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Abstract

In applications requiring uncertain, imprecise, and multi-dimensional data, where traditional distance measures frequently fall short of capturing the full complexity of interactions among elements, a distance measure for complex intuitionistic fuzzy sets (DMCIFSs) becomes essential. Although DMCIFSs have been developed, most of them do not account for the hesitation degree, which is crucial for capturing ambiguity and uncertainty in human reasoning. As extensions of the normalized Hamming and Euclidean distance measures, this work proposes two new measures namely the Hesitance DMCIFSs (HDMCIFSs) and the Euclidean Hesitance DMCIFSs (EHDM-CIFSs). These newly proposed measures provide a more comprehensive framework for modeling uncertainty by explicitly incorporating the hesitancy component. In addition to the proposed measures, several fundamental procedures and theoretical results are also presented. Furthermore, a novel decision-making method utilizing these distance measures is developed and applied to multi-criteria decision-making (MCDM) problems. The effectiveness of the proposed methods is demonstrated through a comparative study, highlighting their potential for improved sensitivity and accuracy in practical decision-making scenarios.

1. Introduction

1.1. Fuzzy set and intuitionistic fuzzy set

Fuzzy set theory was approached by Zadeh [1]. It is a useful mathematical tool for dealing with fuzziness and uncertainty. It may be applied to multi-attribute decision-making (MADM) to manage imprecision and fuzziness between attributes. Based on the basic ideas of fuzzy-set concepts [2], which Pedrycz addressed a variety of real-world issues. A technique for making choices in soil was presented by McBratney and Odeh [3] fuzzy logic and fuzzy measurement-based sciences. Applications of FSs to guarantee security and dependability engineering were studied by Kabir and Papadopoulos [4]. Atanassov [5] then raised intuitionistic fuzzy sets (IFSs), presenting fuzzy set-based non-membership degrees. This foundational work was later expanded through the definition of new operations on IFSs [6] and further theoretical developments presented in [7]. The distance measure for fuzzy sets that are IFSs is defined axiomatically and a few distance metrics are suggested, along with the supporting evidence. Analysis is done on the IFSs' distance and similarity measures. In circumstances where a spoken attribute indicates an issue that seems excessively rough, intuitionistic fuzzy sets are highly helpful, as demonstrated by Szmidt *et al* [8] provided in terms of the benefits of membership. IFS is a technique for more human-consistent reasoning under inadequately described facts because of its adaptability in handling ambiguity. The IFS model's structure makes it more important than the FS model. Numerous fields, like data mining, image processing etc [9], have made use of intuitionistic fuzzy sets. Nevertheless, two-dimensional issues are outside the scope of all three models.

1.2. Complex fuzzy set and complex intuitionistic fuzzy set

The concept of complex fuzzy sets (CFSs) was introduced by Ramot *et al* [10] to address two-dimensional phenomena. A CFS is an extension of fuzzy set with a complex-valued membership function, meaning that its unit disk in a complex plane represented by the range. The complex-valued function uses both magnitude and phase information to represent a member's level of participation in the CFS. Numerous domains, including image processing, decision-making and pattern recognition, have found use for CFSs [11, 12] also offer a strong instrument for imprecisely and uncertainly constructing and assessing complicated structures. A decision making approach based on complex fuzzy entropy measurements and complex fuzzy distance measure (CFDM) was presented by Liu *et al* [13].

1.3. Distance measures

Signal analysis method for the CFDM system. New DM was established in the context of CFSs by Zeeshan *et al* [14]. In the context of CFSs, Khan *et al* [15] presented a novel technique for separating key signals from big signals. Zeeshan *et al* [16] developed a novel technique for locating a reference signal using sophisticated fuzzy soft settings' distances function and the Hamming distance method to calculate the separation between each student and each department. As a result, Chen *et al* [17] created a number of appropriate distance measures for higher order double hesitant fuzzy sets (HODHFS) and proposed the concept of HODHFS. Those who are advantageous in circumstances requiring a more sophisticated approach to membership and non-membership due to ambiguity and uncertainty. In the IFSs [18] environment, Xiao defined a new DM based on the Euclidean and cosine DM [19], Alkan and Kahraman presented a novel intuitionistic fuzzy multi-distance metric. The decision-making process for waste disposal destination selection applications was established by them. He created a pattern categorization technique based upon the IF distance measure (IFDM) that was suggested Gohain *et al* [20], a new IFDM was proposed. In order to enhance decision-making in the face of ambiguity, Khan *et al* [21] suggested new DMCF that incorporate hesitance values. Their approach provided improved discrimination and adaptability, demonstrating efficacy in intricate multi-attribute situations. For decision-making, clustering, and pattern recognition tasks. Multi-attribute making choices [22] has made extensive use of intuitionistic fuzzy measures. However, only a small number of distance measurements consider individual inclinations in the intricate process of making choices. In this study, we provide a unique emergency decision-making approach employing the IFDM, which is based on decision makers' tendencies. According to Dutta *et al* [23], the idea of PFDMs that are not linear. To solve issues with pattern recognition, medical diagnostics, and choosing COVID-19 medications. The concept of a new DM in the context of Pythagorean fuzzy sets (PFSs) was developed by Yin *et al* [24]. To solve MCDM issues, healthcare diagnosis methods, and recognition of patterns algorithms. Compared to traditional set theory, FS, IFS, PFS all provide a more adaptable and subtle depiction of uncertainty and imprecision. These sets have been used in image processing, pattern identification, and decision-making [25]. For challenges involving pattern recognition and medical diagnostics. By adding the standard parameters that characterize PFSs [26], Ejegwa and Awolola presented a few additional DMs. Suggested DM to issues involving pattern recognition in the context of PFSs, Li *et al* [27] created novel DMs and similarity metrics. Also, demonstrated how it may be used to choose advertising platforms Alkouri and Salleh [28] used linguistic variables and hedges to investigate distance measures for complicated fuzzy sets, providing tools for uncertainty-based decision-making.

Zeng *et al* [29] introduced a novel distance measure that accurately represents the link between membership and non-membership degrees, resulting in enhanced pattern recognition performance. Similarly, Mahanta and Panda [30] proposed a generalized distance measure capable of reflecting uncertainty more fully, resulting in better grouping and decision-making results than traditional approaches. Ashraf *et al* [31] developed a difference sequence-based distance measure to highlight variances among fuzzy elements and shown its resilience in decision-making applications. Garg and Rani [32] expanded on geometric viewpoints by developing distance measurements based on the triangle centers of isosceles triangular fuzzy numbers, resulting in improved sensitivity and interpretability in similarity evaluation. Thao *et al* [33] developed a score-function-based distance framework to better quantify the closeness of options, hence increasing the efficiency of fuzzy decision systems. Meanwhile, Xiao *et al* [34] proposed an enhanced distance metric that incorporates weighing and normalization procedures, resulting in increased classification accuracy in pattern recognition tasks. Extending to interval-valued environments, Qin *et al* [35] proposed a novel distance measure combined with a

TOPSIS technique for multi-attribute decision-making, resulting in stable rankings and enhanced flexibility under uncertainty. Collectively, these research demonstrate the ongoing growth of distance measurements for IFSs, emphasizing both theoretical precision and practical application in complicated decision-support and pattern recognition settings. Alkouri and Salleh [36] proposed CIFSs, expanding classical intuitionistic fuzzy sets with complex membership and non-membership values. Alkouri and Salleh [37] introduced operations on CIFSs, such as aggregation and intersection, to enhance the computational foundation for processing complicated fuzzy information. Zhang et al [38] studied the operational properties and δ equalities of complex fuzzy sets, formalizing their behavior under union, intersection, and complement. In [39] discussed the square root interval-valued normal pythagorean fuzzy multi-attribute decision-making model based on Aggregation operators. IF distance measures and their uses in medical analysis and decision-making have been the subject of more recent research. In order to improve classification under uncertainty, Onoja et al [40] presented weighted IF distance metrics for pattern identification and disease diagnosis. Anum et al [41] created a weighted distance measure based on tendency coefficients that improves IF decision models' dependability. Through an admission determination process, Ejegwa et al [42] presented a new distance formulation with higher discrimination capacity. Nwokoro et al [43] successfully captured uncertainty in clinical assessments by using an IF technique to predict maternal outcomes in the healthcare domain. Kılınç [44] used logistic entropy to interpret trabecular bone scores in SLE patients, whereas Cekim et al [45] applied it to evaluate bone mineral density in diabetic and non-diabetic groups. These studies show that advanced fuzzy and entropy-based measures are becoming increasingly important in dealing with uncertainty in both decision-making and healthcare settings.

The primary goal of this study proposes two new DMs based on CIFSs, HDMCIF and EHDMCIF, which are the generalization of NHDMCIF and NEDMCIF. We also discussed the applications of the proposed DMCIFs in solving decision-making problems, presented a new decision making algorithm that integrates DMCIFs into decision-making procedures, offering a reliable methodology for managing real-world complexities, and compared the suggested DMCIFs with normalized DMCIFs.

1.4. Research gap

Numerous distance measures have been created for fuzzy, intuitionistic, and complex fuzzy sets. However, the majority of existing models do not take into account both the hesitance degree and the complex-valued phase component. For example, classic CIFS based distance measurements concentrate primarily on amplitude information, ignoring the uncertainty caused by indecision or missing preference data. Recently proposed hesitance aware frameworks, such as PFSs and q-ROFSs, allow for greater flexibility in modeling uncertainty, but they are limited to real-valued domains and do not model phase dependent connections. As a result, there is an methodological gap in establishing a comprehensive distance measure that combines magnitude, phase, and hesitance information in a single framework. Addressing this gap is critical to improving the interpretability and realism of uncertainty modeling in complicated decision-making settings.

1.5. Motivation

Real-world multi-criteria decision-making (MCDM) situations commonly incorporate imprecise, hesitant, and cyclic information, such as time-dependent preferences or periodic signals. Conventional distance metrics in fuzzy or intuitionistic fuzzy contexts do not capture this complexity, resulting in information loss or biased evaluations. Motivated by these restrictions, this study presents hesitance distance measures for complex intuitionistic fuzzy sets (HDMCIFS and EHDMCIFS), which explicitly include the hesitance degree as well as amplitude and phase parameters. This integration allows for a richer and more accurate measurement of similarity and dissimilarity across complex fuzzy elements, boosting the robustness of decision-making models under high uncertainty.

1.6. Contribution

The HDMCIF and EHDMCIF are the two new distance measures for CIFSs that specifically include hesitance values in addition to membership and non-membership functions, are presented in this work. These metrics offer a more thorough and accurate depiction of doubt and indecision in intricate decision-making situations by include the degree of hesitance. The suggested metrics improve the modeling of ambiguity in CIFS settings by generalizing the current normalized Euclidean and Hamming distances. Furthermore, the study creates a strong algorithm for decision-making that makes use of these novel metrics, enhancing the precision and dependability of MCDM procedures. The usefulness of the suggested approaches is demonstrated by a comparison with current distance models, underscoring its potential uses in pattern identification, medical diagnosis, preference modeling, and other practical decision support systems.

Table 1. Notation table.

Symbol	Meaning
$\mathfrak{L}_{\mathcal{S}}(s)$	Membership function
$\mathfrak{J}_{\mathcal{S}}(s)$	Non membership function
$\mathfrak{h}_{\mathcal{S}}(s)$	Hesitance Degree of Complex Intuitionistic Fuzzy Set
$d_z(\mathcal{S}_1, \mathcal{S}_2)$	Zhang Distance Measure
$d_H(\mathcal{S}_1, \mathcal{S}_2)$	Hamming Distance Measure
$d_{EH}(\mathcal{S}_1, \mathcal{S}_2)$	Euclidean Distance Measure
FS	Fuzzy Set
IFS	Intuionistic Fuzzy Set
CFS	Complex Fuzzy Set
CIFS	Complex Intuitionistic Fuzzy Set
DMCIFSs	Distance Measure for Complex Intuitionistic Fuzzy
HDMCIFSs	Hamming Hesitance Distance Measure for Complex Intuitionistic Fuzzy
EHDMCIFSs	Euclidean Hesitance Distance Measure for Complex Intuitionistic Fuzzy
$p_{\mathcal{S}}(s)$	Amplitude Term of Membership Function
$q_{\mathcal{S}}(s)$	Amplitude Term of Non Membership Function
$m_{\mathcal{S}}(s)$	Phase Term of Membership Function
$n_{\mathcal{S}}(s)$	Phase Term of Non Membership Function

1.7. Novelty of the Study

By adding explicit hesitance components along with phase information, the suggested HDMCIFS and EHDMCIFS measures expand on conventional Hamming and Euclidean distances inside the complex intuitionistic fuzzy framework. The current method enables the joint consideration of amplitude and phase uncertainties, which captures cyclic or rotational characteristics frequently seen in real-world signals, preferences, or expert judgments, in contrast to previous measures that treat hesitance implicitly or limit analysis to real-valued domains. Instead of being a straightforward parameterized extension, the suggested measures are conceptually unique due to their dual-level representation. Additionally, unlike previous distance definitions, hesitance-based normalization guarantees that the new measures satisfy metric axioms under complex-valued conditions.

To enable more flexibility in representing reluctance, a number of extensions of intuitionistic fuzzy theory have been proposed recently, including Pythagorean fuzzy sets and q-rung orthopair fuzzy sets. By reducing the membership and nonmembership restriction to $\mu^2 + \nu^2 \leqslant 1$ for PFS and $\mu^p + \nu^p \leqslant 1$ for q-ROFS, both frameworks provide an extra degree of freedom and enable higher levels of uncertainty representation. However, phase information that appears in complex-valued data is not captured by these models, which are limited to real-valued domains. The proposed HDMCIFS and EHDMCIFS extend beyond these limitations by integrating both magnitude and phase components of membership and non-membership, along with an explicit hesitance term. This combination enables richer representation of uncertainty in two-dimensional or cyclic phenomena, which cannot be effectively modeled using PFS or q-ROFS. Therefore, the proposed framework generalizes the notion of hesitance-aware distance measures to the complex plane, enhancing both interpretability and application scope in decision-making and pattern recognition contexts.

Notation table: To improve the clarity and consistency of mathematical expressions used in this work, the main symbols, parameters, and operators are summarized in the following notation table 1.

2. Basic concepts

Definition 1. [1] Let $\mathcal{P}$ be a non empty set. An FS $\mathcal{S}$ is defined as $\mathcal{S} = [s, \mathfrak{T}_{\mathcal{S}}(s): s \in \mathcal{P}]$. Where $\mathfrak{T}_{\mathcal{S}}(s): \mathcal{P} \to [0, 1]$ indicates truth grade of $\mathcal{S}$ for any element $s \in \mathcal{A}$.

Definition 2. [5] Let $\mathcal{P}$ be a non empty set. An IFS $\mathcal{S}$ is defined as $\mathcal{S} = [(s, (\mathfrak{L}_{\mathcal{S}}(s), \mathfrak{J}_{\mathcal{S}}(s))): s \in \mathcal{P}]$.where $\mathfrak{L}_{\mathcal{S}}(s): \mathcal{P} \to [0, 1]$ and $\mathfrak{J}_{\mathcal{S}}(s): \mathcal{P} \to [0, 1]$ indicates truth and false grade of $\mathcal{S}$ for any element $s \in \mathcal{P}$, and it satisfies the condition $0 \leqslant (\mathfrak{L}_{\mathcal{S}}(s) + \mathfrak{J}_{\mathcal{S}}(s)) \leqslant 1$.

Definition 3. [36] A CIFS $\mathcal{S}$ is characterized by a grade value $\mathfrak{L}_{\mathcal{S}}(s) = p_{\mathcal{S}}(s)e^{im_{\mathcal{S}}(s)}$ and $\mathfrak{J}_{\mathcal{S}}(s) = q_{\mathcal{S}}(s)e^{in_{\mathcal{S}}(s)}$ on a universe of discourse $\mathcal{P}$. In a CIFS $\mathcal{S}$, the GV $\mathfrak{L}_{\mathcal{S}}(s)$ and $\mathfrak{J}_{\mathcal{S}}(s)$ of an element $s \in \mathcal{P}$, is determined by a intuitionistic complex-valued function rather than a simple grade value. In the grade value of CIFS $\mathcal{S}$, the term $p_{\mathcal{S}}(s)$ is said to be an amplitude term of membership function, the term $q_{\mathcal{S}}(s)$ is said to be an amplitude term of non membership function and $m_{\mathcal{S}}(s)$ is called the phase term for membership function and $n_{\mathcal{S}}(s)$ is called the phase term for non membership function. Note the both these functions are real-valued and $p_{\mathcal{S}}(s),\ q_{\mathcal{S}}(s) \in [0,\ 1]$. The hesitance degree of CIFS $\mathcal{S}$ is denoted by $\mathfrak{h}_{\mathcal{S}}(s)$ and is given by for membership function,

$$\mathfrak{h}_{\mathcal{S}}(s) = |(1 - p_{\mathcal{S}}(s)) - \frac{1}{2\pi}(2\pi - m_{\mathcal{S}}(s))|$$

for non membership function,

$$\mathfrak{h}_{\mathcal{S}}(s) = |(1 - q_{\mathcal{S}}(s)) - \frac{1}{2\pi}(2\pi - n_{\mathcal{S}}(s))|$$

mathematically, it can be written as:

$$\begin{aligned}\mathcal{S} &= \{(s,\ (\mathfrak{L}_{\mathcal{S}}(s),\ \mathfrak{J}_{\mathcal{S}}(s))):\ s \in \mathcal{P}\}\\ &= \{p_{\mathcal{S}}(s)e^{im_{\mathcal{S}}(s)},\ q_{\mathcal{S}}(s)e^{in_{\mathcal{S}}(s)}:\ s \in \mathcal{P}\}\end{aligned}$$

Definition 4. [36] Let $\mathcal{S}_n,\ n = 1,\ 2,\ 3,\ldots,N$ be N CIFS defined on $\mathcal{P}$ and $\mathfrak{L}_{\mathcal{S}}(s) = p_{\mathcal{S}}(s)e^{im_{\mathcal{S}}(s)}$ and $\mathfrak{J}_{\mathcal{S}}(s) = q_{\mathcal{S}}(x)e^{in_{\mathcal{S}}(s)}$ their grade values for membership and non membership function. The complex intuitionistic fuzzy(CIF) cartesian product of $\mathcal{S}_n$ represented by $\prod_{i=1}^{n} \mathcal{S}_i = \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3 \times \cdots \times \mathcal{S}_n$ is defined by a function

$$\begin{aligned}\mathfrak{L}_{\prod_{i=1}^{n}\mathcal{S}_i}(s) &= p_{\prod_{i=1}^{n}\mathcal{S}_i}(s)e^{im_{\prod_{i=1}^{n}\mathcal{S}_i}(s)}\\ &= \min(p_{\mathcal{S}_1}(s_1),\ p_{\mathcal{S}_2}(s_2),\ p_{\mathcal{S}_3}(s_3),\ldots,p_{\mathcal{S}_n}(s_n))e^{imin(m_{\mathcal{S}_1}(s_1),m_{\mathcal{S}_2}(s_2),\ldots,m_{\mathcal{S}_n}(s_n))}\\ \mathfrak{J}_{\prod_{i=1}^{n}\mathcal{S}_i}(s) &= q_{\prod_{i=1}^{n}\mathcal{S}_i}(s)e^{in_{\prod_{i=1}^{n}\mathcal{S}_i}(s)}\\ &= \max(q_{\mathcal{S}_1}(s_1),\ q_{\mathcal{S}_2}(s_2),\ q_{\mathcal{S}_3}(s_3),\ldots,q_{\mathcal{S}_n}(s_n))e^{imax(n_{\mathcal{S}_1}(s_1),n_{\mathcal{S}_2}(s_2),\ldots,n_{\mathcal{S}_n}(s_n))}\end{aligned}$$

