mathcal{P}(\mu, \nu, \tau) = \frac{1}{\mathcal{M}(\mu, \nu, \tau)} - 1 + \hbar(\mu, \nu, \tau),$$

for all $\mu, \nu \in Y$ and $\tau > 0$. Then $(Y, \mathcal{P}, \hbar)$ forms a perturbed parametric metric space on Y with respect to the perturbation $\hbar$. The associated exact parametric metric $p : Y \times Y \times (0, +\infty) \to [0, +\infty)$ is given by

$$p(\mu, \nu, \tau) = \frac{1}{\mathcal{M}(\mu, \nu, \tau)} - 1.$$

Proposition 1 *Let $\mathcal{P}, \hbar, h : Y \times Y \to [0, +\infty)$ be three given mappings, and let $\eta > 0$.*

(i) *If $(Y, \mathcal{P}, \hbar)$ and $(Y, \mathcal{P}, h)$ are perturbed parametric metric spaces, then the triplet $(Y, \mathcal{P}, \frac{\hbar + h}{2})$ also constitutes a perturbed parametric metric space.*

(ii) *If $(Y, \mathcal{P}, \hbar)$ is a perturbed parametric metric space, then $(Y, \eta\mathcal{P}, \eta\hbar)$ is likewise a perturbed parametric metric space.*

Proof

(i) Due to the fact that $\mathcal{P} - \hbar$ and $\mathcal{P} - h$ are two metrics on Y, it follows that:

$$\frac{1}{2}\left[(\mathcal{P} - \hbar) + (\mathcal{P} - h)\right] = \mathcal{P} - \frac{\hbar + h}{2}$$

is a metric on Y, proving part (i).

(ii) Since $\mathcal{P} - \hbar$ is a metric on Y and $\eta > 0$, as a result

$$\eta(\mathcal{P} - \hbar) = \eta\mathcal{P} - \eta\hbar, \text{ is a metric on } Y \text{ proving part (ii).}$$ □

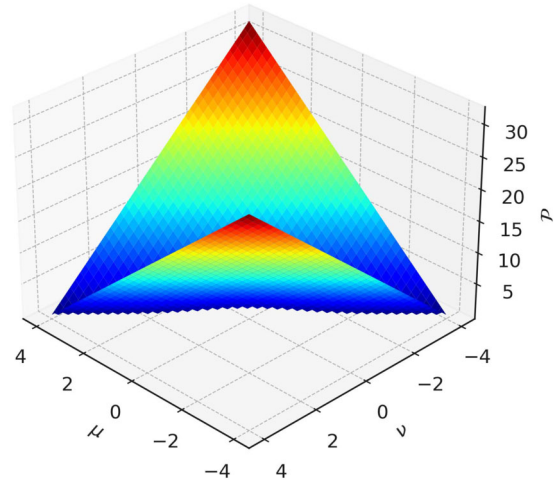


Figure 1 Surface of $\mathcal{P}(\mu, \nu, \tau)$ for $\tau = 0.25$. The exponential term $e^{-\tau(\mu^2+\nu^2)}$ dominates near the origin, generating a noticeable curvature above the exact distance function

4 Numerical and graphical illustration of the exponential-type perturbation

To better visualize the structure of perturbed parametric metrics, consider the exponential-type perturbation introduced in Example 4, given by

$$\mathcal{P}(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau} + e^{-\tau(\mu^2+\nu^2)}, \quad \mu, \nu \in \mathbb{R},\ \tau > 0.$$

The corresponding perturbation and exact parametric metric are

$$\hbar(\mu, \nu, \tau) = e^{-\tau(\mu^2+\nu^2)}, \qquad p(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau}.$$

Figures 1–4 display the numerical surfaces of $\mathcal{P}$ and its associated quantities for different values of the parameter τ.

These visual representations reveal that the exponential perturbation induces a localized and smooth deformation of the underlying exact metric, which progressively diminishes as the parameter τ increases. The elevated surface of $\mathcal{P}$ relative to p measures the extent of this deviation, illustrating how the perturbative influence fades with larger values of τ. Thus, the parameter τ serves as a stabilization factor that governs the gradual convergence of the perturbed structure toward the exact parametric metric framework.

5 Fixed point theorems under different contraction conditions

After establishing the framework of perturbed parametric metric spaces and examining their behavior through analytical examples and numerical visualizations, our attention now turns to the study of fixed points for self-mappings defined within this generalized environment. The investigation of existence and uniqueness results in such spaces extends the classical fixed point theory and provides new insights into mappings influenced by perturbative effects. To lay the groundwork for these developments, we first initiate some topological notions of convergence, completeness, and continuity that arise naturally in the perturbed parametric context.

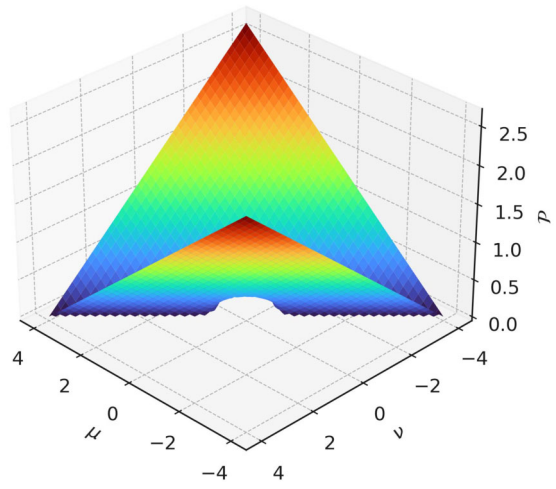


Figure 2 Surface of $\mathcal{P}(\mu, \nu, \tau)$ for $\tau = 3$. As τ increases, the perturbation term vanishes exponentially, and $\mathcal{P}$ converges to the exact metric $p(\mu, \nu, \tau) = |\mu - \nu|/\tau$

