Some fixed point theorems via γ-ψ_S-contractions on S-metric spaces

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Abstract:

Introduction: This paper focuses on extending the theory of fixed points in S-metric spaces by introducing new generalized contractive conditions. These developments aim to enrich the analytical tools available for studying such spaces.

Methods: A variety of fixed-point theorems is established by applying the newly defined contractive conditions. The methodology includes both standard and integral-type contractive mappings. Furthermore, a geometric approach is utilized to obtain novel fixed-circle theorems within the S-metric framework.

Results: Several fixed-point and fixed-circle theorems are proved under the proposed conditions. Illustrative examples are provided to validate the theoretical findings and demonstrate the applicability of the results.

Conclusion: The findings of this study not only broaden the scope of fixed-point theory in S-metric spaces but also offer potential implications for real-world applications. In particular, the results may contribute to developments in computational mathematics and the design of neural network activation functions.

Keywords: S-metric space, fixed-point theorem, generalized contraction, integral-type contraction, fixed-circle, geometric approach, activation function, neural networks.

Introduction and preliminaries

Fixed-point theory has far-reaching implications in mathematics and its applications to other disciplines. One of its most significant contributions is in the realm of dynamical systems, where it is used to analyze equilibrium points or steady states. For example, in economics, fixed points help to model market equilibria, where no participant has an incentive to change their strategy. In game theory, Nash equilibrium points are essentially fixed points of certain types of functions. Furthermore, in computational mathematics, algorithms such as those used in optimization or numerical methods often rely on fixed point results to guarantee convergence. The general idea that under certain conditions a system will reach a state of stability is a profound insight that has shaped much of theoretical and applied research.

Metric spaces are crucial for studying the topological properties of spaces, such as continuity, compactness, and convergence. They provide a framework for understanding distance-based relationships between objects in various fields, from analysis to geometric topology. For example, the concept of convergence in a metric space allows mathematicians to rigorously define limits and study the behavior of sequences and functions. Metric spaces are particularly important in functional analysis and measure theory, where the behavior of functions or sequences within these spaces is a key focus.

On the other hand, generalized metric spaces, including S-metric spaces (Sedghi et al, 2012), broaden the scope by allowing non-separable points, making them useful in settings like group theory or quotient spaces. S-metric spaces extend the traditional concept of metric spaces by incorporating a three-variable distance function, which enables more flexible modeling of geometric and topological structures. These spaces can model scenarios where traditional distance functions fail to capture equivalence relations or where the notion of “closeness” is not strict. In essence, generalized metric spaces offer a more flexible structure, which can be applied to various advanced topics in algebraic topology, category theory, and even computer science, especially in the study of data structures and approximation methods, for more details, see (Fetouci & Radenovic, 2009; Iqbal et al, 2024; Bimol et al, 2024).

Both metric and generalized metric spaces are foundational for many modern branches of mathematics, and their applications extend to fields such as signal processing, machine learning, network theory, and mathematical physics, underscoring their importance beyond theoretical investigations. These concepts help mathematicians and scientists model and understand complex phenomena in an increasingly interconnected world.

The paper by Özgür NY and Taş N., titled "Some Fixed-Circle Theorems on Metric Spaces", published in the Bulletin of the Malaysian Mathematical Sciences Society (Özgür & Taş, 2019a), explores important fixed-point results in the context of metric spaces, specifically focusing on "fixed-circle" theorems. In this study, the authors extend and generalize classical fixed point theorems by introducing new conditions under which fixed points can be guaranteed. Their work contributes to a deeper understanding of the structure of metric spaces and the behavior of certain types of mappings that preserve geometric properties, such as distance. These findings are significant for both theoretical and applied mathematics, as fixed point theorems play a crucial role in the areas such as dynamical systems, game theory, and optimization. By providing novel results in the realm of metric spaces, this research opens new pathways for investigating equilibrium solutions and offers tools for solving complex problems across various mathematical and interdisciplinary domains. The importance of this paper lies in its potential applications to both pure and applied fields, including analysis, topology, and computational mathematics.

By the above motivations, in this paper, we introduce some generalized contractive conditions on S-metric spaces. Using these new contractions, we prove some fixed-point theorem and integral type fixed-point theorems on S-metric spaces. Also, using the geometric approach, we obtain new fixed-circle results on S-metric spaces with necessary examples. Finally, in the conclusion section, we mention the importance of this paper with an example of activation functions.

The following is a definition of the concept of an S-metric space that was provided by Sedghi, Shobe, and Aliouche:

Definition 1. *(Sedghi et al, 2012) Let X be a nonempty set. An S-metric on X is a function $S : X \times X \times X \to [0, +\infty)$ that satisfies the following conditions for all $x, y, z, a \in X$:*

(S1) $S(x, y, z) = 0$ if and only if $x = y = z$,

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X,S) is called an S-metric space.

For an S-metric space, the symmetry condition can be considered as follows:

$$S(x,x,y)=S(y,y,x)$$

for all $x,y\in X$ (Sedghi et al, 2012).

EXAMPLE 1. (Sedghi et al, 2012) Let $\mathbb{R}$ be the real line. Then

$$S(x,y,z)=|x-z|+|y-z|$$

for all $x,y,z\in\mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the usual S-metric.

EXAMPLE 2. Let $X=\mathbb{R}$ and

$$S(x,y,z)=|y+z-2x|+|y-z|$$

for all $x,y,z\in X$. Then, (X,S) is an S-metric space (Sedghi et al, 2012). But this S-metric cannot be generated by any metric (Hieu et al, 2015) .