Definition 5. [36] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two CIFS on $\mathfrak{L}_{\mathcal{S}_1(s)} = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}$ and $\mathfrak{L}_{\mathcal{S}_2(s)} = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}$, $\mathfrak{J}_{\mathcal{S}_1(s)} = q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)}$ and $\mathfrak{J}_{\mathcal{S}_2(s)} = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}$ their grade values of membership and non membership function. The CIF intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ is denoted by $\mathcal{S}_1 \cap \mathcal{S}_2$ and is defined by a function

$$\begin{aligned}\mathcal{S}_1 \cap \mathcal{S}_2 &= p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)} \cap p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}\\ &= \min(p_{\mathcal{S}_1}(s_1),\ p_{\mathcal{S}_2}(s_2))e^{imin(m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s))}\\ \mathcal{S}_1 \cap \mathcal{S}_2 &= q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)} \cap q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}\\ &= \max(q_{\mathcal{S}_1}(s_1),\ q_{\mathcal{S}_2}(s_2))e^{imax(n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s))}\end{aligned}$$

Definition 6. [36] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two CIFS on $\mathfrak{L}_{\mathcal{S}_1(s)} = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}$ and $\mathfrak{L}_{\mathcal{S}_2(s)} = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}$, $\mathfrak{J}_{\mathcal{S}_1(s)} = q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)}$ and $\mathfrak{J}_{\mathcal{S}_2(s)} = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}$ their grade values of membership and non membership function. The CIF union of $\mathcal{S}_1$ and $\mathcal{S}_2$ is denoted by $\mathcal{S}_1 \bigcup\ \mathcal{S}_2$ and is defined by a function

$$\begin{aligned}\mathcal{S}_1 \bigcup\ \mathcal{S}_2 &= p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)} \bigcup\ p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}\\ &= \max(p_{\mathcal{S}_1}(s_1),\ p_{\mathcal{S}_2}(s_2))e^{imax(m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s))}\\ \mathcal{S}_1 \bigcup\ \mathcal{S}_2 &= q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)} \bigcup\ q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}\\ &= \min(q_{\mathcal{S}_1}(s_1),\ q_{\mathcal{S}_2}(s_2))e^{imin(n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s))}\end{aligned}$$

Definition 7. [36] Let $\mathcal{S}$ be a CIFS on $\mathcal{P}$ and $\mathfrak{L}_{\mathcal{S}}(s) = p_{\mathcal{S}}(s)e^{im_{\mathcal{S}}(s)}$ *and* $\mathfrak{J}_{\mathcal{S}}(s) = q_{\mathcal{S}}(s)e^{im_{\mathcal{S}}(s)}$ its grade value of membership and non-membership. The CIF complement of a CIFS $\mathcal{S}$ is expressed by $\mathcal{S}^c$ and is given by for membership,

$$\begin{aligned}\mathcal{S}^c &= p_{\mathcal{S}}^c(s)e^{im_{\mathcal{S}}^c(s)}\\ &= (1 - p_{\mathcal{S}}(s))e^{i(2\pi - m_{\mathcal{S}}(s))}\end{aligned}$$

for non membership,

$$\mathcal{S}^c = q_{\mathcal{S}}^c(s)e^{in_{\mathcal{S}}^c(s)} = (1 - q_{\mathcal{S}}(s))e^{(2\pi - in_{\mathcal{S}}(s))}$$

Theorem 1. *For any three CIFSs $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ on universe $\mathcal{P}$ satisfies*

i. $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S}_2 \cap \mathcal{S}_1$

ii. $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}_2 \cup \mathcal{S}_1$

iii. $\mathcal{S}_1 \cap (\mathcal{S}_2 \cap \mathcal{S}_3) = (\mathcal{S}_1 \cap \mathcal{S}_2) \cap \mathcal{S}_3$

iv. $\mathcal{S}_1 \cup (\mathcal{S}_2 \cup \mathcal{S}_3) = (\mathcal{S}_1 \cup \mathcal{S}_2) \cup \mathcal{S}_3$

v. $\mathcal{S}_1 \cap (\mathcal{S}_2 \cup \mathcal{S}_3) = (\mathcal{S}_1 \cap \mathcal{S}_2) \cup (\mathcal{S}_1 \cap \mathcal{S}_3)$

vi. $\mathcal{S}_1 \cup (\mathcal{S}_2 \cap \mathcal{S}_3) = (\mathcal{S}_1 \cup \mathcal{S}_2) \cap (\mathcal{S}_1 \cup \mathcal{S}_3)$

vii. $(\mathcal{S}_1 \cap \mathcal{S}_2)^c = (\mathcal{S}_2)^c \cup (\mathcal{S}_1)^c$

viii. $(\mathcal{S}_1 \cup \mathcal{S}_2)^c = (\mathcal{S}_2)^c \cap (\mathcal{S}_1)^c$

Definition 8. [37] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two CIFSs and $\mathfrak{L}_{\mathcal{S}_1(s)} = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}$ and $\mathfrak{L}_{\mathcal{S}_2(s)} = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}$, $\mathfrak{J}_{\mathcal{S}_1(s)} = q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)}$ and $\mathfrak{J}_{\mathcal{S}_2(s)} = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}$ represent membership non membership function. The CIF product of $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\mathcal{S}_1 \circ \mathcal{S}_2$ defined by,

$$\mathcal{S}_1 \circ \mathcal{S}_2 = \{p_{\mathcal{S}_1}(s).p_{\mathcal{S}_2}(s)e^{i\left(\frac{m_{\mathcal{S}_1}(s).m_{\mathcal{S}_2}(s)}{2\pi}\right)}, q_{\mathcal{S}_1}(s) + q_{\mathcal{S}_2}(s) - q_{\mathcal{S}_1}(s).q_{\mathcal{S}_2}(s)e^{i(n_{\mathcal{S}_1}(s)+n_{\mathcal{S}_2}(s)-\frac{n_{\mathcal{S}_1}(s).n_{\mathcal{S}_2}(s)}{2\pi})}\}$$

Definition 9. [37] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two CIFSs and $\mathfrak{L}_{\mathcal{S}_1(s)} = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}$ and $\mathfrak{L}_{\mathcal{S}_2(s)} = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}$, $\mathfrak{J}_{\mathcal{S}_1(s)} = q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)}$ and $\mathfrak{J}_{\mathcal{S}_2(s)} = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}$ represent their membership and non membership function, respectively. The CIF probabilistic sum of $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\mathcal{S}_1 \mp \mathcal{S}_2$ defined by,

$$\mathcal{S}_1 \mp \mathcal{S}_2 = \{p_{\mathcal{S}_1}(s) + p_{\mathcal{S}_2}(s) - p_{\mathcal{S}_1}(s).p_{\mathcal{S}_2}(s)e^{i(m_{\mathcal{S}_1}(s)+m_{\mathcal{S}_2}(s)-\frac{m_{\mathcal{S}_1}(s).m_{\mathcal{S}_2}(s)}{2\pi})}, q_{\mathcal{S}_1}(s).q_{\mathcal{S}_2}(s)e^{i\left(\frac{n_{\mathcal{S}_1}(s).n_{\mathcal{S}_2}(s)}{2\pi}\right)}\}$$

Definition 10. [37] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two CIFSs and $\mathfrak{L}_{\mathcal{S}_1(s)} = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}$ and $\mathfrak{L}_{\mathcal{S}_2(s)} = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}$, $\mathfrak{J}_{\mathcal{S}_1(s)} = q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)}$ and $\mathfrak{J}_{\mathcal{S}_2(s)} = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}$ represent their membership and non membership function. Then, $\mathcal{S}_1 \otimes \mathcal{S}_2$ defined by

$$\mathcal{S}_1 \times \mathcal{S}_2 = \{\min[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_2}(s)]e^{i\min[(m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s))]}, \max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_2}(s)]e^{i\max[(m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s))]}\}$$

3. Distance measures

The following list includes some of the distance measures like Zhang, Hamming, Euclidean, Normalized Hamming and Normalized Euclidean DM. Several researchers proposed different DMs, then utilized in various real-world issues. The DMs of CFS can serve as mathematical techniques in the application of DM and communication and networks.

Definition 11. [38] CFSs' Zhang DM is a function d_z: $\mathcal{S}(\mathcal{P}) \times \mathcal{S}(\mathcal{P}) \rightarrow [0, 1]$. For any $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{P})$

i. $0 \leqslant d_z(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$

ii. $d_z(\mathcal{S}_1, \mathcal{S}_2) = 0 \Leftrightarrow \mathcal{S}_1 = \mathcal{S}_2$

iii. $d_z(\mathcal{S}_1, \mathcal{S}_2) = d_z(\mathcal{S}_2, \mathcal{S}_1)$

iv. $d_z(\mathcal{S}_1, \mathcal{S}_3) \leqslant d_z(\mathcal{S}_1, \mathcal{S}_2) + d_z(\mathcal{S}_2, \mathcal{S}_3)$

Zhang DM is

$$d_z(\mathcal{S}_1, \mathcal{S}_2) = \max[sup_{s\in\mathcal{P}}|\mathcal{P}_{\mathcal{S}_1}(s) - \mathcal{P}_{\mathcal{S}_2}(s)|, \frac{1}{2\pi}sup_{s\in\mathcal{P}}|m_{\mathcal{S}_1}(s) - m_{\mathcal{S}_2}(s)|]$$

Definition 12. [28] Hamming DM of CFSs is a function d_H: $\mathcal{S}(\mathcal{P}) \times \mathcal{S}(\mathcal{P}) \rightarrow [0, 1]$. For any $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{P})$

i. $0 \leqslant d_H(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$

ii. $d_H(\mathcal{S}_1, \mathcal{S}_2) = 0 \Leftrightarrow \mathcal{S}_1 = \mathcal{S}_2$

iii. $d_H(\mathcal{S}_1, \mathcal{S}_2) = d_H(\mathcal{S}_2, \mathcal{S}_1)$

iv. $d_H(\mathcal{S}_1, \mathcal{S}_3) \leqslant d_H(\mathcal{S}_1, \mathcal{S}_2) + d_H(\mathcal{S}_2, \mathcal{S}_3)$

Hamming DM is

$$d_H(\mathcal{S}_1, \mathcal{S}_2) = \frac{1}{2}\left[\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s) - \mathcal{P}_{\mathcal{S}_2}(s)| + \frac{1}{2\pi}\sum_{i=1}^{n}|m_{\mathcal{S}_1}(s) - m_{\mathcal{S}_2}(s)|\right]$$

Definition 13. [28] Normalized Hamming DM of CFSs is a function d_N: $\mathcal{S}(\mathcal{P}) \times \mathcal{S}(\mathcal{P}) \to [0, 1]$. For any $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{P})$

i. $0 \leqslant d_N(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$

ii. $d_N(\mathcal{S}_1, \mathcal{S}_2) = 0 \Leftrightarrow \mathcal{S}_1 = \mathcal{S}_2$

iii. $d_N(\mathcal{S}_1, \mathcal{S}_2) = d_N(\mathcal{S}_2, \mathcal{S}_1)$

iv. $d_N(\mathcal{S}_1, \mathcal{S}_3) \leqslant d_N(\mathcal{S}_1, \mathcal{S}_2) + d_N(\mathcal{S}_2, \mathcal{S}_3)$.

Normalized Hamming DM is

$$d_N(\mathcal{S}_1, \mathcal{S}_2) = \frac{1}{2n}\left[\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s) - \mathcal{P}_{\mathcal{S}_2}(s)| + \frac{1}{2\pi}\sum_{i=1}^{n}|m_{\mathcal{S}_1}(s) - m_{\mathcal{S}_2}(s)|\right]$$

Definition 14. [28] Euclidean DM of CFSs is a function d_E: $\mathcal{S}(\mathcal{P}) \times \mathcal{S}(\mathcal{P}) \to [0, 1]$. For any $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{P})$

i. $0 \leqslant d_E(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$

ii. $d_E(\mathcal{S}_1, \mathcal{S}_2) = 0 \Leftrightarrow \mathcal{S}_1 = \mathcal{S}_2$

iii. $d_E(\mathcal{S}_1, \mathcal{S}_2) = d_E(\mathcal{S}_2, \mathcal{S}_1)$

iv. $d_E(\mathcal{S}_1, \mathcal{S}_3) \leqslant d_E(\mathcal{S}_1, \mathcal{S}_2) + d_E(\mathcal{S}_2, \mathcal{S}_3)$.

Euclidean DM is

$$d_E(\mathcal{S}_1, \mathcal{S}_2) = \sqrt{\frac{1}{2}\left[\sum_{i=1}^{n}(\mathcal{P}_{\mathcal{S}_1}(s) - \mathcal{P}_{\mathcal{S}_2}(s))^2 + \frac{1}{2\pi^2}\sum_{i=1}^{n}(m_{\mathcal{S}_1}(s) - m_{\mathcal{S}_2}(s))^2\right]}$$

Definition 15. [28] Normalized Euclidean DM of CFSs is a function d_N: $\mathcal{S}(\mathcal{P}) \times \mathcal{S}(\mathcal{P}) \to [0, 1]$ and d_N: $\mathcal{S}(\mathcal{Q}) \times \mathcal{S}(\mathcal{Q}) \to [0, 1]$. For any $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{P})$ and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{Q})$

i. $0 \leqslant d_N(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$

ii. $d_N(\mathcal{S}_1, \mathcal{S}_2) = 0 \Leftrightarrow \mathcal{S}_1 = \mathcal{S}_2$

iii. $d_N(\mathcal{S}_1, \mathcal{S}_2) = d_N(\mathcal{S}_2, \mathcal{S}_1)$

iv. $d_N(\mathcal{S}_1, \mathcal{S}_3) \leqslant d_N(\mathcal{S}_1, \mathcal{S}_2) + d_N(\mathcal{S}_2, \mathcal{S}_3)$.

Normalized Euclidean DM is

$$d_N(\mathcal{S}_1, \mathcal{S}_2) = \sqrt{\frac{1}{2n}\left[\sum_{i=1}^{n}(\mathcal{P}_{\mathcal{S}_1}(s) - \mathcal{P}_{\mathcal{S}_2}(s))^2 + \frac{1}{2\pi^2}\sum_{i=1}^{n}(m_{\mathcal{S}_1}(s) - m_{\mathcal{S}_2}(s))^2\right]}$$

4. Key findings pertaining to complex intuitionistic fuzzy operations

A number of fundamental findings about CIFS operations, including union, intersection, and complement, are presented in this section. The algebraic relationships and consistency criteria required to define complex-valued distance functions are established by these theorems. These findings provide the theoretical foundation for the hesitance-based distance metrics suggested in section 5 by demonstrating that CIFS operations follow commutative, associative, and distributive rules. In order to guarantee that the suggested HDMCIFS and EHDMCIFS formulations are mathematically coherent within the CIFS framework, the following theorems are not independent.

Theorem 2. *Prove that CIFSs* $\mathcal{S}_i$, $i=1,2,\ldots,n$. $\mathcal{S}_1 \cup (\bigcap_{i=1}^{n} \mathcal{S}_i) = \bigcap_{i=1}^{n}(\mathcal{S}_1 \cup \mathcal{S}_i)$

Proof. Let $\mathcal{S}_1, \mathcal{S}_2,\ldots,\mathcal{S}_n$ be two CIFSs and $\mathfrak{L}_{\mathcal{S}_1}(s) = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}}(s)$, $\mathfrak{L}_{\mathcal{S}_2}(s) = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}}(s),\ldots,$ $\mathfrak{L}_{\mathcal{S}_n(s)} = p_{\mathcal{S}_n}(s)e^{in_{\mathcal{S}_2}}(s)$, $\mathfrak{J}_{\mathcal{S}_1}(s) = q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}}(s)$ $\mathfrak{J}_{\mathcal{S}_2}(s) = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}}(s),\ldots,\mathfrak{J}_{\mathcal{S}_n(s)} = q_{\mathcal{S}_n}(s)e^{in_{\mathcal{S}_n}}(s)$ represent their membership and non membership function, respectively.

$$\begin{aligned}\mathcal{S}_1 \bigcup (\textstyle\bigcap_{i=1}^{n} \mathcal{S}_i) &= \mathcal{S}_1 \bigcup (\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 \cap,\ldots,\bigcap \mathcal{S}_n)\\ &= \mathfrak{L}_{\mathcal{S}_1}(s) \bigcup (\mathfrak{L}_{\mathcal{S}_1}(s) \cap \mathfrak{L}_{\mathcal{S}_2}(s),\ldots,\bigcap \mathfrak{L}_{\mathcal{S}_1}(s))\\ &= \max\{p_{\mathcal{S}_1}(s), \min[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_2}(s),\ldots,p_{\mathcal{S}_n}(s)]\}\, e^{i\max\{m_{\mathcal{S}_1}(s),\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s),\ldots,m_{\mathcal{S}_n}(s)]\}}\end{aligned} \tag{4.1}$$

$$\begin{aligned}\textstyle\bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i) &= (\mathcal{S}_1 \bigcup \mathcal{S}_1) \cap (\mathcal{S}_1 \bigcup \mathcal{S}_2) \cap \ldots (\mathcal{S}_1 \bigcup \mathcal{S}_n)\\ &= (\mathfrak{L}_{\mathcal{S}_1}(s) \bigcup \mathfrak{L}_{\mathcal{S}_1}(s)) \cap (\mathfrak{L}_{\mathcal{S}_1}(s) \bigcup \mathfrak{L}_{\mathcal{S}_2}(s)) \cap \ldots \bigcap (\mathfrak{L}_{\mathcal{S}_1}(s) \bigcup \mathfrak{L}_{\mathcal{S}_n}(s))\\ &= \min\{\max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_1}(s)], \max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_2}(s)],\ldots,\max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_n}(s)]\}\\ &\quad \times e^{i\min\{\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_1}(s)],\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)],\ldots,\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_n}(s)]\}}\end{aligned} \tag{4.2}$$

Case 1: If $p_{\mathcal{S}_1}(s) \leqslant p_{\mathcal{S}_2}(s) \leqslant \ldots \leqslant p_{\mathcal{S}_n}(s)$ and $m_{\mathcal{S}_1}(s) \leqslant m_{\mathcal{S}_2}(s) \leqslant \ldots \leqslant m_{\mathcal{S}_n}(s)$ then equations (4.1) and (4.2) implies

$$\begin{aligned}\mathcal{S}_1 \bigcup (\textstyle\bigcap_{i=1}^{n} \mathcal{S}_i) &= \max\{p_{\mathcal{S}_1}(s), \min[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_2}(s),\ldots,p_{\mathcal{S}_n}(s)]\}\, e^{i\max\{m_{\mathcal{S}_1}(s),\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s),\ldots,m_{\mathcal{S}_n}(s)]\}}\\ &= \max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_1}(s))]\, e^{i\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_1}(s)]}\\ &= p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}\end{aligned} \tag{4.3}$$

also

$$\begin{aligned}\textstyle\bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i) &= \min\{\max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_1}(s)], \max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_2}(s)],\ldots,\max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_n}(s)]\}\\ &\quad \times e^{i\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_1}(s)],[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)],\ldots,[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_n}(s)]}\\ &= p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}\end{aligned} \tag{4.4}$$

From equations (4.3) and (4.4)

$$\mathcal{S}_1 \bigcup (\textstyle\bigcap_{i=1}^{n} \mathcal{S}_i) = \bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i)$$

Case 2: If $p_{\mathcal{S}_n}(s) \leqslant p_{\mathcal{S}_{n-1}}(s) \leqslant \ldots \leqslant p_{\mathcal{S}_1}(s)$ and $m_{\mathcal{S}_n}(s) \leqslant m_{\mathcal{S}_{n-1}}(s) \leqslant \ldots \leqslant m_{\mathcal{S}_1}(s)$ then equations (4.1) and (4.2) implies

$$\begin{aligned}\mathcal{S}_1 \bigcup (\textstyle\bigcap_{i=1}^{n} \mathcal{S}_i) &= \max\{p_{\mathcal{S}_1}(s), \min[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_2}(s),\ldots,p_{\mathcal{S}_n}(s)]\}\, e^{i\max\{m_{\mathcal{S}_1}(s),\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s),\ldots,m_{\mathcal{S}_n}(s)]\}}\\ &= \max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_n}(s))]\, e^{i\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_n}(s)]}\\ &= p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}\end{aligned} \tag{4.5}$$

also

$$\begin{aligned}\textstyle\bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i) &= \min\{\max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_1}(s)], \max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_2}(s)],\ldots,\max[p_{\mathcal{S}_1}(s), p_{\mathcal{S}_n}(s)]\}\\ &\quad e^{i\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_1}(s)],[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)],\ldots,[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_n}(s)]}\\ &= p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}\end{aligned} \tag{4.6}$$