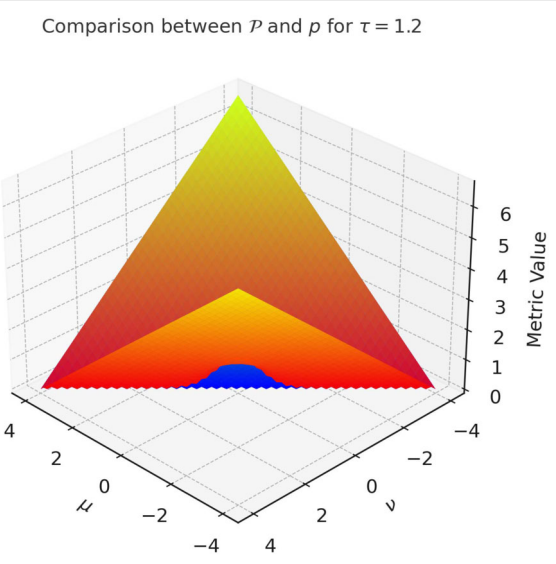


Figure 3 Comparison at $\tau = 1.2$ between the exact metric $p(\mu, \nu, \tau)$ and the perturbed metric $\mathcal{P}(\mu, \nu, \tau)$. The upper surface represents $\mathcal{P}$ and the lower surface p; the visible gap between them corresponds to the perturbation term $e^{-\tau(\mu^2+\nu^2)}$, clearly showing its decaying influence as τ grows

Definition 3 Let $(Y, \mathcal{P}, \hbar)$ be a perturbed parametric metric space, $\{v_n\}$ a sequence in Y.

(i) The sequence $\{v_n\}$ in $(Y, \mathcal{P}, \hbar)$ is *perturbed parametric convergent* if it converges in the parametric metric space (Y, p), where $p = \mathcal{P} - \hbar$.

(ii) The sequence $\{v_n\}$ in $(Y, \mathcal{P}, \hbar)$ is a *perturbed parametric Cauchy sequence* if it is Cauchy in (Y, p).

(iii) The space $(Y, \mathcal{P}, \hbar)$ is *complete* if (Y, p) is complete, or equivalently, every perturbed parametric Cauchy sequence in $(Y, \mathcal{P}, \hbar)$ converges in the same space.

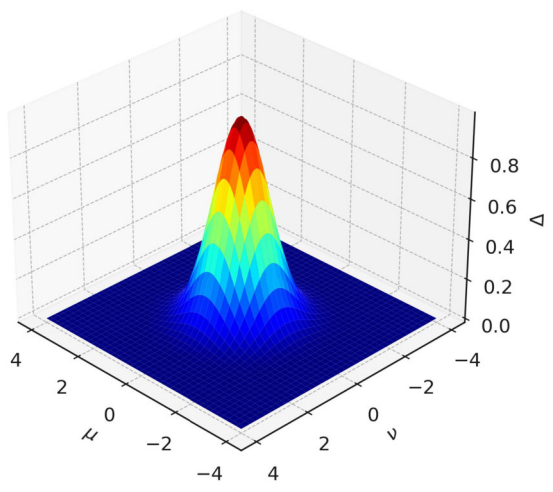


Figure 4 Perturbation surface $\Delta(\mu, \nu, \tau) = e^{-\tau(\mu^2+\nu^2)}$ for $\tau = 0.75$. The dome-shaped decay pattern illustrates the localized nature of the perturbation, concentrated near the origin

(iv) A mapping $\mathcal{G} : Y \to Y$ is *perturbed parametric continuous* if it is continuous with respect to p.

Theorem 2 *Let $(Y, \mathcal{P}, \hbar)$ be a complete perturbed parametric metric space, and let $\mathcal{G} : Y \to Y$ be a perturbed parametric continuous self-map. Assume that for all $\mu, \nu \in Y$ with $\mu \neq \nu$ and every $\tau > 0$, the inequality*

$$\mathcal{P}(\mathcal{G}\mu, \mathcal{G}\nu, \tau) \leq \alpha \max \left\{ \mathcal{P}(\mu, \nu, \tau), \mathcal{P}(\mu, \mathcal{G}\mu, \tau), \mathcal{P}(\nu, \mathcal{G}\nu, \tau), \frac{\mathcal{P}(\mu, \mathcal{G}\mu, \tau)\mathcal{P}(\nu, \mathcal{G}\nu, \tau)}{\mathcal{P}(\mu, \nu, \tau)} \right\}, \tag{5.1}$$

holds for some constant $\alpha \in [0, 1)$. Then $\mathcal{G}$ admits a unique fixed point in Y.

Proof Let $\nu_0 \in Y$ be chosen arbitrarily, and construct a sequence $\{\nu_n\}_{n \in \mathbb{N} \cup \{0\}}$ by the iterative rule

$$\nu_{n+1} = \mathcal{G}(\nu_n), \quad n \in \mathbb{N} \cup \{0\}.$$

If there exists some index $i_0 \in \mathbb{N} \cup \{0\}$ such that $\nu_{i_0} = \nu_{i_0+1} = \mathcal{G}(\nu_{i_0})$, then ν_{i_0} is clearly a fixed point of $\mathcal{G}$.

Assume now that $\nu_n \neq \nu_{n+1}$ for every $n \in \mathbb{N} \cup \{0\}$. Invoking inequality (5.1) and substituting $\mu = \nu_n$ and $\nu = \nu_{n-1}$, we derive

$$\begin{aligned} \mathcal{P}(\nu_{n+1}, \nu_n, \tau) &= \mathcal{P}(\mathcal{G}\nu_n, \mathcal{G}\nu_{n-1}, \tau) \\ &\leq \alpha \max \left\{ \mathcal{P}(\nu_n, \nu_{n-1}, \tau), \mathcal{P}(\nu_n, \mathcal{G}\nu_n, \tau), \mathcal{P}(\nu_{n-1}, \mathcal{G}\nu_{n-1}, \tau), \right. \\ &\qquad \left. \frac{\mathcal{P}(\nu_n, \mathcal{G}\nu_n, \tau)\, \mathcal{P}(\nu_{n-1}, \mathcal{G}\nu_{n-1}, \tau)}{\mathcal{P}(\nu_n, \nu_{n-1}, \tau)} \right\}. \end{aligned}$$

This relation can be reduced to the following simplified form:

$$\begin{aligned}\mathcal{P}(\nu_{n+1},\nu_n,\tau) &\le \alpha \max\left\{\mathcal{P}(\nu_n,\nu_{n-1},\tau),\mathcal{P}(\nu_n,\nu_{n+1},\tau),\mathcal{P}(\nu_{n-1},\nu_n,\tau),\right.\\ &\qquad \left.\frac{\mathcal{P}(\nu_n,\nu_{n+1},\tau)\,\mathcal{P}(\nu_{n-1},\nu_n,\tau)}{\mathcal{P}(\nu_n,\nu_{n-1},\tau)}\right\}\\ &= \alpha\ \max\left\{\mathcal{P}(\nu_n,\nu_{n-1},\tau),\mathcal{P}(\nu_n,\nu_{n+1},\tau)\right\}.\end{aligned}$$

If the maximum value on the right-hand side is realized by $\mathcal{P}(\nu_n,\nu_{n+1},\tau)$, then

$$\begin{aligned}\mathcal{P}(\nu_{n+1},\nu_n,\tau) &\le \alpha\,\mathcal{P}(\nu_n,\nu_{n+1},\tau)\\ &< \mathcal{P}(\nu_n,\nu_{n+1},\tau),\end{aligned}$$

which gives a contradiction since $\alpha < 1$. Hence, the remaining possibility yields

$$\mathcal{P}(\nu_{n+1},\nu_n,\tau) \le \alpha\,\mathcal{P}(\nu_n,\nu_{n-1},\tau), \qquad \forall n \in \mathbb{N},\ \tau > 0.$$

By applying this inequality recursively, we obtain

$$\mathcal{P}(\nu_{n+1},\nu_n,\tau) \le \alpha^n\,\mathcal{P}(\nu_1,\nu_0,\tau).$$

Equivalently,

$$\mathcal{P}(\nu_{n+1},\nu_n,\tau) \le \alpha^n\,\zeta(\tau), \qquad \text{where } \zeta(\tau) = \mathcal{P}(\nu_1,\nu_0,\tau). \tag{5.2}$$

Let $p = \mathcal{P} - \hbar$ denote the associated exact parametric metric. From inequality (5.2), one obtains

$$p(\nu_n,\nu_{n+1},\tau) + \hbar(\nu_n,\nu_{n+1},\tau) \le \alpha^n\,\zeta(\tau), \qquad n \in \mathbb{N},\ \tau > 0.$$

Since $p(\nu_n,\nu_{n+1},\tau) \le p(\nu_n,\nu_{n+1},\tau) + \hbar(\nu_n,\nu_{n+1},\tau)$, we directly infer that

$$p(\nu_n,\nu_{n+1},\tau) \le \alpha^n\,\zeta(\tau), \qquad n \in \mathbb{N},\ \tau > 0. \tag{5.3}$$

Now, for any pair of indices $n > m$, and using (5.3), the triangle inequality for p yields

$$\begin{aligned}p(\nu_n,\nu_m,\tau) &\le p(\nu_n,\nu_{n-1},\tau) + p(\nu_{n-1},\nu_{n-2},\tau) + \cdots + p(\nu_{m+1},\nu_m,\tau)\\ &\le \left(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m\right)\zeta(\tau)\\ &\le \frac{\alpha^m}{1-\alpha}\,\zeta(\tau), \qquad \forall\,\tau > 0.\end{aligned}$$

Taking the limit as $m, n \to \infty$ in the above expression gives

$$\lim_{m,n\to\infty} p(\nu_n,\nu_m,\tau) = 0.$$

Consequently, the sequence $\{\nu_n\}$ is Cauchy in the exact parametric metric space (Y, p). Hence, $\{\nu_n\}$ is also a perturbed parametric Cauchy sequence with respect to the perturbed structure $(Y, \mathcal{P}, \hbar)$. Since $(Y, \mathcal{P}, \hbar)$ is assumed to be complete, there exists a point $\nu^* \in Y$ satisfying

$$\nu_n \longrightarrow \nu^* \quad \text{as } n \to \infty.$$

Because the mapping $\mathcal{G}$ is continuous, the limit relation gives

$$\nu^* = \lim_{n\to\infty} \nu_{n+1} = \lim_{n\to\infty} \mathcal{G}\nu_n = \mathcal{G}\nu^*.$$

Thus, ν^* is a fixed point of $\mathcal{G}$.