From equations (4.5) and (4.6)

$$\mathcal{S}_1 \bigcup (\textstyle\bigcap_{i=1}^{n} \mathcal{S}_i) = \bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i)$$

Non membership function,

$$\mathcal{S}_1 \bigcup (\bigcap_{i=1}^{n} \mathcal{S}_i) = \mathcal{S}_1 \bigcup (\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 \cap, \ldots, \bigcap \mathcal{S}_n)$$
$$= \mathfrak{J}_{\mathcal{S}_1}(s) \bigcup (\mathfrak{J}_{\mathcal{S}_1}(s) \cap \mathfrak{J}_{\mathcal{S}_2}(s), \ldots, \bigcap \mathfrak{J}_{\mathcal{S}_1}(s))$$
$$= \max\{q_{\mathcal{S}_1}(s),\ \min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_2}(s), \ldots, q_{\mathcal{S}_n}(s)]\}\, e^{i\max\{n_{\mathcal{S}_1}(s), \min[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s), \ldots, n_{\mathcal{S}_n}(s)]\}} \tag{4.7}$$

$$\bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i) = (\mathcal{S}_1 \bigcup \mathcal{S}_1) \cap (\mathcal{S}_2 \bigcup \mathcal{S}_2) \cap \ldots (\mathcal{S}_n \bigcup \mathcal{S}_n)$$
$$= (\mathfrak{J}_{\mathcal{S}_1}(s) \bigcup \mathfrak{J}_{\mathcal{S}_1}(s)) \cap (\mathfrak{J}_{\mathcal{S}_1}(s) \bigcup \mathfrak{J}_{\mathcal{S}_2}(s)) \cap \ldots \bigcap (\mathfrak{J}_{\mathcal{S}_1}(s) \bigcup \mathfrak{J}_{\mathcal{S}_n}(s))$$
$$= \min\{\max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_1}(s)],\ \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_2}(s)], \ldots, \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s)]\}$$
$$\times\ e^{\min\{i\max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_1}(s)], [n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s)], \ldots, [n_{\mathcal{S}_1}(s), n_{\mathcal{S}_n}(s)]\}} \tag{4.8}$$

Case 1: If $q_{\mathcal{S}_1}(s) \leqslant q_{\mathcal{S}_2}(s) \leqslant \ldots \leqslant q_{\mathcal{S}_n}(s)$ and $n_{\mathcal{S}_1}(s) \leqslant n_{\mathcal{S}_2}(s) \leqslant \ldots \leqslant n_{\mathcal{S}_n}(s)$ then equation (4.7) and (4.8) implies

$$\mathcal{S}_1 \bigcup (\bigcap_{i=1}^{n} \mathcal{S}_i) = \max\{q_{\mathcal{S}_1}(s),\ \min[q_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s), \ldots, q_{\mathcal{S}_n}(s)]\}\, e^{i\max\{n_{\mathcal{S}_1}(s), \min[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s), \ldots, n_{\mathcal{S}_n}(s)]\}}$$
$$= \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s))]\, e^{i\max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_n}(s)]}$$
$$= q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}(s)} \tag{4.9}$$

also

$$\bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i) = \min\{\max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_1}(s)],\ \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_2}(s)], \ldots, \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s)]\}$$
$$\times\ e^{i\min\{\max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_1}(s)], \max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s)], \ldots, \max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_n}(s)]\}}$$
$$= q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}(s)} \tag{4.10}$$

From equations (4.9) and (4.10)

$$\mathcal{S}_1 \bigcup (\bigcap_{i=1}^{n} \mathcal{S}_i) = \bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i)$$

Case 2: If $q_{\mathcal{S}_n}(s) \leqslant q_{\mathcal{S}_{n-1}}(s) \leqslant \ldots \leqslant q_{\mathcal{S}_1}(s)$ and $n_{\mathcal{S}_n}(s) \leqslant n_{\mathcal{S}_{n-1}}(s) \leqslant \ldots \leqslant n_{\mathcal{S}_1}(s)$ then (4.7) and (4.8) implies

$$\mathcal{S}_1 \bigcup (\bigcap_{i=1}^{n} \mathcal{S}_i) = \max\{q_{\mathcal{S}_1}(s),\ \min[q_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s), \ldots, q_{\mathcal{S}_n}(s)]\}\, e^{i\max\{n_{\mathcal{S}_1}(s), \min[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s), \ldots, n_{\mathcal{S}_n}(s)]\}}$$
$$= \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s))]\, e^{i\max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_n}(s)]}$$
$$= q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}(s)} \tag{4.11}$$

also

$$\bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i) = \min\{\max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_1}(s)],\ \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_2}(s)], \ldots, \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s)]\}$$
$$\times\ e^{i\min\{\max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_1}(s)], \max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s)], \ldots, \max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_n}(s)]\}}$$
$$= q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}(s)} \tag{4.12}$$

From equations (4.11) and (4.12)

$$\mathcal{S}_1 \bigcup (\bigcap_{i=1}^{n} \mathcal{S}_i) = \bigcap_{i=1}^{n}(\mathcal{S}_1 \bigcup \mathcal{S}_i)$$

The proof is similar for other cases. □

Theorem 3. *For CIFSs* $\mathcal{S}_i,\ i = 1, 2, \ldots, n'. \mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) = \bigcup_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i)$

Proof. Let $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ be two CIFSs and $\mathfrak{L}_{\mathcal{S}_1}(s) = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}}(s)$, $\mathfrak{L}_{\mathcal{S}_2}(s) = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}}(s), \ldots, \mathfrak{L}_{\mathcal{S}_n(s)} = p_{\mathcal{S}_n}(s)e^{in_{\mathcal{S}_2}}(s)$, $\mathfrak{J}_{\mathcal{S}_1}(s) = q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}}(s)\mathfrak{J}_{\mathcal{S}_2}(s) = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}}(s), \ldots, \mathfrak{J}_{\mathcal{S}_n(s)} = q_{\mathcal{S}_n}(s)\, e^{in_{\mathcal{S}_n}}(s)$ represent their membership and non membership function, respectively.

Membership function,

$$\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) = \mathcal{S}_1 \cap (\mathcal{S}_1 \bigcup \mathcal{S}_2 \bigcup \mathcal{S}_3 \bigcup, \ldots, \bigcup \mathcal{S}_n)$$
$$= \mathfrak{L}_{\mathcal{S}_1}(s) \cap (\mathfrak{L}_{\mathcal{S}_1}(s) \bigcup \mathfrak{L}_{\mathcal{S}_2}(s), \ldots, \bigcup \mathfrak{L}_{\mathcal{S}_1}(s))$$
$$= \min\{p_{\mathcal{S}_1}(s),\ \max[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s), \ldots, p_{\mathcal{S}_n}(s)]\}\, e^{i\min\{m_{\mathcal{S}_1}(s), \max[m_{\mathcal{S}_1}(s), m_{\mathcal{S}_2}(s), \ldots, m_{\mathcal{S}_n}(s)]\}} \tag{4.13}$$

$$\bigcup_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i) = (\mathcal{S}_1 \cap \mathcal{S}_1) \bigcup (\mathcal{S}_2 \cap \mathcal{S}_2) \bigcup \ldots (\mathcal{S}_n \cap \mathcal{S}_n)$$
$$= (\mathfrak{L}_{\mathcal{S}_1}(s) \cap \mathfrak{L}_{\mathcal{S}_1}(s)) \bigcup (\mathfrak{L}_{\mathcal{S}_1}(s) \cap \mathfrak{L}_{\mathcal{S}_2}(s)) \bigcup \ldots \bigcup (\mathfrak{L}_{\mathcal{S}_1}(s) \cap \mathfrak{L}_{\mathcal{S}_n}(s))$$
$$= \max\{\min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_1}(s)],\ \min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s)], \ldots, \min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_n}(s)]\}$$
$$\times\ e^{i\max\{\min[m_{\mathcal{S}_1}(s), m_{\mathcal{S}_1}(s)], \min[m_{\mathcal{S}_1}(s), m_{\mathcal{S}_2}(s)], \ldots, \min[m_{\mathcal{S}_1}(s), m_{\mathcal{S}_n}(s)]\}} \tag{4.14}$$

Case 1: If $p_{\mathcal{S}_1}(s) \leqslant p_{\mathcal{S}_2}(s) \leqslant \ldots \leqslant p_{\mathcal{S}_n}(s)$ and $m_{\mathcal{S}_1}(s) \leqslant m_{\mathcal{S}_2}(s) \leqslant \ldots \leqslant m_{\mathcal{S}_n}(s)$ then equations (4.13) and (4.14) implies

$$\begin{aligned}\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) &= \min\{p_{\mathcal{S}_1}(s),\ \max[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s),\ldots,p_{\mathcal{S}_n}(s)]\}\, e^{i\min\{m_{\mathcal{S}_1}(s),\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s),\ldots,m_{\mathcal{S}_n}(s)]\}}\\ &= \min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_1}(s))]\, e^{i\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_1}(s)]}\\ &= p_{\mathcal{S}_1}(s)\, e^{im_{\mathcal{S}_1}(s)} \end{aligned} \tag{4.15}$$

also

$$\begin{aligned}\bigcup_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i) &= \max\{\min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_1}(s)],\ \min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s)],\ldots,\min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_n}(s)]\}\\ &\quad\times\ e^{i\max\{\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_1}(s)],\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)],\ldots,\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_n}(s)]\}}\\ &= p_{\mathcal{S}_1}(s)\, e^{im_{\mathcal{S}_1}(s)} \end{aligned} \tag{4.16}$$

From equations (4.15) and (4.16)

$$\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) = \bigcup_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i)$$

Case 2: If $p_{\mathcal{S}_n}(s) \leqslant p_{\mathcal{S}_{n-1}}(s) \leqslant \ldots \leqslant p_{\mathcal{S}_1}(s)$ and $m_{\mathcal{S}_n}(s) \leqslant m_{\mathcal{S}_{n-1}}(s) \leqslant \ldots \leqslant m_{\mathcal{S}_1}(s)$ then equations (4.13) and (4.14) implies

$$\begin{aligned}\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) &= \min\{p_{\mathcal{S}_1}(s),\ \max[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s),\ldots,p_{\mathcal{S}_n}(s)]\}\, e^{i\min\{m_{\mathcal{S}_1}(s),\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s),\ldots,m_{\mathcal{S}_n}(s)]\}}\\ &= \min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_n}(s))]\, e^{i\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_n}(s)]}\\ &= p_{\mathcal{S}_1}(s)\, e^{im_{\mathcal{S}_1}(s)} \end{aligned} \tag{4.17}$$

also

$$\begin{aligned}\bigcup_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i) &= \max\{\min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_1}(s)],\ \min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s)],\ldots,\min[p_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_n}(s)]\}\\ &\quad\times\ e^{i\max\{\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_1}(s)],\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)],\ldots,\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_n}(s)]\}}\\ &= p_{\mathcal{S}_1}(s)\, e^{im_{\mathcal{S}_1}(s)} \end{aligned} \tag{4.18}$$

From equations (4.17)and (4.18)

$$\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) = \bigcup_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i)$$

Non membership function,

$$\begin{aligned}\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) &= \mathcal{S}_1 \cap (\mathcal{S}_1 \bigcup \mathcal{S}_2 \bigcup \mathcal{S}_3 \bigcup ,\ldots,\bigcup \mathcal{S}_n)\\ &= \mathfrak{I}_{\mathcal{S}_1}(s)\cap(\mathfrak{I}_{\mathcal{S}_1}(s) \bigcup \mathfrak{I}_{\mathcal{S}_2}(s),\ldots,\bigcup\mathfrak{I}_{\mathcal{S}_1}(s))\\ &= \min\{q_{\mathcal{S}_1}(s),\ \max[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_2}(s),\ldots,q_{\mathcal{S}_n}(s)]\}\, e^{i\min\{n_{\mathcal{S}_1}(s),\max[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s),\ldots,n_{\mathcal{S}_n}(s)]\}} \end{aligned} \tag{4.19}$$

$$\begin{aligned}\bigcup_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i) &= (\mathcal{S}_1 \cap \mathcal{S}_1) \bigcup (\mathcal{S}_2 \cap \mathcal{S}_2) \bigcup \ldots(\mathcal{S}_n \cap \mathcal{S}_n)\\ &= (\mathfrak{I}_{\mathcal{S}_1}(s)\cap \mathfrak{I}_{\mathcal{S}_1}(s)) \bigcup (\mathfrak{I}_{\mathcal{S}_1}(s)\cap \mathfrak{I}_{\mathcal{S}_2}(s)) \bigcup \ldots\bigcup(\mathfrak{I}_{\mathcal{S}_1}(s)\cap \mathfrak{I}_{\mathcal{S}_n}(s))\\ &= \max\{\min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_1}(s)],\ \min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_2}(s)],\ldots,\min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s)]\}\\ &\quad\times\ e^{i\max\{\min[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_1}(s)],[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s)],\ldots,[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_n}(s)]\}} \end{aligned} \tag{4.20}$$

Case 1: If $q_{\mathcal{S}_1}(s) \leqslant q_{\mathcal{S}_2}(s) \leqslant \ldots \leqslant q_{\mathcal{S}_n}(s)$ and $n_{\mathcal{S}_1}(s) \leqslant n_{\mathcal{S}_2}(s) \leqslant \ldots \leqslant n_{\mathcal{S}_n}(s)$ then equations (4.19) and (4.20) implies

$$\begin{aligned}\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) &= \min\{q_{\mathcal{S}_1}(s),\ \max[q_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s),\ldots,q_{\mathcal{S}_n}(s)]\}\, e^{i\min\{n_{\mathcal{S}_1}(s),\max[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s),\ldots,n_{\mathcal{S}_n}(s)]\}}\\ &= \min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_1}(s))]\, e^{i\min[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_1}(s)]}\\ &= q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}(s)} \end{aligned} \tag{4.21}$$

also

$$\begin{aligned}\bigcap_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i) &= \max\{\min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_1}(s)],\ \min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_2}(s)],\ldots,\min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s)]\}\\ &\quad\times\ e^{i\max\{\min[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_1}(s)],\min[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s)],\ldots,\min[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_n}(s)]\}}\\ &= q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}(s)} \end{aligned} \tag{4.22}$$

From equations (4.21) and (4.22)

$$\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) = \bigcup_{i=1}^{n}(\mathcal{S}_1 \cap \mathcal{S}_i)$$

Case 2: If $q_{\mathcal{S}_n}(s) \leqslant q_{\mathcal{S}_{n-1}}(s) \leqslant \ldots \leqslant q_{\mathcal{S}_1}(s)$ and $n_{\mathcal{S}_n}(s) \leqslant n_{\mathcal{S}_{n-1}}(s) \leqslant \ldots \leqslant n_{\mathcal{S}_1}(s)$ then equations (4.19) and (4.20) implies

$$\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) = \min\{q_{\mathcal{S}_1}(s),\ \max[q_{\mathcal{S}_1}(s),\ p_{\mathcal{S}_2}(s),\ldots,q_{\mathcal{S}_n}(s)]\}\, e^{imin\{n_{\mathcal{S}_1}(s),\max[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s),\ldots,n_{\mathcal{S}_n}(s)]\}}$$
$$= \min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s))]\, e^{imin[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_n}(s)]}$$
$$= q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}(s)} \tag{4.23}$$

also

$$\bigcup_{i=1}^{n} (\mathcal{S}_1 \cap \mathcal{S}_i) = \max\{\min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_1}(s)],\ \min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_2}(s)],\ldots,\min[q_{\mathcal{S}_1}(s),\ q_{\mathcal{S}_n}(s)]\}$$
$$\times\ e^{i\max\{\min[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_1}(s)],\min[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s)],\ldots,\min[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_n}(s)]\}}$$
$$= q_{\mathcal{S}_1}(s)\, e^{in_{\mathcal{S}_1}(s)} \tag{4.24}$$

From equations (4.23) and (4.24)

$$\mathcal{S}_1 \cap (\bigcup_{i=1}^{n} \mathcal{S}_i) = \bigcup_{i=1}^{n} (\mathcal{S}_1 \cap \mathcal{S}_i)$$

□

Theorem 4. *CIFS $\mathcal{S}_1$, $\mathcal{S}_2$ on $\mathcal{V}$ then CIF standard union, standard complement and standard intersection must satisfy $(\mathcal{S}_1{}^c \cup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c = \mathcal{S}_1{}^c$*

Proof. $\mathfrak{L}_{\mathcal{S}_1(s)} = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}$ and $\mathfrak{L}_{\mathcal{S}_2(s)} = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}$, $\mathfrak{J}_{\mathcal{S}_1(s)} = q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)}$ and $\mathfrak{J}_{\mathcal{S}_2(s)} = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}$ are the membership and non membership function of $\mathcal{S}_1$, $\mathcal{S}_2$

$$(\mathcal{S}_1{}^c \bigcup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c = \mathfrak{L}_{\mathcal{S}_1^c(s)} \bigcup \mathfrak{L}_{\mathcal{S}_2^c(s)}$$
$$= p_{\mathcal{S}_1^c}(s)e^{im_{\mathcal{S}_1}(s)} \bigcup p_{\mathcal{S}_2^c}(s)e^{im_{\mathcal{S}_2}(s)}$$
$$= \max[1 - p_{\mathcal{S}_1}(s),\ 1 - p_{\mathcal{S}_2}(s)]\, e^{imax[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)]}$$

$$(\mathcal{S}_1{}^c \bigcup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c = \max[1 - p_{\mathcal{S}_1}(s),\ 1 - p_{\mathcal{S}_2}(s)]\, e^{imax[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)]} \cap (1 - p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)})$$
$$= \min\{\max(1 - p_{\mathcal{S}_1}(s),\ 1 - p_{\mathcal{S}_2}(s)),\ 1 - p_{\mathcal{S}_1}(s)\}\, e^{i\min[\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)],m_{\mathcal{S}_1}(s)]} \tag{4.25}$$

To prove equation (4.25)

Case 1: If $p_{\mathcal{S}_1}(s) \leqslant p_{\mathcal{S}_2}(s)$ and $m_{\mathcal{S}_1}(s) \leqslant m_{\mathcal{S}_2}(s)$ then equation (4.25) implies

$$(\mathcal{S}_1{}^c \bigcup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c = \min\{\max(1 - p_{\mathcal{S}_1}(s),\ 1 - p_{\mathcal{S}_2}(s)),\ 1 - p_{\mathcal{S}_1}(s)\}\, e^{i\min[\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)],m_{\mathcal{S}_1}(s)]}$$
$$= \min(1 - p_{\mathcal{S}_1}(s),\ 1 - p_{\mathcal{S}_2}(s))\, e^{i\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)]}$$
$$= p_{\mathcal{S}_1^c}(s)e^{im_{\mathcal{S}_1}(s)}$$
$$= \mathcal{S}_1{}^c$$