To establish uniqueness, suppose another fixed point $\mu^* \in Y$ exists such that $\mu^* \neq \nu^*$. By applying inequality (5.1), we obtain

$$\begin{aligned}\mathcal{P}(\nu^*, \mu^*, \tau) &= \mathcal{P}(\mathcal{G}\nu^*, \mathcal{G}\mu^*, \tau)\\ &\leq \alpha \max\left\{\mathcal{P}(\nu^*, \mu^*, \tau), \mathcal{P}(\nu^*, \mathcal{G}\nu^*, \tau), \mathcal{P}(\mu^*, \mathcal{G}\mu^*, \tau), \right.\\ &\qquad \left. \frac{\mathcal{P}(\nu^*, \mathcal{G}\nu^*, \tau)\,\mathcal{P}(\mu^*, \mathcal{G}\mu^*, \tau)}{\mathcal{P}(\nu^*, \mu^*, \tau)}\right\}\\ &= \alpha\, \mathcal{P}(\nu^*, \mu^*, \tau).\end{aligned}$$

Consequently,

$$p(\nu^*, \mu^*, \tau) + \hbar(\nu^*, \mu^*, \tau) \leq \alpha\big[p(\nu^*, \mu^*, \tau) + \hbar(\nu^*, \mu^*, \tau)\big].$$

Since $\nu^* \neq \mu^*$, it follows that $p(\nu^*, \mu^*, \tau) + \hbar(\nu^*, \mu^*, \tau) \neq 0$. The previous inequality would then imply $\alpha \geq 1$, contradicting the assumption $0 \leq \alpha < 1$. Hence, the mapping $\mathcal{G}$ admits a unique fixed point in Y. □

Corollary 1 *Let $(Y, \mathcal{P}, \hbar)$ be a complete perturbed parametric metric space, and let $\mathcal{G} : Y \to Y$ be a perturbed parametric continuous mapping satisfying*

$$\mathcal{P}(\mathcal{G}\mu, \mathcal{G}\nu, \tau) \leq \alpha \max\Big\{\mathcal{P}(\mu, \mathcal{G}\mu, \tau), \mathcal{P}(\nu, \mathcal{G}\nu, \tau)\Big\},$$

for every $\mu, \nu \in Y$ and all $\tau > 0$, where the contraction constant α satisfies $0 \leq \alpha < 1$. Then the mapping $\mathcal{G}$ possesses a unique fixed point in Y.

Corollary 2 *Let $(Y, \mathcal{P}, \hbar)$ be a complete perturbed parametric metric space, and let $\mathcal{G} : Y \to Y$ denote a perturbed parametric continuous mapping satisfying*

$$\mathcal{P}(\mathcal{G}\mu, \mathcal{G}\nu, \tau) \leq \delta\, \mathcal{P}(\mu, \nu, \tau) + \gamma\, \frac{\mathcal{P}(\mu, \mathcal{G}\mu, \tau)\,\mathcal{P}(\nu, \mathcal{G}\nu, \tau)}{\mathcal{P}(\mu, \nu, \tau)}, \tag{5.4}$$

for all distinct $\mu, \nu \in Y$ and every $\tau > 0$, where the parameters $\gamma, \delta \geq 0$ satisfy $\gamma + \delta < 1$. Then $\mathcal{G}$ admits a unique fixed point in Y.

Proof Starting from inequality (5.4), we can write

$$\begin{aligned}
&\mathcal{P}(\mathcal{G}\mu, \mathcal{G}\nu, \tau) \\
&\quad \leq \delta\, \mathcal{P}(\mu, \nu, \tau) + \gamma\, \frac{\mathcal{P}(\mu, \mathcal{G}\mu, \tau)\, \mathcal{P}(\nu, \mathcal{G}\nu, \tau)}{\mathcal{P}(\mu, \nu, \tau)} \\
&\quad \leq (\delta + \gamma)\, \max\left\{ \mathcal{P}(\mu, \nu, \tau), \frac{\mathcal{P}(\mu, \mathcal{G}\mu, \tau)\, \mathcal{P}(\nu, \mathcal{G}\nu, \tau)}{\mathcal{P}(\mu, \nu, \tau)} \right\} \\
&\quad \leq (\delta + \gamma)\, \max\left\{ \mathcal{P}(\mu, \nu, \tau), \mathcal{P}(\mu, \mathcal{G}\mu, \tau), \mathcal{P}(\nu, \mathcal{G}\nu, \tau), \frac{\mathcal{P}(\mu, \mathcal{G}\mu, \tau)\, \mathcal{P}(\nu, \mathcal{G}\nu, \tau)}{\mathcal{P}(\mu, \nu, \tau)} \right\}.
\end{aligned}$$

Thus, all the assumptions of Theorem 2 are fulfilled, which ensures that $\mathcal{G}$ admits a unique fixed point in the space Y. □

Theorem 3 *Let $(Y, \mathcal{P}, \hbar)$ be a complete perturbed parametric metric space, and let $\mathcal{G} : Y \to Y$ be a self-mapping. Suppose that for every pair $\mu, \nu \in Y$ and for all $\tau > 0$, the following inequality is satisfied*:

$$\mathcal{P}(\mathcal{G}\mu, \mathcal{G}\nu, \tau) \leq \left(\frac{\mathcal{P}(\mu, \mathcal{G}\nu, \tau) + \mathcal{P}(\nu, \mathcal{G}\mu, \tau)}{F(\tau) + \mathcal{P}(\mu, \mathcal{G}\nu, \tau) + \mathcal{P}(\nu, \mathcal{G}\mu, \tau)} \right) \mathcal{P}(\mu, \nu, \tau), \tag{5.5}$$

where $F : (0, \infty) \to (0, \infty)$ denotes a positive function. Then the mapping $\mathcal{G}$ admits a unique fixed point in Y.

Proof Let $\nu_0 \in Y$ be chosen arbitrarily and construct a sequence $\{\nu_n\} \subset Y$ by defining

$$\nu_{n+1} = \mathcal{G}\nu_n, \qquad n \in \mathbb{N}.$$

Using inequality (5.5) and substituting $\mu = \nu_n$ and $\nu = \nu_{n-1}$, we obtain

$$\begin{aligned}
\mathcal{P}(\nu_{n+1}, \nu_n, \tau) &= \mathcal{P}(\mathcal{G}\nu_n, \mathcal{G}\nu_{n-1}, \tau) \\
&\leq \left(\frac{\mathcal{P}(\nu_n, \mathcal{G}\nu_{n-1}, \tau) + \mathcal{P}(\nu_{n-1}, \mathcal{G}\nu_n, \tau)}{F(\tau) + \mathcal{P}(\nu_n, \mathcal{G}\nu_{n-1}, \tau) + \mathcal{P}(\nu_{n-1}, \mathcal{G}\nu_n, \tau)} \right) \mathcal{P}(\nu_n, \nu_{n-1}, \tau).
\end{aligned}$$

This leads to the estimate

$$\begin{aligned}
\mathcal{P}(\nu_{n+1}, \nu_n, \tau) &\leq \left(\frac{\mathcal{P}(\nu_{n-1}, \nu_{n+1}, \tau)}{F(\tau) + \mathcal{P}(\nu_{n-1}, \nu_{n+1}, \tau)} \right) \mathcal{P}(\nu_n, \nu_{n-1}, \tau) \\
&\leq \left(\frac{\mathcal{P}(\nu_{n-1}, \nu_{n+1}, \tau) + \mathcal{P}(\nu_n, \nu_{n+1}, \tau)}{F(\tau) + \mathcal{P}(\nu_{n-1}, \nu_{n+1}, \tau) + \mathcal{P}(\nu_n, \nu_{n+1}, \tau)} \right) \mathcal{P}(\nu_n, \nu_{n-1}, \tau).
\end{aligned}$$

Hence, the sequence $\{\mathcal{P}(\nu_n, \nu_{n+1}, \tau)\}$ is monotone non-increasing. Consequently, we have for all $n \geq 2$,

$$\begin{aligned}
\frac{\mathcal{P}(\nu_{n-1}, \nu_n, \tau) + \mathcal{P}(\nu_n, \nu_{n+1}, \tau)}{F(\tau) + \mathcal{P}(\nu_{n-1}, \nu_n, \tau) + \mathcal{P}(\nu_n, \nu_{n+1}, \tau)} &\leq \frac{\mathcal{P}(\nu_{n-2}, \nu_{n-1}, \tau) + \mathcal{P}(\nu_{n-1}, \nu_n, \tau)}{F(\tau) + \mathcal{P}(\nu_{n-2}, \nu_{n-1}, \tau) + \mathcal{P}(\nu_{n-1}, \nu_n, \tau)} \\
&\leq \cdots
\end{aligned}$$

$$\leq \frac{\mathcal{P}(\nu_0, \nu_1, \tau) + \mathcal{P}(\nu_1, \nu_2, \tau)}{F(\tau) + \mathcal{P}(\nu_0, \nu_1, \tau) + \mathcal{P}(\nu_1, \nu_2, \tau)}$$
$$= \alpha < 1.$$

Therefore, for every $n \in \mathbb{N}$ and $\tau > 0$,

$$\mathcal{P}(\nu_{n+1}, \nu_n, \tau) \leq \left(\frac{\mathcal{P}(\nu_{n-1}, \nu_n, \tau) + \mathcal{P}(\nu_n, \nu_{n+1}, \tau)}{F(\tau) + \mathcal{P}(\nu_{n-1}, \nu_n, \tau) + \mathcal{P}(\nu_n, \nu_{n+1}, \tau)} \right) \mathcal{P}(\nu_n, \nu_{n-1}, \tau)$$
$$\leq \alpha\, \mathcal{P}(\nu_n, \nu_{n-1}, \tau).$$

For all $n \geq 2$ and $\tau > 0$, with $0 \leq \alpha < 1$, we obtain the following estimate:

$$\mathcal{P}(\nu_{n+1}, \nu_n, \tau) \leq \alpha\, \mathcal{P}(\nu_n, \nu_{n-1}, \tau), \qquad n \in \mathbb{N},\ \tau > 0.$$

Iterating this inequality yields

$$\mathcal{P}(\nu_{n+1}, \nu_n, \tau) \leq \alpha^n\, \mathcal{P}(\nu_1, \nu_0, \tau).$$

Hence,

$$\mathcal{P}(\nu_{n+1}, \nu_n, \tau) \leq \alpha^n\, \aleph(\tau), \qquad \text{where } \aleph(\tau) = \mathcal{P}(\nu_1, \nu_0, \tau). \tag{5.6}$$

Let $p = \mathcal{P} - \hbar$ represent the exact parametric metric. From inequality (5.6), it follows that

$$p(\nu_n, \nu_{n+1}, \tau) + \hbar(\nu_n, \nu_{n+1}, \tau) \leq \alpha^n\, \aleph(\tau), \qquad n \in \mathbb{N},\ \tau > 0.$$

Since $p(\nu_n, \nu_{n+1}, \tau) \leq p(\nu_n, \nu_{n+1}, \tau) + \hbar(\nu_n, \nu_{n+1}, \tau)$, we immediately obtain

$$p(\nu_n, \nu_{n+1}, \tau) \leq \alpha^n\, \aleph(\tau), \qquad n \in \mathbb{N},\ \tau > 0.$$

By standard arguments, the preceding estimate ensures that the sequence $\{\nu_n\}$ is Cauchy with respect to the exact parametric metric p on Y. Consequently, $\{\nu_n\}$ is also a perturbed parametric Cauchy sequence in the space $(Y, \mathcal{P}, \hbar)$. Since $(Y, \mathcal{P}, \hbar)$ is complete, there exists an element $\nu^* \in Y$ such that $\nu_n \to \nu^*$ as $n \to \infty$.