Case 2: If $p_{\mathcal{S}_1}(s) \geqslant p_{\mathcal{S}_2}(s)$ and $m_{\mathcal{S}_1}(s) \geqslant m_{\mathcal{S}_2}(s)$ then equation (4.25) implies

$$(\mathcal{S}_1{}^c \bigcup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c = \min\{\max(1 - p_{\mathcal{S}_1}(s),\ 1 - p_{\mathcal{S}_2}(s)),\ 1 - p_{\mathcal{S}_1}(s)\}\, e^{i\min[\max[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)],m_{\mathcal{S}_1}(s)]}$$
$$= \min(1 - p_{\mathcal{S}_1}(s),\ 1 - p_{\mathcal{S}_2}(s))\, e^{i\min[m_{\mathcal{S}_1}(s),m_{\mathcal{S}_2}(s)]}$$
$$= p_{\mathcal{S}_1^c}(s)e^{im_{\mathcal{S}_1}(s)}$$
$$= \mathcal{S}_1{}^c$$

for non membership function

$$(\mathcal{S}_1{}^c \bigcup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c = \mathfrak{J}_{\mathcal{S}_1^c(s)} \bigcup \mathfrak{J}_{\mathcal{S}_2^c(s)}$$
$$= q_{\mathcal{S}_1^c}(s)e^{in_{\mathcal{S}_1}(s)} \bigcup q_{\mathcal{S}_2^c}(s)e^{in_{\mathcal{S}_2}(s)}$$
$$= \max[1 - q_{\mathcal{S}_1}(s),\ 1 - q_{\mathcal{S}_2}(s)]\, e^{imax[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s)]}$$

$$(\mathcal{S}_1{}^c \bigcup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c = \max[1 - q_{\mathcal{S}_1}(s),\ 1 - q_{\mathcal{S}_2}(s)]\, e^{imax[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s)]} \cap (1 - q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)})$$
$$= \min\{\max(1 - q_{\mathcal{S}_1}(s),\ 1 - q_{\mathcal{S}_2}(s)),\ 1 - q_{\mathcal{S}_1}(s)\}\, e^{i\min[\max[n_{\mathcal{S}_1}(s),n_{\mathcal{S}_2}(s)],n_{\mathcal{S}_1}(s)]} \tag{4.26}$$

To prove equation (4.26)

Case 1. If $q_{\mathcal{S}_1}(s) \leqslant q_{\mathcal{S}_2}(s)$ and $n_{\mathcal{S}_1}(s) \leqslant n_{\mathcal{S}_2}(s)$ then equation (4.26) implies

$$\begin{aligned}(\mathcal{S}_1{}^c \bigcup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c &= \min\{\max(1 - q_{\mathcal{S}_1}(s),\, 1 - q_{\mathcal{S}_2}(s)),\, 1 - q_{\mathcal{S}_1}(s)\}\, e^{i\min[\max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s)], n_{\mathcal{S}_1}(s)]}\\ &= \min(1 - q_{\mathcal{S}_1}(s),\, 1 - q_{\mathcal{S}_2}(s))\, e^{i\min[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s)]}\\ &= q_{\mathcal{S}_1^c}(s) e^{in_{\mathcal{S}_1}(s)}\\ &= \mathcal{S}_1{}^c\end{aligned}$$

Case 2. If $q_{\mathcal{S}_1}(s) \geqslant q_{\mathcal{S}_2}(s)$ and $n_{\mathcal{S}_1}(s) \geqslant n_{\mathcal{S}_2}(s)$ then equation (4.26) implies

$$\begin{aligned}(\mathcal{S}_1{}^c \bigcup \mathcal{S}_2{}^c) \cap \mathcal{S}_1{}^c &= \min\{\max(1 - q_{\mathcal{S}_1}(s),\, 1 - q_{\mathcal{S}_2}(s)),\, 1 - q_{\mathcal{S}_1}(s)\}\, e^{i\min[\max[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s)], n_{\mathcal{S}_1}(s)]}\\ &= \min(1 - q_{\mathcal{S}_1}(s),\, 1 - q_{\mathcal{S}_2}(s))\, e^{i\min[n_{\mathcal{S}_1}(s), n_{\mathcal{S}_2}(s)]}\\ &= q_{\mathcal{S}_1^c}(s) e^{in_{\mathcal{S}_1}(s)}\\ &= \mathcal{S}_1{}^c\end{aligned}$$

The proof is similar for other cases □

Theorem 5. *CIFS $\mathcal{S}_1$, $\mathcal{S}_2$ on $\mathcal{V}$ then CIF standard union, standard complement and standard intersection must satisfy $\mathcal{S}_2 \cup (\mathcal{S}_2 \cap \mathcal{S}_1) = \mathcal{S}_2$*

Proof. $\mathfrak{L}_{\mathcal{S}_1(s)} = p_{\mathcal{S}_1}(s)e^{im_{\mathcal{S}_1}(s)}$ and $\mathfrak{L}_{\mathcal{S}_2(s)} = p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}$, $\mathfrak{J}_{\mathcal{S}_1(s)} = q_{\mathcal{S}_1}(s)e^{in_{\mathcal{S}_1}(s)}$ and $\mathfrak{J}_{\mathcal{S}_2(s)} = q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}$ are the membership and non membership function of $\mathcal{S}_1$, $\mathcal{S}_2$. For Membership function

$$\begin{aligned}\mathcal{S}_2 \bigcup (\mathcal{S}_2 \cap \mathcal{S}_1) &= \mathfrak{L}_{\mathcal{S}_1(s)} \bigcup (\mathfrak{L}_{\mathcal{S}_1(s)} \cap \mathfrak{L}_{\mathcal{S}_1(s)})\\ &= \max(p_{\mathcal{S}_2}(s),\, \min(p_{\mathcal{S}_2}(s),\, p_{\mathcal{S}_1}(s)))e^{i}max(m_{\mathcal{S}_2}(s),\, (m_{\mathcal{S}_2}(s),\, m_{\mathcal{S}_1}(s)))\end{aligned} \tag{4.27}$$

To prove this, we take some cases.

Case 1. If $p_{\mathcal{S}_1}(s) \leqslant p_{\mathcal{S}_2}(s)$ and $m_{\mathcal{S}_1}(s) \leqslant m_{\mathcal{S}_2}(s)$ then equation (4.27) implies

$$\begin{aligned}\mathcal{S}_2 \bigcup (\mathcal{S}_2 \cap \mathcal{S}_1) &= \max(p_{\mathcal{S}_2}(s),\, \min(p_{\mathcal{S}_2}(s),\, p_{\mathcal{S}_1}(s)))e^{i}\max(m_{\mathcal{S}_2}(s),\, (m_{\mathcal{S}_2}(s),\, m_{\mathcal{S}_1}(s)))\\ &= \max(p_{\mathcal{S}_2}(s),\, p_{\mathcal{S}_1}(s))e^{i}\max(m_{\mathcal{S}_2}(s),\, m_{\mathcal{S}_1}(s))\\ &= p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}\\ &= \mathcal{S}_2\end{aligned} \tag{4.28}$$

Case 2. If $p_{\mathcal{S}_2}(s) \leqslant p_{\mathcal{S}_1}(s)$ and $m_{\mathcal{S}_2}(s) \leqslant m_{\mathcal{S}_1}(s)$ then equation (4.27) implies

$$\begin{aligned}\mathcal{S}_2 \bigcup (\mathcal{S}_2 \cap \mathcal{S}_1) &= \max(p_{\mathcal{S}_2}(s),\, \min(p_{\mathcal{S}_2}(s),\, p_{\mathcal{S}_1}(s)))e^{i}\max(m_{\mathcal{S}_2}(s),\, (m_{\mathcal{S}_2}(s),\, m_{\mathcal{S}_1}(s)))\\ &= \max(p_{\mathcal{S}_2}(s),\, p_{\mathcal{S}_1}(s))e^{i}\max(m_{\mathcal{S}_2}(s),\, m_{\mathcal{S}_1}(s))\\ &= p_{\mathcal{S}_2}(s)e^{im_{\mathcal{S}_2}(s)}\\ &= \mathcal{S}_2\end{aligned} \tag{4.29}$$

From equations (4.28) and (4.29)

$$\mathcal{S}_2 \bigcup (\mathcal{S}_2 \cap \mathcal{S}_1) = \mathcal{S}_2$$

for non membership function

$$\begin{aligned}\mathcal{S}_2 \bigcup (\mathcal{S}_2 \cap \mathcal{S}_1) &= \mathfrak{J}_{\mathcal{S}_1(s)} \bigcup (\mathfrak{J}_{\mathcal{S}_1(s)} \cap \mathfrak{J}_{\mathcal{S}_1(s)})\\ &= \min(q_{\mathcal{S}_2}(s),\, \max(q_{\mathcal{S}_2}(s),\, q_{\mathcal{S}_1}(s)))e^{i}min(n_{\mathcal{S}_2}(s),\, \max(n_{\mathcal{S}_2}(s),\, n_{\mathcal{S}_1}(s)))\end{aligned} \tag{4.30}$$

To prove this, we take some cases.

Case 1:If $q_{\mathcal{S}_1}(s) \leqslant q_{\mathcal{S}_2}(s)$ and $n_{\mathcal{S}_1}(s) \leqslant n_{\mathcal{S}_2}(s)$ then equation (4.30) implies

$$\begin{aligned}\mathcal{S}_2 \bigcup (\mathcal{S}_2 \cap \mathcal{S}_1) &= \min(q_{\mathcal{S}_2}(s),\, \max(q_{\mathcal{S}_2}(s),\, q_{\mathcal{S}_1}(s)))e^{i}\min(n_{\mathcal{S}_2}(s),\, (n_{\mathcal{S}_2}(s),\, n_{\mathcal{S}_1}(s)))\\ &= \min(q_{\mathcal{S}_2}(s),\, q_{\mathcal{S}_2}(s))e^{i}\min(n_{\mathcal{S}_2}(s),\, n_{\mathcal{S}_2}(s))\\ &= q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}\\ &= \mathcal{S}_2\end{aligned} \tag{4.31}$$

Case 2: If $q_{\mathcal{S}_2}(s) \leqslant q_{\mathcal{S}_1}(s)$ and $n_{\mathcal{S}_2}(s) \leqslant n_{\mathcal{S}_1}(s)$ then equation (4.30) implies

$$\begin{aligned}\mathcal{S}_2 \bigcup (\mathcal{S}_2 \cap \mathcal{S}_1) &= \min(q_{\mathcal{S}_2}(s), \max(q_{\mathcal{S}_2}(s), q_{\mathcal{S}_1}(s)))e^{imin(n_{\mathcal{S}_2}(s),\max(n_{\mathcal{S}_2}(s),n_{\mathcal{S}_1}(s)))}\\ &= \min(q_{\mathcal{S}_2}(s), q_{\mathcal{S}_1}(s))e^{i}min(n_{\mathcal{S}_2}(s), n_{\mathcal{S}_1}(s))\\ &= q_{\mathcal{S}_2}(s)e^{in_{\mathcal{S}_2}(s)}\\ &= \mathcal{S}_2\end{aligned} \tag{4.32}$$

From equations (4.31) and (4.32)

$$\mathcal{S}_2 \bigcup (\mathcal{S}_2 \cap \mathcal{S}_1) = \mathcal{S}_2$$

□

5. New complex intuitionistic fuzzy distance

Complex intuitionistic fuzzy metrics for distance that are new in fuzzy reasoning and making decisions, the hesitance degree is frequently used to express the degree of doubt or indecision about a certain option or preference. When taking hesitance degree is frequently used to express the degree of doubt or indecision about a certain option or preference. When taking hesitance values into account together with DMs, it usually refers to circumstances in which likenesses or preferences are not absolute but rather range. When making decisions involving several criteria when the decision-maker is unsure regarding the relative relevance or each criterion's weight, hesitancy values can be added to DMs. While working using data points that have some degree of membership in other clusters but don't quite fit into one values can be added to DMs. Determining an appropriate measure that accurately reflects the lack of clarity or delay in the setting of the particular problem or solution is essential when hesitation values are combined with DMs. To deal with imprecision and uncertainty in the data,fuzzy logic techniques or comparable methods are frequently used. The CIFDMs with hesitance degree however are important in preference modeling, pattern identification, medical diagnosis, interaction between humans and computers and decision-making. Hesitancy metrics are often used to show the degree of match doubt when evaluating string or sequences. This is particularly important in situations where there may be uncertainty or incomplete matches, resulting in incorrect linking or information retrieval. Here, we describe some unique DMs known as DMCIF with a hesitance degree in the context of CIFSs.

5.1. Intuitionistic complex fuzzy hesitance distance measures

Definition 16. A Hesitance is a function of CIFSs is a function d_H: $\mathcal{S}(\mathcal{P}) \times \mathcal{S}(\mathcal{P}) \rightarrow [0, 1]$ and d_H: $\mathcal{S}(\mathcal{Q}) \times \mathcal{S}(\mathcal{Q}) \rightarrow [0, 1]$. For any $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{P})$ and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{Q})$

i. $0 \leqslant d_H(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$, $d_H(\mathcal{S}_1, \mathcal{S}_2) = 0 \Leftrightarrow \mathcal{S}_1 = \mathcal{S}_2$

ii. $d_H(\mathcal{S}_1, \mathcal{S}_2) = d_H(\mathcal{S}_2, \mathcal{S}_1)$

iii. $d_H(\mathcal{S}_1, \mathcal{S}_3) \leqslant d_H(\mathcal{S}_1, \mathcal{S}_2) + d_H(\mathcal{S}_2, \mathcal{S}_3)$

$$d_H^P(\mathcal{S}_1, \mathcal{S}_2) = \frac{\left[\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n} \tag{5.1}$$

$$d_H^Q(\mathcal{S}_1, \mathcal{S}_2) = \frac{\left[\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n} \tag{5.2}$$

Here, P and Q denote the complex membership and non-membership functions of the CIFS.

Example 1.

$$\mathcal{S}_1 = \left(\frac{0.8e^{i0.8\pi}}{\mathcal{S}_1}, \frac{0.5e^{i0.7\pi}}{\mathcal{S}_1}\right) + \left(\frac{0.3e^{i0.3\pi}}{\mathcal{S}_2}, \frac{0.2e^{i0.7\pi}}{\mathcal{S}_2}\right) + \left(\frac{0.4e^{i0.5\pi}}{\mathcal{S}_3}, \frac{0.7e^{i0.8\pi}}{\mathcal{S}_3}\right)$$
$$\mathcal{S}_2 = \left(\frac{0.8e^{i0.4\pi}}{\mathcal{S}_1}, \frac{0.1e^{i0.9\pi}}{\mathcal{S}_1}\right) + \left(\frac{0.6e^{i0.7\pi}}{\mathcal{S}_2}, \frac{0.7e^{i0.6\pi}}{\mathcal{S}_2}\right) + \left(\frac{0.5e^{i0.3\pi}}{\mathcal{S}_3}, \frac{0.6e^{i0.7\pi}}{\mathcal{S}_3}\right)$$

$$d_H^P(\mathcal{S}_1, \mathcal{S}_2) = \frac{\begin{bmatrix} |0.8 - 0.8| + \frac{1}{2\pi}|0.8\pi - 0.4\pi| + |0.4 - 0.6| \\ +|0.3 - 0.6| + \frac{1}{2\pi}|0.3\pi - 0.7\pi| + |0.15 - 0.25| \\ +|0.4 - 0.5| + \frac{1}{2\pi}|0.5\pi - 0.3\pi| + |0.15 - 0.35| \end{bmatrix}}{3n}$$

$$= \frac{[0 + 0.2 + 0.2 + 0.3 + 0.2 + 0.1 + 0.1 + 0.1 + 0.2]}{3(3)} = 0.15$$

$$d_H^Q(\mathcal{S}_1, \mathcal{S}_2) = \frac{\begin{bmatrix} |0.5 - 0.1| + \frac{1}{2\pi}|0.7\pi - 0.9\pi| + |015 - 0.55| \\ +|0.2 - 0.7| + \frac{1}{2\pi}|0.7\pi - 0.6\pi| + |0.15 - 0.4| \\ +|0.7 - 0.6| + \frac{1}{2\pi}|0.8\pi - 0.7\pi| + |0.3 - 0.25| \end{bmatrix}}{3n}$$

$$= \frac{[0.4 + 0.1 + 0.2 + 0.5 + 0.05 + 0.25 + 0.1 + 0.05 + 0.05]}{3(3)} = 0.18$$

$$d_H(\mathcal{S}_1, \mathcal{S}_2) = (0.15, 0.18)$$

Theorem 6. *The function defined in equations (5.1) and (5.2) is a distance function*

Proof. For Membership function,

i. The condition $d_H(\mathcal{S}_1, \mathcal{S}_2) \geqslant 0$ obviously true. Next,

$$d_H^P(\mathcal{S}_1, \mathcal{S}_2) = \frac{\left[\sum_{i=1}^n |\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}$$

$$= \sum_{i=1}^n \frac{[1 + 1 + 1]}{3n} = \frac{3}{3n} = 1,\ 0 \leqslant d_H^P(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$$

ii.

$$d_H^P(\mathcal{S}_1, \mathcal{S}_2) = \frac{\left[\sum_{i=1}^n |\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}$$

$$= \frac{\left[\sum_{i=1}^n |\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}$$

$$= d_H^P(\mathcal{S}_2, \mathcal{S}_1)$$

iii.

$$d_H^P(\mathcal{S}_1, \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^n [|\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|] \end{bmatrix}$$

$$= \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^n |\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_3}(s_i)| + |\mathcal{P}_{\mathcal{S}_3}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_3}(s_i)| + |m_{\mathcal{S}_3}(s_i) - m_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_3}(s_i)| + |h_{\mathcal{S}_3}(s_i) - h_{\mathcal{S}_2}(s_i)| \end{bmatrix}$$

$$= \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^n |\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_3}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_3}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_3}(s_i)| \end{bmatrix} + \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^n |\mathcal{P}_{\mathcal{S}_2}(s_i) - \mathcal{P}_{\mathcal{S}_3}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_2}(s_i) - m_{\mathcal{S}_3}(s_i)| \\ +|h_{\mathcal{S}_2}(s_i) - h_{\mathcal{S}_3}(s_i)| \end{bmatrix}$$

$$= d_H^P(\mathcal{S}_1, \mathcal{S}_3) + d_H^P(\mathcal{S}_3, \mathcal{S}_2).$$

For Non membership function

i. The condition $d_H^Q(\mathcal{S}_1, \mathcal{S}_2) \geqslant 0$ obviously true. Next,

$$d_H^Q(\mathcal{S}_1, \mathcal{S}_2) = \frac{\left[\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}$$
$$= \sum_{i=1}^{n}\frac{[1+1+1]}{3n} = \frac{3}{3n} = 1,\ 0 \leqslant d_H^Q(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$$

ii.

$$d_H^Q(\mathcal{S}_1, \mathcal{S}_2) = \frac{\left[\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}$$
$$= \frac{\left[\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}$$
$$= d_H^Q(\mathcal{S}_2, \mathcal{S}_1)$$

iii.

$$d_H^Q(\mathcal{S}_1, \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)| \end{bmatrix}$$

$$= \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_3}(s_i)| + |\mathcal{Q}_{\mathcal{S}_3}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_3}(s_i)| + |n_{\mathcal{S}_3}(s_i) - n_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_3}(s_i)| + |h_{\mathcal{S}_3}(s_i) - h_{\mathcal{S}_2}(s_i)| \end{bmatrix}$$

$$= \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_3}(s_i)| \\ +\frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_3}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_3}(s_i)| \end{bmatrix} + \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_2}(s_i) - \mathcal{Q}_{\mathcal{S}_3}(s_i)| \\ +\frac{1}{2\pi}|n_{\mathcal{S}_2}(s_i) - n_{\mathcal{S}_3}(s_i)| \\ +|h_{\mathcal{S}_2}(s_i) - h_{\mathcal{S}_3}(s_i)| \end{bmatrix}$$

$$= d_H^Q(\mathcal{S}_1, \mathcal{S}_3) + d_H^Q(\mathcal{S}_3, \mathcal{S}_2)$$

□

Definition 17. A Euclidean Hesitance DM of CIFSs is a function d_{EH}: $\mathcal{S}(\mathcal{P}) \times \mathcal{S}(\mathcal{P}) \rightarrow [0, 1]$ and d_{EH}: $\mathcal{S}(\mathcal{Q}) \times \mathcal{S}(\mathcal{Q}) \rightarrow [0, 1]$. For any $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{P})$ and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in \mathcal{S}(\mathcal{Q})$

i. $0 \leqslant d_{EH}(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$, $d_{EH}(\mathcal{S}_1, \mathcal{S}_2) = 0 \Leftrightarrow \mathcal{S}_1 = \mathcal{S}_2$

ii. $d_{EH}(\mathcal{S}_1, \mathcal{S}_2) = d_{EH}(\mathcal{S}_2, \mathcal{S}_1)$

iii. $d_{EH}(\mathcal{S}_1, \mathcal{S}_3) \leqslant d_{EH}(\mathcal{S}_1, \mathcal{S}_2) + d_{EH}(\mathcal{S}_2, \mathcal{S}_3)$

$$d_{EH}^P(\mathcal{S}_1, \mathcal{S}_2) = \sqrt{\frac{\sum_{i=1}^{n}[(\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i))^2 + \frac{1}{2\pi}(m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i))^2 + (h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i))^2]}{3n}} \tag{5.3}$$

$$d_{EH}^Q(\mathcal{S}_1, \mathcal{S}_2) = \sqrt{\frac{\sum_{i=1}^{n}[(\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i))^2 + \frac{1}{2\pi}(n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i))^2 + (h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i))^2]}{3n}} \tag{5.4}$$

Here, P and Q denote the complex membership and non-membership functions of the CIFS.

Example 2.

$$\mathcal{S}_1 = \left(\frac{0.8e^{i0.8\pi}}{\mathcal{S}_1}, \frac{0.5e^{i0.7\pi}}{\mathcal{S}_1}\right) + \left(\frac{0.3e^{i0.3\pi}}{\mathcal{S}_2}, \frac{0.2e^{i0.7\pi}}{\mathcal{S}_2}\right) + \left(\frac{0.4e^{i0.5\pi}}{\mathcal{S}_3}, \frac{0.7e^{i0.8\pi}}{\mathcal{S}_3}\right)$$

$$\mathcal{S}_2 = \left(\frac{0.8e^{i0.4\pi}}{\mathcal{S}_1}, \frac{0.1e^{i0.9\pi}}{\mathcal{S}_1}\right) + \left(\frac{0.6e^{i0.7\pi}}{\mathcal{S}_2}, \frac{0.7e^{i0.6\pi}}{\mathcal{S}_2}\right) + \left(\frac{0.5e^{i0.3\pi}}{\mathcal{S}_3}, \frac{0.6e^{i0.7\pi}}{\mathcal{S}_3}\right)$$

then for membership function,

$$\begin{aligned} d_{EH}^{P}(S_1, S_2) &= \sqrt{\frac{\begin{bmatrix} (0.8 - 0.8)^2 + \frac{1}{2\pi}(0.8\pi - 0.4\pi)^2 + (0.4 - 0.6)^2 \\ +(0.3 - 0.6)^2 + \frac{1}{2\pi}(0.3\pi - 0.7\pi)^2 + (0.15 - 0.25)^2 \\ +(0.4 - 0.5)^2 + \frac{1}{2\pi}(0.5\pi - 0.3\pi)^2 + (0.15 - 0.35)^2 \end{bmatrix}}{3n}} \\ &= \sqrt{\frac{0 + 0.04 + 0.04 + 0.09 + 0.04 + 0.01 + 0.01 + 0.01 + 0.04}{9}} \\ &= \sqrt{\frac{0.28}{9}} = 0.17 \end{aligned}$$

For non membership function

$$\begin{aligned} d_{EH}^{Q}(S_1, S_2) &= \sqrt{\frac{\begin{bmatrix} (0.5 - 0.1)^2 + \frac{1}{2\pi}(0.7\pi - 0.9\pi)^2 + (015 - 0.35)^2 \\ +(0.2 - 0.7)^2 + \frac{1}{2\pi}(0.7\pi - 0.6\pi)^2 + (0.15 - 0.4)^2 \\ +(0.4 - 0.5)^2 + \frac{1}{2\pi}(0.8\pi - 0.7\pi)^2 + (0.3 - 0.25)^2 \end{bmatrix}}{3n}} \\ &= \sqrt{\frac{[0.16 + 0.01 + 0.04 + 0.25 + 0.0025 + 0.0625 + 0.01 + 0.0025 + 0.0025]}{9}} \\ &= 0.24 \\ d_{EH}(S_1, S_2) &= (0.17, 0.24) \end{aligned}$$

Theorem 7. *The function defined in equations (5.3) and (5.4) is a distance function*

Proof. For membership function,

i. The condition $d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2) \geqslant 0$ obviously true. Next,

$$\begin{aligned} d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2) &= \sqrt{\frac{\left[\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}} \\ &= \sqrt{\sum_{i=1}^{n}\frac{[1 + 1 + 1]}{3n}} = \sqrt{\frac{3}{3n}} = \sqrt{\frac{1}{n}} \end{aligned}$$

If $n = 1$ then $d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2) = 1$otherwise $d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2) < 1, 0 \leqslant d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$

ii.

$$\begin{aligned} d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2) &= \sqrt{\frac{\left[\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}} \\ &= \sqrt{\frac{\left[\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}} \\ &= d_{EH}^{P}(\mathcal{S}_2, \mathcal{S}_1) \end{aligned}$$

iii.

$$d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2) = \sqrt{\frac{1}{3n}\left[\begin{array}{c}\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\end{array}\right]}$$

$$= \sqrt{\frac{1}{3n}\left[\begin{array}{c}\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_3}(s_i) + \mathcal{P}_{\mathcal{S}_3}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) + m_{\mathcal{S}_3}(s_i) - m_{\mathcal{S}_3}(s_i) - m_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i)h_{\mathcal{S}_3}(s_i)h_{\mathcal{S}_3}(s_i) - h_{\mathcal{S}_2}(s_i)|\end{array}\right]}$$

$$= \sqrt{\frac{1}{3n}\left[\begin{array}{c}\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_3}(s_i)| + |\mathcal{P}_{\mathcal{S}_3}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_3}(s_i)| + |m_{\mathcal{S}_3}(s_i) - m_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_3}(s_i)| + |h_{\mathcal{S}_3}(s_i) - h_{\mathcal{S}_2}(s_i)|\end{array}\right]}$$

$$= \sqrt{\frac{1}{3n}\left[\begin{array}{c}\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_3}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_3}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_3}(s_i)|\end{array}\right]} + \sqrt{\frac{1}{3n}\left[\begin{array}{c}\sum_{i=1}^{n}|\mathcal{P}_{\mathcal{S}_2}(s_i) - \mathcal{P}_{\mathcal{S}_3}(s_i)| \\ +\frac{1}{2\pi}|m_{\mathcal{S}_2}(s_i) - m_{\mathcal{S}_3}(s_i)| \\ +|h_{\mathcal{S}_2}(s_i) - h_{\mathcal{S}_3}(s_i)|\end{array}\right]}$$

$$= d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_3) + d_{EH}^{P}(\mathcal{S}_3, \mathcal{S}_2).$$

For non membership function

i. The condition $d_H(\mathcal{S}_1, \mathcal{S}_2) \geqslant 0$ obviously true. Next,

$$d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_2) = \sqrt{\frac{\left[\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}}$$

$$= \sqrt{\sum_{i=1}^{n}\frac{[1+1+1]}{3n}} = \sqrt{\frac{3}{3n}} = \sqrt{\frac{1}{n}}$$

If $n = 1$ then $d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_2) = 1$otherwise $d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_2) < 1,\ 0 \leqslant d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_2) \leqslant 1$

ii.

$$d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_2) = \sqrt{\frac{\left[\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}}$$

$$= \sqrt{\frac{\left[\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| + \frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| + |h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\right]}{3n}}$$

$$= d_{EH}^{Q}(\mathcal{S}_2, \mathcal{S}_1)$$

iii.

$$d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_2) = \sqrt{\frac{1}{3n}\left[\begin{array}{c}\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i)|\end{array}\right]}$$

$$= \sqrt{\frac{1}{3n}\left[\begin{array}{c}\sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_3}(s_i)| + |\mathcal{Q}_{\mathcal{S}_3}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i)| \\ +\frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_3}(s_i)| + |n_{\mathcal{S}_3}(s_i) - n_{\mathcal{S}_2}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_3}(s_i)| + |h_{\mathcal{S}_3}(s_i) - h_{\mathcal{S}_2}(s_i)|\end{array}\right]}$$

$$= \sqrt{\frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_3}(s_i)| \\ +\frac{1}{2\pi}|n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_3}(s_i)| \\ +|h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_3}(s_i)| \end{bmatrix}} + \sqrt{\frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|\mathcal{Q}_{\mathcal{S}_2}(s_i) - \mathcal{Q}_{\mathcal{S}_3}(s_i)| \\ +\frac{1}{2\pi}|n_{\mathcal{S}_2}(s_i) - n_{\mathcal{S}_3}(s_i)| \\ +|h_{\mathcal{S}_2}(s_i) - h_{\mathcal{S}_3}(s_i)| \end{bmatrix}}$$

$$= d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_3) + d_{EH}^{Q}(\mathcal{S}_3, \mathcal{S}_2)$$ □

Based on their union and intersection operators, theorems 2 and 3 establish the basic set theoretic relationships of CIFSs. Although the proofs in this paper employ case-based reasoning for clarity, the algebraic properties of complex-valued membership and non-membership functions like commutativity, associativity, and distributivity ensure that the same relationships hold for all CIFS tuples. Thus, the validity of the theorems extends to the general algebraic formulation of CIFSs without dependence on case distinctions. Furthermore, the proposed distance measures HDMCIFS and EHDMCIFS satisfy the essential metric axioms. Because each component is constructed using absolute differences, non-negativity, identity, and symmetry follow directly. The triangle inequality is satisfied under the standard assumption in CIFS literature that phase values are treated as real-valued quantities in $[0, 2\pi]$ without circular wrap-around. Under this linearized phase representation, the additive and Euclidean-type terms used in the definitions preserve the triangle inequality, ensuring that HDMCIFS and EHDMCIFS behave as valid real metrics within the CIFS framework.

Definition 18. Let d_H be the hesitance measure of CIFSs. Then the complement hesitance distance measure $(d_H(\mathcal{S}_1)^c, d_H(\mathcal{S}_2)^c)$ of two CIFs is defined as

$$(d_H^P(\mathcal{S}_1)^c, d_H^P(\mathcal{S}_2)^c) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|(1 - \mathcal{P}_{\mathcal{S}_1}(s_i)) - (1 - \mathcal{P}_{\mathcal{S}_2}(s_i))| \\ +\frac{1}{2\pi}|(m_{\mathcal{S}_1}(s_i)) - (m_{\mathcal{S}_2}(s_i))| \\ +|(1 - h_{\mathcal{S}_1}(s_i)) - (1 - h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$(d_H^Q(\mathcal{S}_1)^c, d_H^Q(\mathcal{S}_2)^c) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|(1 - \mathcal{Q}_{\mathcal{S}_1}(s_i)) - (1 - \mathcal{Q}_{\mathcal{S}_2}(s_i))| \\ +\frac{1}{2\pi}|(n_{\mathcal{S}_1}(s_i)) - (n_{\mathcal{S}_2}(s_i))| \\ +|(1 - h_{\mathcal{S}_1}(s_i)) - (1 - h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

Definition 19. Let d_H be the hesitance measure of CIFSs. Then the max hesitance distance measure $\mathcal{S}_1$ and $\mathcal{S}_2$ of two CIFs is defined as

$$d_H^P(\mathcal{S}_1 \bigcup \mathcal{S}_2, \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|(\mathcal{P}_{\mathcal{S}_1}(s_i) \cup \mathcal{P}_{\mathcal{S}_2}(s_i) - \mathcal{P}_{\mathcal{S}_1}(s_i)| \\ +\frac{1}{2\pi}|(m_{\mathcal{S}_1}(s_i) \cup m_{\mathcal{S}_2}(s_i) - m_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i) - h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$d_H^Q(\mathcal{S}_1 \bigcup \mathcal{S}_2, \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|(\mathcal{Q}_{\mathcal{S}_1}(s_i) \cup \mathcal{Q}_{\mathcal{S}_2}(s_i) - \mathcal{Q}_{\mathcal{S}_1}(s_i))| \\ +\frac{1}{2\pi}|(n_{\mathcal{S}_1}(s_i) \cup n_{\mathcal{S}_2}(s_i) - n_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i) - h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$d_H^P(\mathcal{S}_1, \mathcal{S}_2 \bigcup \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|(\mathcal{P}_{\mathcal{S}_1}(s_i) - \mathcal{P}_{\mathcal{S}_2}(s_i) \cup \mathcal{P}_{\mathcal{S}_1}(s_i))| \\ +\frac{1}{2\pi}|(m_{\mathcal{S}_1}(s_i) - m_{\mathcal{S}_2}(s_i) \cup m_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i) \cup h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$d_H^Q(\mathcal{S}_1, \mathcal{S}_2 \bigcup \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|(\mathcal{Q}_{\mathcal{S}_1}(s_i) - \mathcal{Q}_{\mathcal{S}_2}(s_i) \cup \mathcal{Q}_{\mathcal{S}_1}(s_i))| \\ +\frac{1}{2\pi}|(n_{\mathcal{S}_1}(s_i) - n_{\mathcal{S}_2}(s_i) \cup n_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) - h_{\mathcal{S}_2}(s_i) \cup h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

where $\cup$ is a max operator of CIFSs

Definition 20. Let d_H be the hesitance measure of CIFSs. Then the min hesitance distance measure $\mathcal{S}_1$ and $\mathcal{S}_2$ of two CIFs is defined as

$$d_H^P(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n}|(\mathcal{P}_{\mathcal{S}_1}(s_i) \cap \mathcal{P}_{\mathcal{S}_2}(s_i)) - \mathcal{P}_{\mathcal{S}_1}(s_i))| \\ +\frac{1}{2\pi}|(m_{\mathcal{S}_1}(s_i) \cap m_{\mathcal{S}_2}(s_i) - m_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i) - h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$d_H^Q(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} |(\mathcal{Q}_{\mathcal{S}_1}(s_i) \cap \mathcal{Q}_{\mathcal{S}_2}(s_i) - \mathcal{Q}_{\mathcal{S}_1}(s_i))| \\ +\frac{1}{2\pi}|(n_{\mathcal{S}_1}(s_i) \cap n_{\mathcal{S}_2}(s_i) - n_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i) - h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$d_H^P(\mathcal{S}_1, \mathcal{S}_2 \cap \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} |(\mathcal{P}_{\mathcal{S}_1}(s_i) - (\mathcal{P}_{\mathcal{S}_2}(s_i) \cap \mathcal{P}_{\mathcal{S}_1}(s_i)))| \\ +\frac{1}{2\pi}|(m_{\mathcal{S}_1}(s_i) - (m_{\mathcal{S}_2}(s_i) \cap m_{\mathcal{S}_2}(s_i)))| \\ +|(h_{\mathcal{S}_1}(s_i) - (h_{\mathcal{S}_2}(s_i) \cap h_{\mathcal{S}_2}(s_i)))| \end{bmatrix}$$

$$d_H^Q(\mathcal{S}_1, \mathcal{S}_2 \cap \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} |(\mathcal{Q}_{\mathcal{S}_1}(s_i) - (\mathcal{Q}_{\mathcal{S}_2}(s_i) \cap \mathcal{Q}_{\mathcal{S}_1}(s_i)))| \\ +\frac{1}{2\pi}|(n_{\mathcal{S}_1}(s_i) - (n_{\mathcal{S}_2}(s_i) \cap n_{\mathcal{S}_2}(s_i)))| \\ +|(h_{\mathcal{S}_1}(s_i) - (h_{\mathcal{S}_2}(s_i) \cap h_{\mathcal{S}_2}(s_i)))| \end{bmatrix}$$

where $\cap$ is a min operator of CIFSs

Definition 21. Let d_H be the hesitance measure of CIFSs. Then the max-min hesitance distance measure $\mathcal{S}_1$ and $\mathcal{S}_2$ of two CIFs is defined as

$$d_H^P(\mathcal{S}_1 \bigcup \mathcal{S}_2, \mathcal{S}_1 \cap \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} |(\mathcal{P}_{\mathcal{S}_1}(s_i) \cup \mathcal{P}_{\mathcal{S}_2}(s_i)) - (\mathcal{P}_{\mathcal{S}_1}(s_i) \cap \mathcal{P}_{\mathcal{S}_2}(s_i))| \\ +\frac{1}{2\pi}|(m_{\mathcal{S}_1}(s_i) \cup m_{\mathcal{S}_2}(s_i)) - (m_{\mathcal{S}_1}(s_i) \cap m_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i)) - (h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$d_H^P(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_1 \bigcup \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} |(\mathcal{P}_{\mathcal{S}_1}(s_i) \cap \mathcal{P}_{\mathcal{S}_2}(s_i)) - (\mathcal{P}_{\mathcal{S}_1}(s_i) \cup \mathcal{P}_{\mathcal{S}_2}(s_i))| \\ +\frac{1}{2\pi}|(m_{\mathcal{S}_1}(s_i) \cap m_{\mathcal{S}_2}(s_i)) - (m_{\mathcal{S}_1}(s_i) \cup m_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i)) - (h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$d_H^Q(\mathcal{S}_1 \bigcup \mathcal{S}_2, \mathcal{S}_1 \cap \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} |(\mathcal{Q}_{\mathcal{S}_1}(s_i) \cup \mathcal{Q}_{\mathcal{S}_2}(s_i)) - (\mathcal{Q}_{\mathcal{S}_1}(s_i) \cap \mathcal{Q}_{\mathcal{S}_2}(s_i))| \\ +\frac{1}{2\pi}|(n_{\mathcal{S}_1}(s_i) \cup n_{\mathcal{S}_2}(s_i)) - (n_{\mathcal{S}_1}(s_i) \cap n_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i)) - (h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

$$d_H^Q(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_1 \bigcup \mathcal{S}_2) = \frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} |(\mathcal{Q}_{\mathcal{S}_1}(s_i) \cap \mathcal{Q}_{\mathcal{S}_2}(s_i)) - (\mathcal{Q}_{\mathcal{S}_1}(s_i) \cup \mathcal{Q}_{\mathcal{S}_2}(s_i))| \\ +\frac{1}{2\pi}|(n_{\mathcal{S}_1}(s_i) \cap n_{\mathcal{S}_2}(s_i)) - (n_{\mathcal{S}_1}(s_i) \cup n_{\mathcal{S}_2}(s_i))| \\ +|(h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i)) - (h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i))| \end{bmatrix}$$

where $\cup$ and $\cap$ is a max and min operator of CIFSs

Definition 22. Let d_{EH} be the Euclidean Hesitance measure of CIFSs. Then the complement Hesitance distance measure $(d_{EH}(\mathcal{S}_1)^c, d_{EH}(\mathcal{S}_2)^c)$ of two CIFs is defined as

$$(d_{EH}^P(\mathcal{S}_1)^c, d_{EH}^P(\mathcal{S}_2)^c) = \sqrt{\frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} ((1 - \mathcal{P}_{\mathcal{S}_1}(s_i)) - (1 - \mathcal{P}_{\mathcal{S}_2}(s_i)))^2 \\ +\frac{1}{2\pi^2}(((m_{\mathcal{S}_1}(s_i)) - m_{\mathcal{S}_2}(s_i)))^2 \\ +((1 - h_{\mathcal{S}_1}(s_i)) - (1 - h_{\mathcal{S}_2}(s_i)))^2 \end{bmatrix}}$$

$$(d_{EH}^Q(\mathcal{S}_1)^c, d_{EH}^Q(\mathcal{S}_2)^c) = \sqrt{\frac{1}{3n}\begin{bmatrix} \sum_{i=1}^{n} ((1 - \mathcal{Q}_{\mathcal{S}_1}(s_i)) - (1 - \mathcal{Q}_{\mathcal{S}_2}(s_i)))^2 \\ +\frac{1}{2\pi^2}((n_{\mathcal{S}_1}(s_i)) - (n_{\mathcal{S}_2}(s_i)))^2 \\ +((1 - h_{\mathcal{S}_1}(s_i)) - (1 - h_{\mathcal{S}_2}(s_i)))^2 \end{bmatrix}}$$