Applying inequality (5.5) and taking $\mu = \nu_n$ and $\nu = \nu^*$, we obtain

$$\mathcal{P}(\nu_{n+1}, \mathcal{G}\nu^*, \tau) = \mathcal{P}(\mathcal{G}\nu_n, \mathcal{G}\nu^*, \tau)$$
$$\leq \left(\frac{\mathcal{P}(\nu_n, \mathcal{G}\nu^*, \tau) + \mathcal{P}(\nu^*, \mathcal{G}\nu_n, \tau)}{F(\tau) + \mathcal{P}(\nu_n, \mathcal{G}\nu^*, \tau) + \mathcal{P}(\nu^*, \mathcal{G}\nu_n, \tau)} \right) \mathcal{P}(\nu_n, \nu^*, \tau). \tag{5.7}$$

Taking the limit as $n \to \infty$ in inequality (5.7), and using the convergence $\nu_n \to \nu^*$, we obtain

$$\mathcal{P}(\nu^*, \mathcal{G}\nu^*, \tau) \leq 0.$$

Consequently,

$$p(\nu^*, \mathcal{G}\nu^*, \tau) + \hbar(\nu^*, \mathcal{G}\nu^*, \tau) \leq 0,$$

which immediately yields $p(\nu^*, \mathcal{G}\nu^*, \tau) = 0$. Hence, we conclude that $\nu^* = \mathcal{G}\nu^*$, and thus ν^* is a fixed point of $\mathcal{G}$. □

6 An application to the fixed-circle problem

In this section, we present an application of the preceding theoretical results to the study of the *fixed-circle problem* in the framework of perturbed parametric metric spaces. This problem originates from the observation that the set of fixed points of a self-mapping may exhibit substantial structural variability: a mapping may possess no fixed point, a unique fixed point, or even infinitely many. Such variability motivates a natural geometric question: when a mapping admits more than one fixed point, can the collection of all such points be described by a geometric locus, and in particular, by a circle? The investigation of this question constitutes the essence of the fixed-circle problem.

The fixed-circle concept was first introduced by Özgür and Taş in the setting of standard metric spaces [31]. Since that initial contribution, the topic has attracted considerable attention, leading to a variety of extensions and refinements within both classical metric spaces and more generalized metric frameworks (see, for instance, [12–14, 37–39] and the references therein).

In the present section, we extend this line of inquiry to the class of perturbed parametric metric spaces. By adapting analytical techniques developed in the earlier sections, we derive several fixed-circle results that demonstrate how perturbations within the parametric structure influence the geometry and existence of such circles.

Definition 4 Let $(Y, \mathcal{P}, \hbar)$ be a perturbed parametric metric space, and let $\mu_0 \in Y$ and $r > 0$. The *circle of center* μ_0 *and radius* r in the space $(Y, \mathcal{P}, \hbar)$ is defined by

$$C^{\mathcal{P}}_{\mu_0, r} = \left\{\mu \in Y : \mathcal{P}(\mu, \mu_0, \tau) = r\right\},$$

where $\tau > 0$ denotes the parametric variable.

Example 10 Consider the perturbed parametric metric space endowed with the mapping

$$\mathcal{P}(\mu, \nu, \tau) = \frac{|\mu - \nu|}{\tau} + \frac{|\mu\nu|}{1 + \tau^2}, \qquad \tau > 0,$$

as introduced in Example 7. Let $\mu_0 = 0$ and $r = 3$. Then the corresponding circle is given by

$$\begin{aligned} C^{\mathcal{P}}_{0,3} &= \{\mu \in Y : \mathcal{P}(\mu, 0, \tau) = 3\} \\ &= \{\mu \in Y : |\mu| = 3\tau\}. \end{aligned}$$

Hence, the number of the circle $C^{\mathcal{P}}_{0,3}$ is infinite.

Definition 5 Let $(Y, \mathcal{P}, \hbar)$ be a perturbed parametric metric space, and let $C^{\mathcal{P}}_{\mu_0,r}$ denote a circle in Y. Consider a self-mapping $\mathcal{G} : Y \to Y$. If $\mathcal{G}\mu = \mu$ for every $\mu \in C^{\mathcal{P}}_{\mu_0,r}$, then the circle $C^{\mathcal{P}}_{\mu_0,r}$ is said to be a *fixed circle* of $\mathcal{G}$.

Theorem 4 *Let $(Y, \mathcal{P}, \hbar)$ be a perturbed parametric metric space, and let $C^{\mathcal{P}}_{\mu_0,r}$ be a circle in Y. Define the function $\varphi : Y \times (0,\infty) \to [0,\infty)$ by*

$$\varphi(\mu,\tau) = \mathcal{P}(\mu,\mu_0,\tau), \qquad \mu \in Y,\ \tau > 0.$$

Assume that there exists a self-mapping $\mathcal{G} : Y \to Y$ satisfying:

(P_1) $(\mathcal{P} - \hbar)(\mu, \mathcal{G}\mu, \tau) \le \varphi(\mu,\tau) - \varphi(\mathcal{G}\mu,\tau)$,

(P_2) $\mathcal{P}(\mathcal{G}\mu,\mu_0,\tau) \ge r$,

for all $\mu \in C^{\mathcal{P}}_{\mu_0,r}$ and all $\tau > 0$. Then $C^{\mathcal{P}}_{\mu_0,r}$ is a fixed circle of $\mathcal{G}$.