Definition 23. Let d_{EH} be the Euclidean Hesitance measure of CIFSs. Then the max hesitance distance measure $\mathcal{S}_1$ and $\mathcal{S}_2$ of two CIFs is defined as

$$d_{EH}^P(\mathcal{S}_1 \bigcup \mathcal{S}_2, \mathcal{S}_2) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} ((\mathcal{P}_{\mathcal{S}_1}(s_i) \cup \mathcal{P}_{\mathcal{S}_2}(s_i)) - \mathcal{P}_{\mathcal{S}_2}(s_i))^2 \\ +\frac{1}{2\pi^2}((m_{\mathcal{S}_1}(s_i) \cup m_{\mathcal{S}_2}(s_i)) - m_{\mathcal{S}_2}(s_i))^2 \\ +((h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i)) - h_{\mathcal{S}_2}(s_i))^2 \end{bmatrix}}$$

$$d_{EH}^Q(\mathcal{S}_1 \bigcup \mathcal{S}_2, \mathcal{S}_2) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} ((\mathcal{Q}_{\mathcal{S}_1}(s_i) \cup \mathcal{Q}_{\mathcal{S}_2}(s_i)) - \mathcal{Q}_{\mathcal{S}_2}(s_i))^2 \\ +\frac{1}{2\pi^2}((n_{\mathcal{S}_1}(s_i) \cup n_{\mathcal{S}_2}(s_i)) - n_{\mathcal{S}_2}(s_i))^2 \\ +((h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i)) - h_{\mathcal{S}_2}(s_i))^2 \end{bmatrix}}$$

$$d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2 \bigcup \mathcal{S}_1) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} (\mathcal{P}_{\mathcal{S}_1}(s_i) - (\mathcal{P}_{\mathcal{S}_2}(s_i) \cup \mathcal{P}_{\mathcal{S}_1}(s_i)))^2 \\ +\frac{1}{2\pi^2}(m_{\mathcal{S}_1}(s_i) - (m_{\mathcal{S}_2}(s_i) \cup m_{\mathcal{S}_1}(s_i)))^2 \\ +(h_{\mathcal{S}_1}(s_i) - (h_{\mathcal{S}_2}(s_i) \cup h_{\mathcal{S}_1}(s_i)))^2 \end{bmatrix}}$$

$$d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_2 \bigcup \mathcal{S}_1) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} (\mathcal{Q}_{\mathcal{S}_1}(s_i) - (\mathcal{Q}_{\mathcal{S}_2}(s_i) \cup \mathcal{Q}_{\mathcal{S}_1}(s_i)))^2 \\ +\frac{1}{2\pi^2}(n_{\mathcal{S}_1}(s_i) - (n_{\mathcal{S}_2}(s_i) \cup n_{\mathcal{S}_1}(s_i)))^2 \\ +(h_{\mathcal{S}_1}(s_i) - (h_{\mathcal{S}_2}(s_i) \cup h_{\mathcal{S}_1}(s_i)))^2 \end{bmatrix}}$$

where $\cup$ is a max operator of CIFSs

Definition 24. Let d_{EH} be the Euclidean Hesitance measure of CIFSs. Then the max desitance distance measure $\mathcal{S}_1$ and $\mathcal{S}_2$ of two CIFs is defined as

$$d_{EH}^{P}(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_2) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} |(\mathcal{P}_{\mathcal{S}_1}(s_i) \cap \mathcal{P}_{\mathcal{S}_2}(s_i)) - \mathcal{P}_{\mathcal{S}_1}(s_i)| \\ +\frac{1}{2\pi^2}|(m_{\mathcal{S}_1}(s_i) \cap m_{\mathcal{S}_2}(s_i)) - m_{\mathcal{S}_2}(s_i)| \\ +|(h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i)) - h_{\mathcal{S}_2}(s_i)| \end{bmatrix}}$$

$$d_{EH}^{Q}(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_2) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} |(\mathcal{Q}_{\mathcal{S}_1}(s_i) \cap \mathcal{Q}_{\mathcal{S}_2}(s_i)) - \mathcal{Q}_{\mathcal{S}_1}(s_i)| \\ +\frac{1}{2\pi^2}|(n_{\mathcal{S}_1}(s_i) \cap n_{\mathcal{S}_2}(s_i)) - n_{\mathcal{S}_2}(s_i)| \\ +|(h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i)) - h_{\mathcal{S}_2}(s_i)| \end{bmatrix}}$$

$$d_{EH}^{P}(\mathcal{S}_1, \mathcal{S}_2 \cap \mathcal{S}_1) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} |\mathcal{P}_{\mathcal{S}_1}(s_i) - (\mathcal{P}_{\mathcal{S}_2}(s_i) \cap \mathcal{P}_{\mathcal{S}_1}(s_i))| \\ +\frac{1}{2\pi^2}|m_{\mathcal{S}_1}(s_i) - (m_{\mathcal{S}_2}(s_i) \cap m_{\mathcal{S}_1}(s_i))| \\ +|h_{\mathcal{S}_1}(s_i) - (h_{\mathcal{S}_2}(s_i) \cap h_{\mathcal{S}_1}(s_i))| \end{bmatrix}}$$

$$d_{EH}^{Q}(\mathcal{S}_1, \mathcal{S}_2 \cap \mathcal{S}_1) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} |\mathcal{Q}_{\mathcal{S}_1}(s_i) - (\mathcal{Q}_{\mathcal{S}_2}(s_i) \cap \mathcal{Q}_{\mathcal{S}_1}(s_i))| \\ +\frac{1}{2\pi^2}|n_{\mathcal{S}_1}(s_i) - (n_{\mathcal{S}_2}(s_i) \cap n_{\mathcal{S}_1}(s_i))| \\ +|h_{\mathcal{S}_1}(s_i) - (h_{\mathcal{S}_2}(s_i) \cap h_{\mathcal{S}_1}(s_i))| \end{bmatrix}}$$

where $\cap$ is a max operator of CIFSs

Definition 25. Let d_{EH} be CIFSs' Euclidean hesitancy measure. Next, the maximum-min Hesitance distance measurements of two CIFs, $\mathcal{S}_1$ and $\mathcal{S}_2$, are specified as

$$d_{EH}^{P}(\mathcal{S}_1 \bigcup \mathcal{S}_2, \mathcal{S}_1 \cap \mathcal{S}_2) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} ((\mathcal{P}_{\mathcal{S}_1}(s_i) \cup \mathcal{P}_{\mathcal{S}_2}(s_i)) - (\mathcal{P}_{\mathcal{S}_1}(s_i) \cap \mathcal{P}_{\mathcal{S}_2}(s_i)))^2 \\ +\frac{1}{2\pi^2}((m_{\mathcal{S}_1}(s_i) \cup m_{\mathcal{S}_2}(s_i)) - (m_{\mathcal{S}_1}(s_i) \cap m_{\mathcal{S}_2}(s_i)))^2 \\ +((h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i)) - (h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i)))^2 \end{bmatrix}}$$

$$d_{EH}^{Q}(\mathcal{S}_1 \bigcup \mathcal{S}_2, \mathcal{S}_1 \cap \mathcal{S}_2) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} ((\mathcal{Q}_{\mathcal{S}_1}(s_i) \cup \mathcal{Q}_{\mathcal{S}_2}(s_i)) - (\mathcal{Q}_{\mathcal{S}_1}(s_i) \cap \mathcal{Q}_{\mathcal{S}_2}(s_i)))^2 \\ +\frac{1}{2\pi^2}((n_{\mathcal{S}_1}(s_i) \cup n_{\mathcal{S}_2}(s_i)) - (n_{\mathcal{S}_1}(s_i) \cap n_{\mathcal{S}_2}(s_i)))^2 \\ +((h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i)) - (h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i)))^2 \end{bmatrix}}$$

$$d_{EH}^{P}(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_1 \bigcup \mathcal{S}_2) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} ((\mathcal{P}_{\mathcal{S}_1}(s_i) \cap \mathcal{P}_{\mathcal{S}_2}(s_i)) - (\mathcal{P}_{\mathcal{S}_1}(s_i) \cup \mathcal{P}_{\mathcal{S}_2}(s_i)))^2 \\ +\frac{1}{2\pi^2}((m_{\mathcal{S}_1}(s_i) \cap m_{\mathcal{S}_2}(s_i)) - (m_{\mathcal{S}_1}(s_i) \cup m_{\mathcal{S}_2}(s_i)))^2 \\ +((h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i)) - (h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i)))^2 \end{bmatrix}}$$

$$d_{EH}^{Q}(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_1 \bigcup \mathcal{S}_2) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\begin{bmatrix} ((\mathcal{Q}_{\mathcal{S}_1}(s_i) \cap \mathcal{Q}_{\mathcal{S}_2}(s_i)) - (\mathcal{Q}_{\mathcal{S}_1}(s_i) \cup \mathcal{Q}_{\mathcal{S}_2}(s_i)))^2 \\ +\frac{1}{2\pi^2}((n_{\mathcal{S}_1}(s_i) \cap n_{\mathcal{S}_2}(s_i)) - (n_{\mathcal{S}_1}(s_i) \cup n_{\mathcal{S}_2}(s_i)))^2 \\ +((h_{\mathcal{S}_1}(s_i) \cap h_{\mathcal{S}_2}(s_i)) - (h_{\mathcal{S}_1}(s_i) \cup h_{\mathcal{S}_2}(s_i)))^2 \end{bmatrix}}$$

where $\cup$ and $\cap$ is a max and min operator of CIFSs

Table 2. Collection of attributes.

E_k	δ_i	b_1	b_2	$\dots$	b_j
E_1	δ_1	$(sB_{11}(b_1)e^{imB_{11}(b_1)}, sB_{11}(b_1)e^{inB_{11}(b_1)})$	$(sB_{12}(b_2)e^{imB_{12}(b_2)}, sB_{12}(b_2)e^{inB_{12}(b_2)})$	$\dots$	$(sB_{1j}(b_j)e^{imB_{1j}(b_j)}, sB_{1j}(b_j)e^{inB_{1j}(b_j)})$
	δ_2	$(sB_{21}(b_1)e^{imB_{21}(b_1)}, sB_{21}(b_1)e^{inB_{21}(b_1)})$	$(sB_{22}(b_2)e^{imB_{22}(b_2)}, sB_{22}(b_2)e^{inB_{22}(b_2)})$	$\dots$	$(sB_{2j}(b_j)e^{imB_{2j}(b_j)}, sB_{2j}(b_j)e^{inB_{2j}(b_j)})$
	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
	δ_i	$(sB_{i1}(b_1)e^{imB_{i1}(b_1)}, sB_{i1}(b_1)e^{inB_{i1}(b_1)})$	$(sB_{i2}(b_2)e^{imB_{i2}(b_2)}, sB_{i2}(b_2)e^{inB_{i2}(b_2)})$	$\dots$	$(sB_{ij}(b_j)e^{imB_{ij}(b_j)}, sB_{ij}(b_j)e^{inB_{ij}(b_j)})$
E_2	δ_1	$(sB_{11}(b_1)e^{imB_{11}(b_1)}, sB_{11}(b_1)e^{inB_{11}(b_1)})$	$(sB_{12}(b_2)e^{imB_{12}(b_2)}, sB_{12}(b_2)e^{inB_{12}(b_2)})$	$\dots$	$(sB_{1j}(b_j)e^{imB_{1j}(b_j)} sB_{1j}(b_j)e^{inB_{1j}(b_j)})$
	δ_2	$(sB_{21}(b_1)e^{imB_{21}(b_1)}, sB_{21}(b_1)e^{inB_{21}(b_1)})$	$(sB_{22}(b_2)e^{imB_{22}(b_2)}, sB_{22}(b_2)e^{inB_{22}(b_2)})$	$\dots$	$(sB_{2j}(b_j)e^{imB_{2j}(b_j)}, sB_{2j}(b_j)e^{inB_{2j}(b_j)})$
	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
	δ_i	$(sB_{i1}(b_1)e^{imB_{i1}(b_1)}, sB_{i1}(b_1)e^{inB_{i1}(b_1)})$	$(sB_{i2}(b_2)e^{imB_{i2}(b_2)}, sB_{i2}(b_2)e^{inB_{i2}(b_2)})$	$\dots$	$(sB_{ij}(b_j)e^{imB_{ij}(b_j)}, sB_{ij}(b_j)e^{inB_{ij}(b_j)})$
E_k	δ_1	$(sB_{11}(b_1)e^{imB_{11}(b_1)}, sB_{11}(b_1)e^{inB_{11}(b_1)})$	$(sB_{12}(b_2)e^{imB_{12}(b_2)}, sB_{12}(b_2)e^{inB_{12}(b_2)})$	$\dots$	$(sB_{1j}(b_j)e^{imB_{1j}(b_j)}, sB_{1j}(b_j)e^{inB_{1j}(b_j)})$
	δ_2	$(sB_{21}(b_1)e^{imB_{21}(b_1)}, sB_{21}(b_1)e^{inB_{21}(b_1)})$	$(sB_{22}(b_2)e^{imB_{22}(b_2)}, sB_{22}(b_2)e^{inB_{22}(b_2)})$	$\dots$	$(sB_{2j}(b_j)e^{imB_{2j}(b_j)}, sB_{2j}(b_j)e^{inB_{2j}(b_j)})$
	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
	δ_i	$(sB_{i1}(b_1)e^{imB_{i1}(b_1)}, sB_{i1}(b_1)e^{inB_{i1}(b_1)})$	$(sB_{i2}(b_2)e^{imB_{i2}(b_2)}, sB_{i2}(b_2)e^{inB_{i2}(b_2)})$	$\dots$	$(sB_{ij}(b_j)e^{imB_{ij}(b_j)}, sB_{ij}(b_j)e^{inB_{ij}(b_j)})$

6. Applications for complex intuitionistic fuzzy sets

Different algorithms for making decisions are created for particular situations and issues. A new method for making decisions based on CIFSs and DMCIFSs is suggested in this section. By adding complex numbers, CIFS theory expands on the potential of conventional fuzzy set theory and provides a more sophisticated way to describe and manipulate uncertainty in decision-making scenarios. It provides decision-makers with a strong foundation for handling challenging and unpredictable situations and coming to wise decisions. CIFSs can be used to more accurately simulate competing criteria, preferences, and imprecise information in decision-making situations. Decision-makers can take into account a wider variety of options and make better decisions because of the complexity of the figures. Furthermore, the algorithm probably offers a workable way to include intricate fuzzy distance measures into the decision-making process, as the paper states.

Definition 26. Let $\delta_i = (\delta_1, \delta_2, \dots, \delta_i)$ be range of i alternatives and $b_j = (b_1, b_2, \dots, b_j)$ be range of j attributes. Let $E = E_1, E_2, \dots, E_k$ is a collection of k experts, where E_k represents the kth expert involved in the decision making process. The evaluated value supplied by the expert E_k for alternative δ_i with regard to the criterion b_j is $B_{ij} = sB_{ij}(b_j)e^{imB_{ij}(b_j)}$, which is the attribute value of alternatives δ_i for index b_j. Thus, the table that was produced using these assessed values is given in table 2.

Definition 27. Let $\delta_i = (\delta_1, \delta_2, \dots, \delta_i)$ be range of i alternatives and $b_j = (b_1, b_2, \dots, b_j)$ be range of j attributes. Let $E = E_1, E_2, \dots, E_k$ is a collection of k experts, where E_k represents the kth expert involved in the decision making process. The evaluated value supplied by the expert E_k for alternative δ_i with regard to the criterion b_j is $B_{ij} = sB_{ij}(b_j)e^{imB_{ij}(b_j)}$, which is the attribute value of alternatives δ_i for index b_j. Then, as specified by table 2, is the elements of the distance matrix.

Definition 28. The score function $S(\delta_i)$ of i alternative(δ_i);i = 1,2,…,n

$$[DM]_{3\times 3} = \begin{bmatrix} (d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_2)}), d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_2)})) & (d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_3)}), d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_3)})) & ,\dots,(d(\mathcal{P}_1^{(E_{k-1})}, \mathcal{P}_1^{(E_k)}), (\mathcal{Q}_1^{(E_{k-1})}, \mathcal{Q}_1^{(E_k)})) \\ (d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_2)}), d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_2)})) & (d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_3)}), d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_3)})) & ,\dots,(d(\mathcal{P}_2^{(E_{k-1})}, \mathcal{P}_2^{(E_k)}), (\mathcal{Q}_2^{(E_{k-1})}, \mathcal{Q}_2^{(E_k)})) \\ (d(\mathcal{P}_i^{(E_1)}, \mathcal{P}_i^{(E_2)}), d(\mathcal{Q}_i^{(E_1)}, \mathcal{Q}_i^{(E_2)})) & (d(\mathcal{P}_i^{(E_1)}, \mathcal{P}_i^{(E_3)}), d(\mathcal{Q}_i^{(E_1)}, \mathcal{Q}_i^{(E_3)})) & ,\dots,(d(\mathcal{P}_i^{(E_{k-1})}, \mathcal{P}_i^{(E_k)}), (\mathcal{Q}_i^{(E_{k-1})}, \mathcal{Q}_i^{(E_k)})) \end{bmatrix}$$

$$S(\delta_i) = \frac{\sum_{i=1}^{n}(d(\mathcal{P}_i^{(E_j)}, \mathcal{P}_i^{(E_k)}), d(\mathcal{Q}_i^{(E_j)}, \mathcal{Q}_i^{(E_k)}))}{n}$$

6.1. MCDM algorithm

The following is a general outline of the steps involved in the proposed MCDM algorithms. These steps are based on DMCIFSs, Step 1: Create the decision-making table, as defined in 2, in the context of CIFSs.

Step 2 : Determine the HDMCIFS and NHDMCIFS between the values evaluated by various experts corresponding to alternatives.

Step 3 : Create the distance matrix, based on the values of HDMCIFS and NHDMCIFS.

Step 4 : Determine the score functions corresponding to each alternatives.

Step 5 : Rank the alternatives according to the score functions and choose the most desirable one.

6.2. Problem description

6.2.1. Retail and E-commerce decision-making framework

Let $(\mathcal{P}, \mathcal{Q}) = (\mathcal{P}_1, \mathcal{Q}_1) = (Walmart, eBay)$, $(\mathcal{P}_2, \mathcal{Q}_2) = (Costco$, Local Supermarkets), $(\mathcal{P}_3, \mathcal{Q}_3) =$ (Amazon Prime, Regular Amazon Shopping) be the collection of Retail and E-commerce, $B = (B_1, B_2, B_3, B_4, B_5)$ be a collection of attributes where b_1, b_2, b_3, b_4, b_5 represents Pricing, Shipping Speed, Product Availability, Customer Perks, Purchase Flexibility are the common attributes in Retail and E-commerce.

Step 1 : Suppose that $E = E_1, E_2, E_3$ is a panel of three professionals taking part in decision-making.. Assume Mr. John is faced with a decision-making dilemma and want to launch an industry. Mr. John assigns three people to carry out the evaluation and selection process, particularly to examine options in light of five choice criteria experts in decision-making, E = E1,E2,E3. Five qualitative factors pertaining to industry are taken into consideration through conversations with the three specialists. The following table 2 provides sophisticated fuzzy information regarding internet business based on the three experts' areas of expertise.

Step 2 : Determine the HDMCIFSs between the various have professionals assessed and that correspond to the options. $L_i; i = 1, 2, 3$.

$$d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_2)}) = \frac{\begin{bmatrix} |0.8 - 0.8| + \frac{1}{2\pi}|0.8\pi - 0.4\pi| + |0.4 - 0.6| \\ +|0.3 - 0.6| + \frac{1}{2\pi}|0.3\pi - 0.7\pi| + |0.15 - 0.25| \\ +|0.4 - 0.5| + \frac{1}{2\pi}|0.5\pi - 0.3\pi| + |0.15 - 0.35| \\ +|0.6 - 0.3| + \frac{1}{2\pi}|0.7\pi - 0.1.2\pi| + |0.25 - 0.3| \\ +|0.5 - 0.8| + \frac{1}{2\pi}|0.3\pi - 2\pi| + |0.35 - 0.2| \end{bmatrix}}{15}$$
$$= \frac{[0 + 0.2 + 0.2 + 0.3 + 0.2 + 0.1 + 0.1 + 0.1 + 0.2 + 0.3 + 0.25 + 0.05 + 0.3 + 0.85 + 0.15]}{15}$$
$$= 0.22$$

$$d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_2)}) = \frac{\begin{bmatrix} |0.5 - 0.1| + \frac{1}{2\pi}|0.7\pi - 0.9\pi| + |015 - 0.55| \\ +|0.2 - 0.7| + \frac{1}{2\pi}|0.7\pi - 0.6\pi| + |0.15 - 0.4| \\ +|0.4 - 0.5| + \frac{1}{2\pi}|0.8\pi - 0.7\pi| + |0.3 - 0.25| \\ +|0.7 - 0.5| + \frac{1}{2\pi}|0.6\pi - 1.3\pi| + |0.4 - 0.15| \\ +|0.6 - 0.3| + \frac{1}{2\pi}|0.7\pi - 0.2\pi| + |0.25 - 0.2| \end{bmatrix}}{15}$$
$$= \frac{[0.4 + 0.1 + 0.2 + 0.5 + 0.05 + 0.25 + 0.1 + 0.05 + 0.05 + 0.2 + 0.35 + 0.25 + 0.3 + 0.25 + 0.05]}{15}$$
$$= 0.20$$

$$d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_3)}) = \frac{\begin{bmatrix} |0.8 - 0.5| + \frac{1}{2\pi}|0.8\pi - 1.5\pi| + |0.4 - 0.25| \\ +|0.3 - 0.3| + \frac{1}{2\pi}|0.3\pi - 1.2\pi| + |0.15 - 0.3| \\ +|0.4 - 0.8| + \frac{1}{2\pi}|0.5\pi - 2\pi| + |0.15 - 0.2| \\ +|0.3 - 0.6| + \frac{1}{2\pi}|0.3\pi - 0.7\pi| + |0.15 - 0.25| \\ +|0.4 - 0.5| + \frac{1}{2\pi}|0.5\pi - 0.3\pi| + |0.15 - 0.35| \end{bmatrix}}{15}$$
$$= \frac{[0.3 + 0.35 + 0.15 + 0 + 0.45 + 0.15 + 0.4 + 0.75 + 0.05 + 0.3 + 0.2 + 0.1 + 0.1 + 0.1 + 0.2]}{15}$$
$$= 0.24$$

$$d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_3)}) = \frac{\begin{bmatrix} |0.5 - 0.9| + \frac{1}{2\pi}|0.7\pi - 0.7\pi| + |0.15 - 0.35| \\ +|0.2 - 0.5| + \frac{1}{2\pi}|0.7\pi - 1.3\pi| + |0.15 - 0.15| \\ +|0.7 - 0.3| + \frac{1}{2\pi}|0.8\pi - 0.2\pi| + |0.3 - 0.6| \\ +|0.2 - 0.7| + \frac{1}{2\pi}|0.7\pi - 0.6\pi| + |0.15 - 0.4| \\ +|0.4 - 0.5| + \frac{1}{2\pi}|0.8\pi - 0.7\pi| + |0.3 - 0.25| \end{bmatrix}}{9}$$
$$= \frac{[0.4 + 0 + 0.4 + 0.3 + 0.3 + 0 + 0.4 + 0.3 + 0.1 + 0.5 + 0.05 + 0.25 + 0.1 + 0.05 + 0.05]}{15}$$
$$= 0.21$$

$$\begin{aligned} d(\mathcal{P}_1^{(E_2)}, \mathcal{P}_1^{(E_3)}) &= 0.30,\ d(\mathcal{Q}_1^{(E_2)}, \mathcal{Q}_1^{(E_3)}) = 0.26, \\ d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_2)}) &= 0.15,\ d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_2)}) = 0.26, \\ d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_3)}) &= 0.24,\ d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_3)}) = 0.20, \\ d(\mathcal{P}_2^{(E_2)}, \mathcal{P}_2^{(E_3)}) &= 0.29,\ d(\mathcal{Q}_2^{(E_2)}, \mathcal{Q}_2^{(E_3)}) = 0.18, \\ d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_2)}) &= 0.31,\ d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_2)}) = 0.34, \\ d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_3)}) &= 0.32,\ d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_3)}) = 0.29, \\ d(\mathcal{P}_3^{(E_2)}, \mathcal{P}_3^{(E_3)}) &= 0.38,\ d(\mathcal{Q}_3^{(E_2)}, \mathcal{Q}_3^{(E_3)}) = 0.36. \end{aligned}$$

$$[DM]_{3\times 3} = \begin{bmatrix} (d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_2)}), d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_2)})) & (d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_3)}), d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_3)})) & (d(\mathcal{P}_1^{(E_2)}, \mathcal{P}_1^{(E_3)}), (\mathcal{Q}_1^{(E_2)}, \mathcal{Q}_1^{(E_3)})) \\ (d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_2)}), d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_2)})) & (d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_3)}), d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_3)})) & (d(\mathcal{P}_2^{(E_2)}, \mathcal{P}_2^{(E_3)}), (\mathcal{Q}_2^{(E_2)}, \mathcal{Q}_2^{(E_3)})) \\ (d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_2)}), d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_2)})) & (d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_3)}), d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_3)})) & (d(\mathcal{P}_3^{(E_2)}, \mathcal{P}_3^{(E_3)}), (\mathcal{Q}_3^{(E_2)}, \mathcal{Q}_3^{(E_3)})) \end{bmatrix}$$

$$[DM]_{3\times 3} = \begin{bmatrix} (0.22, 0.20) & (0.24, 0.21) & (0.30, 0.26) \\ (0.15, 0.26) & (0.24, 0.20) & (0.29, 0.18) \\ (0.31, 0.34) & (0.32, 0.29) & (0.38, 0.36) \end{bmatrix}$$

$$\begin{aligned} S(L_i) &= \frac{\sum_{i=1}^{3}(d(\mathcal{P}_i^{(E_j)}, \mathcal{P}_i^{(E_j)}), d(\mathcal{Q}_i^{(E_k)}, \mathcal{Q}_i^{(E_k)}))}{3} \\ S(L_1) &= \left(\sum_{i=1}^{3} \frac{(0.22 + 0.24 + 0.30)}{3}, \frac{(0.20 + 0.21 + 0.26)}{3}\right) = (0.25, 0.22) \\ S(L_2) &= \left(\sum_{i=1}^{3} \frac{(0.15 + 0.24 + 0.29)}{3}, \frac{(0.26 + 0.20 + 0.18)}{3}\right) = (0.24, 0.21) \\ S(L_3) &= \left(\sum_{i=1}^{3} \frac{(0.31 + 0.32 + 0.38)}{3}, \frac{(0.34 + 0.29 + 0.36)}{3}\right) = (0.33, 0.33) \\ &(0.24, 0.21) < (0.25, 0.22) < (0.33, 0.33) \\ &S(L_2) < S(L_1) < S(L_3) \end{aligned}$$

$S(L_3) = (\mathcal{P}_3, \mathcal{Q}_3) =$ (Amazon Prime, Regular Amazon Shopping)is thus selected as the top Retail and E-commerce by the score functions based on HDMCIFSs.

Euclidean hesitance complex intuitionistic fuzzy distance measures

$$d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_2)}) = \sqrt{\frac{\begin{bmatrix} (0.8 - 0.8)^2 + \frac{1}{2\pi^2}(0.8\pi - 0.4\pi)^2 + (0.4 - 0.6)^2 \\ +(0.3 - 0.6)^2 + \frac{1}{2\pi^2}(0.3\pi - 0.7\pi)^2 + (0.15 - 0.25)^2 \\ +(0.4 - 0.5)^2 + \frac{1}{2\pi^2}(0.5\pi - 0.3\pi)^2 + (0.15 - 0.35)^2 \\ +(0.6 - 0.3)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.1.2\pi)^2 + (0.25 - 0.3)^2 \\ +(0.5 - 0.8)^2 + \frac{1}{2\pi^2}(0.3\pi - 2\pi)^2 + (0.35 - 0.2)^2 \end{bmatrix}}{15}}$$
$$= 0.29$$

$$d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_2)}) = \sqrt{\frac{\begin{bmatrix} (0.5 - 0.1)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.9\pi)^2 + (015 - 0.35)^2 \\ +(0.2 - 0.7)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.6\pi)^2 + (0.15 - 0.4)^2 \\ +(0.4 - 0.5)^2 + \frac{1}{2\pi^2}(0.8\pi - 0.7\pi)^2 + (0.3 - 0.25)^2 \\ +(0.7 - 0.5)^2 + \frac{1}{2\pi^2}(0.6\pi - 1.3\pi)^2 + (0.4 - 0.15)^2 \\ +(0.6 - 0.3)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.2\pi)^2 + (0.25 - 0.2)^2 \end{bmatrix}}{15}}$$
$$= 0.25$$

$$d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_3)}) = \sqrt{\frac{\begin{bmatrix} (0.8 - 0.5)^2 + \frac{1}{2\pi^2}(0.8\pi - 1.5\pi)^2 + (0.4 - 0.25)^2 \\ +(0.3 - 0.3)^2 + \frac{1}{2\pi^2}(0.3\pi - 1.2\pi)^2 + (0.15 - 0.3)^2 \\ +(0.4 - 0.8)^2 + \frac{1}{2\pi^2}(0.5\pi - 2\pi)^2 + (0.15 - 0.2)^2 \\ +(0.3 - 0.6)^2 + \frac{1}{2\pi^2}(0.3\pi - 0.7\pi)^2 + (0.15 - 0.25)^2 \\ +(0.4 - 0.5)^2 + \frac{1}{2\pi^2}(0.5\pi - 0.3\pi)^2 + (0.15 - 0.35)^2 \end{bmatrix}}{15}}$$
$$= 0.36$$

$$d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_3)}) = \sqrt{\frac{\begin{bmatrix} (0.5 - 0.9)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.7\pi)^2 + (0.15 - 0.35)^2 \\ +(0.2 - 0.5)^2 + \frac{1}{2\pi^2}(0.7\pi - 1.3\pi)^2 + (0.4 - 0.15)^2 \\ +(0.7 - 0.3)^2 + \frac{1}{2\pi^2}(0.8\pi - 0.2\pi)^2 + (0.3 - 0.6)^2 \\ +(0.2 - 0.7)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.6\pi)^2 + (0.15 - 0.4)^2 \\ +(0.4 - 0.5)^2 + \frac{1}{2\pi^2}(0.8\pi - 0.7\pi)^2 + (0.3 - 0.25)^2 \end{bmatrix}}{15}}$$
$$= 0.27$$

$$\begin{aligned} &d(\mathcal{P}_1^{(E_2)}, \mathcal{P}_1^{(E_3)}) = 0.40, d(\mathcal{Q}_1^{(E_2)}, \mathcal{Q}_1^{(E_3)}) = 0.32, \\ &d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_2)}) = 0.23, d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_2)}) = 0.28, \\ &d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_3)}) = 0.29, d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_3)}) = 0.32, \\ &d(\mathcal{P}_2^{(E_2)}, \mathcal{P}_2^{(E_3)}) = 0.36, d(\mathcal{Q}_2^{(E_2)}, \mathcal{Q}_2^{(E_3)}) = 0.25, \\ &d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_2)}) = 0.36, d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_2)}) = 0.41, \\ &d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_3)}) = 0.38, d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_3)}) = 0.39, \\ &d(\mathcal{P}_3^{(E_2)}, \mathcal{P}_3^{(E_3)}) = 0.43, d(\mathcal{Q}_3^{(E_2)}, \mathcal{Q}_3^{(E_3)}) = 0.38 \end{aligned}$$

$$[DM]_{3\times 3} = \begin{bmatrix} (d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_2)}), d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_2)})) & (d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_3)}), d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_3)})) & (d(\mathcal{P}_1^{(E_2)}, \mathcal{P}_1^{(E_3)}), (\mathcal{Q}_1^{(E_2)}, \mathcal{Q}_1^{(E_3)})) \\ (d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_2)}), d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_2)})) & (d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_3)}), d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_3)})) & (d(\mathcal{P}_2^{(E_2)}, \mathcal{P}_2^{(E_3)}), (\mathcal{Q}_2^{(E_2)}, \mathcal{Q}_2^{(E_3)})) \\ (d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_2)}), d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_2)})) & (d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_3)}), d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_3)})) & (d(\mathcal{P}_3^{(E_2)}, \mathcal{P}_3^{(E_3)}), (\mathcal{Q}_3^{(E_2)}, \mathcal{Q}_3^{(E_3)})) \end{bmatrix}$$

$$[DM]_{3\times 3} = \begin{bmatrix} (0.29, 0.25) & (0.30, 0.27) & (0.40, 0.32) \\ (0.23, 0.28) & (0.29, 0.32) & (0.36, 0, 25) \\ (0.36, 0.41) & (0.38, 0.39) & (0.43, 0.38) \end{bmatrix}$$

$$\begin{aligned} S(L_i) &= \frac{\sum_{i=1}^{3}(d(\mathcal{P}_i^{(E_j)}, \mathcal{P}_i^{(E_j)}), d(\mathcal{Q}_i^{(E_k)}, \mathcal{Q}_i^{(E_k)}))}{3} \\ S(L_1) &= \left(\sum_{i=1}^{3} \frac{(0.29 + 0.30 + 0.40)}{3}, \frac{(0.25 + 0.27 + 0.32)}{3}\right) = (0.31, 0.29) \\ S(L_2) &= \left(\sum_{i=1}^{3} \frac{(0.23 + 0.29 + 0.36)}{3}, \frac{(0.28 + 0.32 + 0.25)}{3}\right) = (0.29, 0.28) \\ S(L_3) &= \left(\sum_{i=1}^{3} \frac{(0.36 + 0.38 + 0.43)}{3}, \frac{(0.41 + 0.37 + 0.42)}{3}\right) = (0.37, 0.40) \\ &(0.29, 0.28) < (0.31, 0.29) < (0.37, 0.40) \\ &S(L_2) < S(L_1) < S(L_3) \end{aligned}$$

$S(L_3) = (\mathcal{P}_3, \mathcal{Q}_3) = (\text{Amazon Prime}, \text{ Regular Amazon Shopping})$ is thus selected as the top Retail and E-commerce by the score functions based on EHDMCIFSs.

6.2.2. Healthcare decision-making framework

An additional healthcare-based example is also included. Let $(\mathcal{P}_1, \mathcal{Q}_1) =$ (Government Hospital, Private Hospital), $(\mathcal{P}_2, \mathcal{Q}_2) =$ (Generic Medication, Branded Medication), $(\mathcal{P}_3, \mathcal{Q}_3) =$ (Telemedicine Consultation, In-person Clinic Visit). Let $B = (B_1, B_2, B_3, B_4, B_5)$ be a collection of attributes where b_1, b_2, b_3, b_4, b_5 represent treatment costs, service speed, specialist availability, patient convenience, and diagnostic reliability. These characteristics are frequently employed when evaluating healthcare alternatives, and they give a more comprehensive, real-world decision context for analyzing the effectiveness of the proposed distance measures.

The same computational steps used in the previous example can be applied here to construct the decision matrix, compute the distance values, obtain the aggregated scores, and determine the final ranking.

6.3. Comparison analysis

The comparison of the HDMCIF and EHDMCIF with the NHDMCIF and NEDMCIF was covered in this part. The suggested DMCIFs are reduced to NHDMCIF and NEDMCIF if we take certain limitations on HDMCIF and EHDMCIF. The comparison is established below:

Note 1 We are going to consider the Hesitancy degree as zero then, the HDMCIF is reduced to NHDMCIF.

Note 2 Similarly, we are going to consider the Hesitancy degree as zero then, the EHDMCIF is reduced to NEHDMCIF

6.3.1. Normalized hesitance distance measures of complex intuitionistic fuzzy

The decision-making issues based on other DMCIF without hesitation degree have also been covered only for the three attributes b_1, b_2, b_3 given in table 3. The following is the comparative study:

$$\begin{aligned} d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_2)}) &= \frac{\begin{bmatrix} |0.8 - 0.8| + \frac{1}{2\pi}|0.8\pi - 0.4\pi| \\ +|0.3 - 0.6| + \frac{1}{2\pi}|0.3\pi - 0.7\pi| \\ +|0.4 - 0.5| + \frac{1}{2\pi}|0.5\pi - 0.3\pi| \end{bmatrix}}{9} \\ &= \frac{[0 + 0.2 + 0.3 + 0.2 + 0.1 + 0.1+]}{9} = 0.15 \end{aligned}$$

Table 3. Evaluated values of alternatives.

E_k	$(\mathcal{P}_i,\ \mathcal{Q}_i)$	b_1	b_2	b_3	b_4	b_5
E_1	$(\mathcal{P}_1,\ \mathcal{Q}_1)$	$(0.8e^{0.8\pi}, 0.5e^{0.7\pi})$	$(0.3e^{0.3\pi}, 0.2e^{0.7\pi})$	$(0.4e^{0.5\pi}, 0.7e^{0.8\pi})$	$(0.6e^{0.7\pi}, 0.7e^{0.6\pi})$	$(0.5e^{0.3\pi}, 0.6e^{0.7\pi})$
	$(\mathcal{P}_2,\ \mathcal{Q}_2)$	$(0.1e^{0.9\pi}, 0.2e^{0.8\pi})$	$(0.3e^{0.7\pi}, 0.5e^{0.8\pi})$	$(0.4e^{0.5\pi}, 0.6e^{0.3\pi})$	$(0.6e^{0.9\pi}, 0.9e^{0.7\pi})$	$(0.5e^{0.3\pi}, 0.6e^{0.7\pi})$
	$(\mathcal{P}_3,\ \mathcal{Q}_3)$	$(0.2e^{0.7\pi}, 0.5e^{0.6\pi})$	$(0.5e^{0.3\pi}, 0.3e^{0.3\pi})$	$(0.4e^{0.6\pi}, 0.8e^{0.5\pi})$	$(0.3e^{1.5\pi}, 0.9e^{1.6\pi})$	$(0.8e^{0.9\pi}, 0.9e^{0.1\pi})$
E_2	$(\mathcal{P}_1,\ \mathcal{Q}_1)$	$(0.8e^{0.4\pi}, 0.1e^{0.9\pi})$	$(0.6e^{0.7\pi}, 0.7e^{0.6\pi})$	$(0.5e^{0.3\pi}, 0.6e^{0.7\pi})$	$(0.3e^{1.2\pi}, 0.5e^{1.3\pi})$	$(0.8e^{0.9\pi}, 0.9e^{0.1\pi})$
	$(\mathcal{P}_2,\ \mathcal{Q}_2)$	$(0e^{0.5\pi}, 0.7e^{0.7\pi})$	$(0.6e^{0.9\pi}, 0.9e^{0.7\pi})$	$(0.5e^{0.3\pi}, 0.6e^{0.7\pi})$	$(0.6e^{1.5\pi}, 0.9e^{0.7\pi})$	$(0.7e^{1.5\pi}, 0.5e^{0.7\pi})$
	$(\mathcal{P}_3,\ \mathcal{Q}_3)$	$(0.4e^{0.5\pi}, 0.4e^{2\pi})$	$(0.3e^{1.5\pi}, 0.9e^{1.6\pi})$	$(0.8e^{0.9\pi}, 0.9e^{0.1\pi})$	$(0.9e^{0.3\pi}, 0.5e^{0.2\pi})$	$(0.3e^{1.8\pi}, 0.9e^{1.7\pi})$
E_3	$(\mathcal{P}_1,\ \mathcal{Q}_1)$	$(0.5e^{1.5\pi}, 0.9e^{0.7\pi})$	$(0.3e^{1.2\pi}, 0.5e^{1.3\pi})$	$(0.8e^{0.9\pi}, 0.9e^{0.1\pi})$	$(0.3e^{0.3\pi}, 0.2e^{0.7\pi})$	$(0.4e^{0.5\pi}, 0.7e^{0.8\pi})$
	$(\mathcal{P}_1,\ \mathcal{Q}_1)$	$(0e^{1.8\pi}, 0.7e^{1.6\pi})$	$(0.6e^{1.5\pi}, 0.9e^{0.7\pi})$	$(0.7e^{1.5\pi}, 0.5e^{0.7\pi})$	$(0.4e^{0.5\pi}, 0.6e^{0.3\pi})$	$(0.6e^{0.9\pi}, 0.9e^{0.7\pi})$
	$(\mathcal{P}_1,\ \mathcal{Q}_1)$	$(0.8e^{1.7\pi}, 0.9e^{1.8\pi})$	$(0.9e^{0.3\pi}, 0.5e^{0.2\pi})$	$(0.3e^{1.8\pi}, 0.9e^{1.7\pi})$	$(0.4e^{0.6\pi}, 0.8e^{0.5\pi})$	$(0.3e^{1.5\pi}, 0.9e^{1.6\pi})$