Proof Let $\mu \in C^{\mathcal{P}}_{\mu_0,r}$. From conditions ($\mathrm{P}_1$) and ($\mathrm{P}_2$), it follows that

$$\begin{aligned}(\mathcal{P} - \hbar)(\mu, \mathcal{G}\mu, \tau) &\le \varphi(\mu,\tau) - \varphi(\mathcal{G}\mu,\tau)\\ &= \mathcal{P}(\mu,\mu_0,\tau) - \mathcal{P}(\mathcal{G}\mu,\mu_0,\tau)\\ &= r - \mathcal{P}(\mathcal{G}\mu,\mu_0,\tau)\\ &\le r - r = 0.\end{aligned}$$

Since $(\mathcal{P} - \hbar)(\mu, \mathcal{G}\mu, \tau) = 0$ if and only if $\mu = \mathcal{G}\mu$, we deduce that every point $\mu \in C^{\mathcal{P}}_{\mu_0,r}$ is a fixed point of $\mathcal{G}$. Hence, the circle $C^{\mathcal{P}}_{\mu_0,r}$ is a fixed circle of $\mathcal{G}$. □

Theorem 5 *Let $(Y, \mathcal{P}, \hbar)$ be a perturbed parametric metric space, and let $C^{\mathcal{P}}_{\mu_0,r}$ denote a circle in Y. Define the function $\varphi : Y \times (0,\infty) \to [0,\infty)$ as in Theorem 4. Assume that there exists a self-mapping $\mathcal{G} : Y \to Y$ satisfying the following conditions*:

(P_3) $(\mathcal{P} - \hbar)(\mu, \mathcal{G}\mu, \tau) \le \varphi(\mu,\tau) + \varphi(\mathcal{G}\mu,\tau) - 2r$,

(P_4) $\mathcal{P}(\mathcal{G}\mu,\mu_0,\tau) \le r$,

for every $\mu \in C^{\mathcal{P}}_{\mu_0,r}$ and all $\tau > 0$. Then the circle $C^{\mathcal{P}}_{\mu_0,r}$ is a fixed circle of $\mathcal{G}$.

Proof Let $\mu \in C^{\mathcal{P}}_{\mu_0,r}$. From conditions ($\mathrm{P}_3$) and ($\mathrm{P}_4$), we have

$$\begin{aligned}(\mathcal{P} - \hbar)(\mu, \mathcal{G}\mu, \tau) &\le \varphi(\mu,\tau) + \varphi(\mathcal{G}\mu,\tau) - 2r\\ &= \mathcal{P}(\mu,\mu_0,\tau) + \mathcal{P}(\mathcal{G}\mu,\mu_0,\tau) - 2r\\ &\le 2r - 2r = 0.\end{aligned}$$

Hence,

$$(\mathcal{P} - \hbar)(\mu, \mathcal{G}\mu, \tau) = 0 \iff \mu = \mathcal{G}\mu.$$

Therefore, every element $\mu \in C^{\mathcal{P}}_{\mu_0,r}$ is a fixed point of $\mathcal{G}$, and thus the circle $C^{\mathcal{P}}_{\mu_0,r}$ is a fixed circle of $\mathcal{G}$. □

Example 11 Let $(Y, \mathcal{P}, \hbar)$ be a perturbed parametric metric space, and let $C^{\mathcal{P}}_{\mu_0,r}$ denote a circle in Y. Define a self-mapping $\mathcal{G} : Y \to Y$ by

$$\mathcal{G}\mu = \begin{cases} \mu, & \text{if } \mu \in C^{\mathcal{P}}_{\mu_0,r}, \\ \mu_0, & \text{otherwise,} \end{cases}$$

for all $\mu \in Y$. It is straightforward to verify that $\mathcal{G}$ satisfies the conditions (P_1)–(P_4) with the function $\varphi : Y \times (0,\infty) \to [0,\infty)$ defined as in Theorem 4. Consequently, the circle $C^{\mathcal{P}}_{\mu_0,r}$ is a fixed circle of the mapping $\mathcal{G}$.

Example 12 Let $(Y, \mathcal{P}, \hbar)$ be a perturbed parametric metric space, and let $C^{\mathcal{P}}_{\mu_1,r_1}$ and $C^{\mathcal{P}}_{\mu_2,r_2}$ be two circles in Y. Define a self-mapping $\mathcal{G} : Y \to Y$ by

$$\mathcal{G}\mu = \begin{cases} \mu, & \text{if } \mu \in C^{\mathcal{P}}_{\mu_1,r_1} \cup C^{\mathcal{P}}_{\mu_2,r_2}, \\ \mu_0, & \text{otherwise,} \end{cases}$$

for all $\mu \in Y$.

Then $\mathcal{G}$ satisfies the conditions (P_1)–(P_4) with the functions $\varphi_1, \varphi_2 : Y \times (0,\infty) \to [0,\infty)$ defined respectively by

$$\varphi_1(\mu,\tau) = \mathcal{P}(\mu,\mu_1,\tau), \qquad \varphi_2(\mu,\tau) = \mathcal{P}(\mu,\mu_2,\tau).$$

Consequently, both circles $C^{\mathcal{P}}_{\mu_1,r_1}$ and $C^{\mathcal{P}}_{\mu_2,r_2}$ are fixed circles of the mapping $\mathcal{G}$.

If we examine Theorems 4 and 5, it becomes clear that they serve as *existence theorems* for the notion of a fixed circle. Furthermore, Example 12 illustrates that the fixed circle associated with a given self-mapping need not be unique. This observation naturally motivates the study of conditions ensuring *uniqueness* in the context of fixed-circle results. To this end, we establish the following theorem.

Theorem 6 *Let $(Y, \mathcal{P}, \hbar)$ be a perturbed parametric metric space, and let $C^{\mathcal{P}}_{\mu_0,r}$ be a circle in Y. Suppose that $\mathcal{G} : Y \to Y$ is a self-mapping satisfying*

$$(\mathcal{P}-\hbar)(\mathcal{G}\mu,\mathcal{G}\nu,\tau) \le \alpha \max\left\{ \begin{array}{c} (\mathcal{P}-\hbar)(\mu,\nu,\tau),\ (\mathcal{P}-\hbar)(\mu,\mathcal{G}\mu,\tau),\ (\mathcal{P}-\hbar)(\nu,\mathcal{G}\nu,\tau), \\ \dfrac{(\mathcal{P}-\hbar)(\mu,\mathcal{G}\mu,\tau)\,(\mathcal{P}-\hbar)(\nu,\mathcal{G}\nu,\tau)}{(\mathcal{P}-\hbar)(\mu,\nu,\tau)} \end{array} \right\},$$

for all $\mu \in C^{\mathcal{P}}_{\mu_0,r}$ and $\nu \in Y \setminus C^{\mathcal{P}}_{\mu_0,r}$, where $0 \le \alpha < 1$. Then the circle $C^{\mathcal{P}}_{\mu_0,r}$ is the unique fixed circle *of the mapping $\mathcal{G}$.*

Proof Assume that there exist two fixed circles $C^{\mathcal{P}}_{\mu_1,r_1}$ and $C^{\mathcal{P}}_{\mu_2,r_2}$ of the self-mapping $\mathcal{G}$. Let $\mu \in C^{\mathcal{P}}_{\mu_1,r_1}$ and $\nu \in C^{\mathcal{P}}_{\mu_2,r_2}$ be distinct points, i.e., $\mu \neq \nu$. Since both points are fixed under $\mathcal{G}$, we have $\mathcal{G}\mu = \mu$ and $\mathcal{G}\nu = \nu$. Using the contractive condition in the hypothesis, we obtain

$$(\mathcal{P}-\hbar)(\mathcal{G}\mu,\mathcal{G}\nu,\tau) = (\mathcal{P}-\hbar)(\mu,\nu,\tau)$$

$$\leq \alpha \max\left\{(\mathcal{P}-\hbar)(\mu,\nu,\tau),\ (\mathcal{P}-\hbar)(\mu,\mathcal{G}\mu,\tau),\ (\mathcal{P}-\hbar)(\nu,\mathcal{G}\nu,\tau),\right.$$
$$\left.\frac{(\mathcal{P}-\hbar)(\mu,\mathcal{G}\mu,\tau)\,(\mathcal{P}-\hbar)(\nu,\mathcal{G}\nu,\tau)}{(\mathcal{P}-\hbar)(\mu,\nu,\tau)}\right\}$$
$$= \alpha \max\{(\mathcal{P}-\hbar)(\mu,\nu,\tau),0,0,0\}$$
$$= \alpha\,(\mathcal{P}-\hbar)(\mu,\nu,\tau).$$

Because $0 \leq \alpha < 1$, this inequality implies that $(\mathcal{P}-\hbar)(\mu,\nu,\tau) = 0$, and hence $\mu = \nu$. This contradicts the assumption that $\mu \neq \nu$. Therefore, the fixed circle of $\mathcal{G}$ must be unique. □

Further generalizations can be provided by extending the fixed-circle approach to settings governed by integral-type contractive conditions in the sense of Branciari [3].

Corollary 3 *Let $(Y,\mathcal{P},\hbar)$ be a perturbed parametric metric space, and let $C^{\mathcal{P}}_{\mu_0,r}$ denote a circle in Y. Define the function $\varphi : Y \times (0,\infty) \to [0,\infty)$ as in Theorem 4. Assume that there exists a self-mapping $\mathcal{G} : Y \to Y$ satisfying*

(IP_1)

$$\int_0^{(\mathcal{P}-\hbar)(\mu,\mathcal{G}\mu,\tau)} \psi(s)\,ds \ \leq\ \int_0^{\varphi(\mu,\tau)-\varphi(\mathcal{G}\mu,\tau)} \psi(s)\,ds,$$

(IP_2)

$$\mathcal{P}(\mathcal{G}\mu,\mu_0,\tau) \geq r,$$

for all $\mu \in C^{\mathcal{P}}_{\mu_0,r}$ and $\tau > 0$, where $\psi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable function that is summable on every compact subset of $[0,\infty)$, nonnegative, and satisfies

$$\int_0^{\varepsilon} \psi(s)\,ds > 0 \quad \textit{for each } \varepsilon > 0.$$

Then the circle $C^{\mathcal{P}}_{\mu_0,r}$ is a fixed circle of $\mathcal{G}$.

7 Conclusion

In this paper, we introduced the notion of a Perturbed Parametric Metric Space (PPMS) as a flexible generalization of classical metric structures, in which distance measurements are influenced by perturbation effects and an additional non-negative parameter. This framework is designed to model perturbative variations and measurement imperfections in complex systems, thereby providing a more realistic and adaptable analytical setting.

We established the fundamental structural and topological properties of PPMS and illustrated the proposed framework through representative examples. Within this setting, several fixed point results were derived for self-mappings satisfying generalized contractive conditions. The application to the fixed-circle problem demonstrated the effectiveness of the PPMS framework in providing refined analytical tools and deeper insight into fixed-circle phenomena beyond classical metric spaces. The results obtained unify and extend a variety of existing metric-type structures, highlighting the versatility and robustness of the proposed approach. The PPMS framework thus opens new avenues for further

investigations, including extensions to fuzzy, probabilistic, or graphical settings, as well as applications to nonlinear integral equations, optimization problems, and dynamical systems under perturbation.

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