$$
\begin{aligned}
d(\mathcal{Q}_1^{(E_1)},\ \mathcal{Q}_1^{(E_2)}) &= \frac{\begin{bmatrix} |0.5 - 0.1| + \frac{1}{2\pi}|0.7\pi - 0.9\pi| \\ +|0.2 - 0.7| + \frac{1}{2\pi}|0.7\pi - 0.6\pi| \\ +|0.4 - 0.5| + \frac{1}{2\pi}|0.8\pi - 0.7\pi| \end{bmatrix}}{9} \\
&= \frac{[0.4 + 0.1 + 0.5 + 0.05 + 0.1 + 0.05+]}{9} = 0.2
\end{aligned}
$$

$$
\begin{aligned}
d(\mathcal{P}_1^{(E_1)},\ \mathcal{P}_1^{(E_3)}) &= \frac{\begin{bmatrix} |0.8 - 0.5| + \frac{1}{2\pi}|0.8\pi - 1.5\pi| \\ +|0.3 - 0.3| + \frac{1}{2\pi}|0.3\pi - 1.2\pi| \\ +|0.4 - 0.8| + \frac{1}{2\pi}|0.5\pi - 2\pi| \end{bmatrix}}{9} \\
&= \frac{[0.3 + 0.35 + 0 + 0.45 + 0.4 + 0.75+]}{9} = 0.37
\end{aligned}
$$

$$
\begin{aligned}
d(\mathcal{Q}_1^{(E_1)},\ \mathcal{Q}_1^{(E_3)}) &= \frac{\begin{bmatrix} |0.5 - 0.9| + \frac{1}{2\pi}|0.7\pi - 0.7\pi| \\ +|0.2 - 0.5| + \frac{1}{2\pi}|0.7\pi - 1.3\pi| \\ +|0.7 - 0.3| + \frac{1}{2\pi}|0.8\pi - 0.2\pi| \end{bmatrix}}{9} \\
&= \frac{[0.4 + 0 + 0.3 + 0.3 + 0.4 + 0.3]}{9} = 0.28
\end{aligned}
$$

$$
[DM]_{3\times 3} = \begin{bmatrix}
(d(\mathcal{P}_1^{(E_1)},\ \mathcal{P}_1^{(E_2)}),\ d(\mathcal{Q}_1^{(E_1)},\ \mathcal{Q}_1^{(E_2)})) & (d(\mathcal{P}_1^{(E_1)},\ \mathcal{P}_1^{(E_3)}),\ d(\mathcal{Q}_1^{(E_1)},\ \mathcal{Q}_1^{(E_3)})) & (d(\mathcal{P}_1^{(E_2)},\ \mathcal{P}_1^{(E_3)}),\ (\mathcal{Q}_1^{(E_2)},\ \mathcal{Q}_1^{(E_3)})) \\
(d(\mathcal{P}_2^{(E_1)},\ \mathcal{P}_2^{(E_2)}),\ d(\mathcal{Q}_2^{(E_1)},\ \mathcal{Q}_2^{(E_2)})) & (d(\mathcal{P}_2^{(E_1)},\ \mathcal{P}_2^{(E_3)}),\ d(\mathcal{Q}_2^{(E_1)},\ \mathcal{Q}_2^{(E_3)})) & (d(\mathcal{P}_2^{(E_2)},\ \mathcal{P}_2^{(E_3)}),\ (\mathcal{Q}_2^{(E_2)},\ \mathcal{Q}_2^{(E_3)})) \\
(d(\mathcal{P}_3^{(E_1)},\ \mathcal{P}_3^{(E_2)}),\ d(\mathcal{Q}_3^{(E_1)},\ \mathcal{Q}_3^{(E_2)})) & (d(\mathcal{P}_3^{(E_1)},\ \mathcal{P}_3^{(E_3)}),\ d(\mathcal{Q}_3^{(E_1)},\ \mathcal{Q}_3^{(E_3)})) & (d(\mathcal{P}_3^{(E_2)},\ \mathcal{P}_3^{(E_3)}),\ (\mathcal{Q}_3^{(E_2)},\ \mathcal{Q}_3^{(E_3)}))
\end{bmatrix}
$$

$$
\begin{aligned}
&d(\mathcal{P}_1^{(E_2)},\ \mathcal{P}_1^{(E_3)}) = 0.42,\ d(\mathcal{Q}_1^{(E_2)},\ \mathcal{Q}_1^{(E_3)}) = 0.33, \\
&d(\mathcal{P}_2^{(E_1)},\ \mathcal{P}_2^{(E_2)}) = 0.15,\ d(\mathcal{Q}_2^{(E_1)},\ \mathcal{Q}_2^{(E_2)}) = 0.29, \\
&d(\mathcal{P}_2^{(E_1)},\ \mathcal{P}_2^{(E_3)}) = 0.34,\ d(\mathcal{Q}_2^{(E_1)},\ \mathcal{Q}_2^{(E_3)}) = 0.27, \\
&d(\mathcal{P}_2^{(E_2)},\ \mathcal{P}_2^{(E_3)}) = 0.29,\ d(\mathcal{Q}_2^{(E_2)},\ \mathcal{Q}_2^{(E_3)}) = 0.15, \\
&d(\mathcal{P}_3^{(E_1)},\ \mathcal{P}_3^{(E_2)}) = 0.27,\ d(\mathcal{Q}_3^{(E_1)},\ \mathcal{Q}_3^{(E_2)}) = 0.39, \\
&d(\mathcal{P}_3^{(E_1)},\ \mathcal{P}_3^{(E_3)}) = 0.36,\ d(\mathcal{Q}_3^{(E_1)},\ \mathcal{Q}_3^{(E_3)}) = 0.32, \\
&d(\mathcal{P}_3^{(E_2)},\ \mathcal{P}_3^{(E_3)}) = 0.52,\ d(\mathcal{Q}_3^{(E_2)},\ \mathcal{Q}_3^{(E_3)}) = 0.41.
\end{aligned}
$$

$$
[DM]_{3\times 3} = \begin{bmatrix}
(0.15,\ 0.2) & (0.37,\ 0.28) & (0.42,\ 0.33) \\
(0.15,\ 0.29) & (0.34,\ 0.27) & (0.29,\ 0.15) \\
(0.27,\ 0.39) & (0.36,\ 0.32) & (0.52,\ 0.41)
\end{bmatrix}
$$

$$S(L_i) = \frac{\sum_{i=1}^{3}(d(\mathcal{P}_i^{(E_j)}, \mathcal{P}_i^{(E_j)}), d(\mathcal{Q}_i^{(E_k)}, \mathcal{Q}_i^{(E_k)}))}{3}$$

$$S(L_1) = \left(\sum_{i=1}^{3} \frac{(0.15 + 0.37 + 0.42)}{3}, \frac{(0.2 + 0.28 + 0.33)}{3}\right) = (0.31, 0.27)$$

$$S(L_2) = \left(\sum_{i=1}^{3} \frac{(0.15 + 0.34 + 0.29)}{3}, \frac{(0.29 + 0.27 + 0.15)}{3}\right) = (0.26, 0.23)$$

$$S(L_3) = \left(\sum_{i=1}^{3} \frac{(0.27 + 0.36 + 0.52)}{3}, \frac{(0.39 + 0.32 + 0.41)}{3}\right) = (0.38, 0.37)$$

$$(0.26, 0.23) \leqslant (0.31, 0.27) \leqslant (0.38, 0.37)$$

$$S(L_2) \leqslant S(L_1) \leqslant S(L_3)$$

$S(L_3) = (\mathcal{P}_3, \mathcal{Q}_3) =$ (Amazon Prime, Regular Amazon Shopping)is thus selected as the top Retail and E-commerce by the score functions based on NHDMCIFSs.

6.3.2. Normalized euclidean hesitance complex intuitionistic fuzzy distance measures

$$d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_2)}) = \sqrt{\frac{\begin{bmatrix} (0.8 - 0.8)^2 + \frac{1}{2\pi^2}(0.8\pi - 0.4\pi)^2 \\ +(0.3 - 0.6)^2 + \frac{1}{2\pi^2}(0.3\pi - 0.7\pi)^2 \\ +(0.4 - 0.5)^2 + \frac{1}{2\pi^2}(0.5\pi - 0.3\pi)^2 \end{bmatrix}}{6}}$$
$$= \sqrt{\frac{0 + 0.04 + +0.09 + 0.04 + +0.01 + 0.01+}{6}}$$
$$= 0.17$$

$$d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_2)}) = \sqrt{\frac{\begin{bmatrix} (0.5 - 0.1)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.9\pi)^2 \\ +(0.2 - 0.7)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.6\pi)^2 \\ +(0.4 - 0.5)^2 + \frac{1}{2\pi^2}(0.8\pi - 0.7\pi)^2 \end{bmatrix}}{6}}$$
$$= 0.26$$

$$d(\mathcal{P}_1^{(E_1)}, \mathcal{P}_1^{(E_3)}) = \sqrt{\frac{\begin{bmatrix} (0.8 - 0.5)^2 + \frac{1}{2\pi^2}(0.8\pi - 1.5\pi)^2 \\ +(0.3 - 0.3)^2 + \frac{1}{2\pi^2}(0.3\pi - 1.2\pi)^2 \\ +(0.4 - 0.8)^2 + \frac{1}{2\pi^2}(0.5\pi - 2\pi)^2 \end{bmatrix}}{6}}$$
$$= 0.43$$

$$d(\mathcal{Q}_1^{(E_1)}, \mathcal{Q}_1^{(E_3)}) = \sqrt{\frac{\begin{bmatrix} (0.5 - 0.9)^2 + \frac{1}{2\pi^2}(0.7\pi - 0.7\pi)^2 \\ +(0.2 - 0.5)^2 + \frac{1}{2\pi^2}(0.7\pi - 1.3\pi)^2 \\ +(0.7 - 0.3)^2 + \frac{1}{2\pi^2}(0.8\pi - 0.2\pi)^2 \end{bmatrix}}{6}}$$
$$= 0.29$$

$$d(\mathcal{P}_1^{(E_2)}, \mathcal{P}_1^{(E_3)}) = 0.47, d(\mathcal{Q}_1^{(E_2)}, \mathcal{Q}_1^{(E_3)}) = 0.4,$$
$$d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_2)}) = 0.16, d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_2)}) = 0.34,$$
$$d(\mathcal{P}_2^{(E_1)}, \mathcal{P}_2^{(E_3)}) = 0.36, d(\mathcal{Q}_2^{(E_1)}, \mathcal{Q}_2^{(E_3)}) = 0.32,$$
$$d(\mathcal{P}_2^{(E_2)}, \mathcal{P}_2^{(E_3)}) = 0.24, d(\mathcal{Q}_2^{(E_2)}, \mathcal{Q}_2^{(E_3)}) = 0.24,$$
$$d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_2)}) = 0.32, d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_2)}) = 0.48,$$
$$d(\mathcal{P}_3^{(E_1)}, \mathcal{P}_3^{(E_3)}) = 0.43, d(\mathcal{Q}_3^{(E_1)}, \mathcal{Q}_3^{(E_3)}) = 0.39,$$
$$d(\mathcal{P}_3^{(E_2)}, \mathcal{P}_3^{(E_3)}) = 0.53, d(\mathcal{Q}_3^{(E_2)}, \mathcal{Q}_3^{(E_3)}) = 0.30.$$

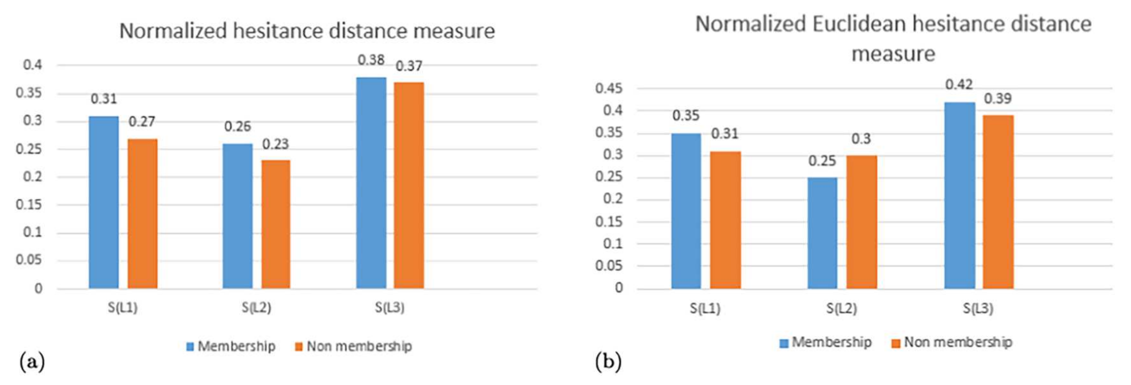

Figure 1. (a) NHDM (b) NEHDM.

Table 4. Ranking of complex intuitionistic distance measures.

	Score Value	Ranking	Best optimal
HDMCIF	$(0.24, 0.21) < (0.25, 0.22) < (0.33, 0.33)$	$S(L2) < S(L1) < S(L3)$	$S(L3)$
EHDMCIF	$(0.29, 0.28) < (0.31, 0.29) < (0.37, 0.40)$	$S(L2) < S(L1) < S(L3)$	$S(L3)$
NHDMCIF	$(0.26, 0.23) < (0.31, 0.27) < (0.38, 0.37)$	$S(L2) < S(L1) < S(L3)$	$S(L3)$
NEDMCIF	$(0.25, 0.30) < (0.35, 0.31) < (0.42, 0.39)$	$S(L2) < S(L1) < S(L3)$	$S(L3)$

$$[DM]_{3\times 3} = \begin{bmatrix} (0.17,\ 0.26) & (0.43,\ 0.29) & (0.47,\ 0.40) \\ (0.16,\ 0.34) & (0.36,\ 0.32) & (0.24,\ 0,\ 24) \\ (0.32,\ 0.48) & (0.43,\ 0.39) & (0.53,\ 0.30) \end{bmatrix}$$

$$S(L_i) = \frac{\sum_{i=1}^{3}(d(\mathcal{P}_i^{(E_j)},\ \mathcal{P}_i^{(E_j)}),\ d(\mathcal{Q}_i^{(E_k)},\ \mathcal{Q}_i^{(E_k)}))}{3}$$

$$S(L_1) = \left(\sum_{i=1}^{3}\frac{(0.17 + 0.43 + 0.47)}{3},\ \frac{(0.26 + 0.29 + 0.40)}{3}\right) = (0.35,\ 0.31)$$

$$S(L_2) = \left(\sum_{i=1}^{3}\frac{(0.16 + 0.36 + 0.24)}{3},\ \frac{(0.34 + 0.32 + 0.24)}{3}\right) = (0.25,\ 0.30)$$

$$S(L_3) = \left(\sum_{i=1}^{3}\frac{(0.32 + 0.43 + 0.53)}{3},\ \frac{(0.48 + 0.39 + 0.30)}{3}\right) = (0.42,\ 0.39)$$

$$(0.25,\ 0.30) \leqslant (0.35,\ 0.31) \leqslant (0.42,\ 0.39)$$
$$S(L_2) \leqslant S(L_1) \leqslant S(L_3)$$

$S(L_3) = \quad (\mathcal{P}_3,\ \mathcal{Q}_3) =$ (Amazon Prime, Regular Amazon Shopping) is thus selected as the top Retail and E-commerce by the score functions based on NEHDMCIFSs.

Table 4 illustrates the comparative ranking results achieved utilizing the new HDMCIF and EHDMCIF distance measurements alongside the existing NHDMCIF and NEDMCIF approaches. For each strategy, the relevant score values of the three alternatives are provided in ascending order, followed by the resulting ranking and the determined best choice. Although all four measures consistently identify S(L3) as the ideal alternative, the numerical disparities in the score values reflect how each technique interprets amplitude, phase, and hesitance information. The NHDMCIF and NEDMCIF yield relatively closer values because there is no hesitance information, whereas the proposed HDMCIF and EHDMCIF measures show a more pronounced distinction among the alternatives, showing improved discriminatory potential.

Overall, the table clearly illustrates how the proposed methods improve choice differentiation in complicated intuitionistic situations. Thus, the presented decision-making scenario demonstrates the suggested measures flexibility and robustness in capturing both hesitance and phase-based uncertainty. Unlike existing CIFS or IF distance models, which only extend standard measures in real-valued domains, the proposed framework provides a conceptually different formulation capable of dealing with complex-valued and hesitant information. This distinction emphasizes the practical and theoretical originality of the current application. Figure 1 presents the schematic representation of the proposed complex intuitionistic fuzzy decision-making framework.

6.4. Sensitivity analysis

The viability of using the proposed HDMCIFS and EHDMCIFS models in an MCDM setting is demonstrated in the application section. Although a simplified example was provided for clarity, the suggested framework can readily be extended to more realistic decision scenarios involving a larger number of alternatives and conflicting criteria, such as cost-benefit trade-offs or performance-risk assessments. In such complex environments, the inclusion of both amplitude and phase information provides richer modeling flexibility, enabling the representation of correlated or cyclic decision parameters that cannot be effectively captured by real-valued fuzzy techniques. Furthermore, variations in amplitude and phase can be conceptually interpreted as a form of sensitivity behavior in the decision-making process. Small changes in these components reflect differing degrees of hesitance or uncertainty among decision-makers. Because the proposed distance measures are designed to respond smoothly to such fluctuations, minor adjustments in amplitude or phase do not lead to disproportionate changes in the final ranking of alternatives. This property highlights the stability and robustness of the proposed models when applied to complex or conflicting decision situations.

6.5. Advantages of hesitance distance measures of complex intuitionistic fuzzy

Hesitance values of CIF provides a strong foundation for addressing and simulating decision-making uncertainty. In complex and uncertain decision situations, these metrics provide a flexible and trustworthy way to represent ambiguous preferences, assisting decision-makers in making more confident and informed decisions. Hesitance in complex distance measurements allows decision-makers to evaluate their unpredictability or hesitation towards other options. Decision-makers can express their level of doubt on the membership of fuzzy set elements by giving them hesitancy values, which gives uncertainty a more accurate portrayal. Furthermore, a framework for expressing ambiguous preferences is provided by DMCIF with hesitance values. Decision-makers can represent the intricate and multidimensional character of unknowns in decision-making situations by expressing different levels of uncertain for various variables or criteria.

7. Conclusion

Applications of DMCIFSs in decision-making procedures are numerous and significant, covering a range of fields where experts must deal with ambiguity, uncertainty, and intricate interactions between options and presented HDMCIFS and EHDMCIFS in this paper. These are extensions of NHDMCIFS, and by including the hesitancy value, EDMCIFS may be included. In managing the complexity of the actual world, the capacity to take membership and hesitancy values into account while assessing choices has proven crucial. In the context of suggested DMCIFSs and complex fuzzy operations, a few novel operations and key findings are examined. Additionally, we created an algorithm for decision-making based on HDMCIFSs and EHDMCIFSs. Compute complexity, which prevents real-time or resource-constrained applications, and interpretability issues brought on by sophisticated mathematical calculations are the anticipated drawbacks of HDMCIF. These measurements frequently call for adjusting a number of parameters, which increases subjectivity and sensitivity in parameter selection. Applications in a variety of fields where managing uncertainty, imprecision, and complexity in data interactions is essential may be covered by the suggested hesitance distance measurements in complex intuitionistic fuzzy in the future. To improve decision accuracy, more research may look at incorporating these metrics into multi-expert group decision-making frameworks and merging them with other fuzzy MCDM approaches like TOPSIS, EDAS. To increase the algorithm's practical usefulness, it would also be beneficial to investigate its computational complexity and optimize it for large-scale data sets.

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Conflicts interest

The authors declare that they have no competing financial interests and personal relationships that could have appeared to influence the research in this paper.

Data availability statement

No new data were created or analysed in this study.

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