EXAMPLE 3. (Özgür & Taş, 2017) Let $X=\mathbb{R}$ and define the function

$$S(x,y,z)=|x-z|+|x+z-2y|$$

for all $x,y,z\in X$. Then, (X,S) is an S-metric space. But this S-metric cannot be generated by any metric.

DEFINITION 2. *(Sedghi et al, 2012) Let (X,S) be an S-metric space.*

1. A sequence $\{x_n\}\subset X$ converges to $x\in X$ if $S(x_n,x_{n,x})\to 0$ as $n\to+\infty$.
2. A sequence $\{x_n\}\subset X$ is a Cauchy sequence if $S(x_n,x_n,x_m)\to 0$ as $n,m\to+\infty$.
3. The S-metric space (X,S) is complete if every Cauchy sequence is a convergent sequence.

DEFINITION 3. *(Sedghi et al, 2014) Let $f:X\to Y$ be a map from an S-metric space X to an S-metric space. Then f is continuous at $x\in X$ if and only if $f(x_n)\to f(x)$ whenever $x_n\to x$.*

As demonstrated in (Hieu et al, 2015) and (Özgür & Taş, 2017), certain S-metrics cannot be derived from standard metrics. Consequently, investigating new fixed-point theorems within the context of S-metric spaces is essential.

A recent innovative approach to geometrically interpretations for fixed points, known as the fixed-circle problem (Özgür & Taş, 2019a), has emerged. The following concepts are now recalled, as they were defined in (Mlaiki et al, 2018), and (Özgür & Taş, 2019b).

Let (X, S) be an S-metric space and $T : X \to X$ a self-mapping. A circle $C^S_{x_0,r}$ and a disc $D^S_{x_0,r}$ are defined by

$$C^S_{x_0,r} = \{u \in X : S(u, u, x_0) = r\}$$

and

$$D^S_{x_0,r} = \{u \in X : S(u, u, x_0) \leq r\},$$

with the center $x_0 \in X$ and the radius $r > 0$.

If $Tx = x$ for all $x \in C^S_{x_0,r}$ (resp. $x \in D^S_{x_0,r}$) then the circle $C^S_{x_0,r}$ (resp. the disc $D^S_{x_0,r}$) is called the fixed circle (resp. fixed disc) of T.

For every function $\Psi : [0, +\infty) \to [0, +\infty)$, let ψ^n be the nth iterate of ψ. Then the following holds:

If Ψ is non-decreasing, then for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ implies $\Psi(t) < t$ (Raji et al, 2024).

In 2012, Samet et al. (Samet et al, 2012) introduced the class of α-admissible mappings.

Definition 4. *(Samet et al, 2012) Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$, we say that T is an α-admissible mapping if*

$$x, y \in X, \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Main results

In this section, we give some fixed-point theorems on S-metric spaces inspiring the approaches used in (Raji et al, 2024). Also, we present the integral versions of our obtained results using the technique given in (Branciari, 2002). As a geometric approach, we obtain new fixed-circle results on S-metric spaces.

DEFINITION 5. *Let (X,S) be an S-metric space and $f : X \to X$ be a self-mapping. f is called a γ-ψ_S-contractive mapping if there exist $\gamma : X \times X \times X \to [0,+\infty)$ and $\psi \in \Psi$ such that*

$$\gamma(x,x,y)\,S(fx,fx,fy) \leq \psi(\Delta(x,y)),$$

for all $x,y \in X$, where

$$\Delta(x,y) = \max\left\{\begin{array}{c} S(x,x,y),S(x,x,fx),S(y,y,fy), \\ \frac{S(x,x,fx)S(y,y,fy)}{S(x,x,y)}, \\ \frac{S(x,x,fx)S(y,y,fy)}{S(x,x,y)+S(x,x,fy)+S(y,y,fx)}, \\ \frac{S(x,x,fx)S(x,x,fy)+S(y,y,fx)S(y,y,fy)}{S(y,y,fx)+S(x,x,fy)} \end{array}\right\}.$$

THEOREM 1. *Let (X,S) be a complete S-metric space and $f : X \to X$ be a γ-ψ_S-contractive mapping. If the following conditions are satisfied*:

(i) f is γ-admissible,

(ii) there exists $x_0 \in X$ such that $\gamma(x_0,x_0,fx_0) \geq 1$, and

(iii) f is continuous,
then there exists $a \in X$ such that $fa = a$.

Proof. By (ii), we say that there exists a point $x_0 \in X$ such that

$$\gamma(fx_0,fx_0,fx_0) \geq 1.$$

Let us define a sequence $\{x_n\}$ in X by $fx_n = x_{n+1}$ for all $n \geq 0$. If for some $n \geq 0$,

$$x_n = x_{n+1},$$

then

$$fx_n = x_{n+1} = x_n,$$

and so x_n is a fixed point of f. On the contrary, we suppose that

$$x_n \neq x_{n+1},$$

for all $n \geq 0$. Since f is γ-admissible, we get

$$\gamma(x_0,x_0,fx_0) = \gamma(x_0,x_0,x_1) \geq 1$$

$$\Longrightarrow \gamma\left(fx_0, fx_0, fx_1\right) = \gamma\left(x_1, x_1, x_2\right) \geq 1.$$

If we continue this process, we have

$$\gamma\left(x_n, x_n, x_{n+1}\right) \geq 1,$$

for all $n \geq 0$. Using the γ-ψ_S-contractive mapping definition, we have

$$\begin{aligned} S\left(x_{n+1}, x_{n+1}, x_n\right) &= S\left(fx_n, fx_n, fx_{n-1}\right) \\ &\leq \gamma\left(x_n, x_n, x_{n-1}\right) S\left(fx_n, fx_n, fx_{n-1}\right) \\ &\leq \psi\left(\Delta\left(x_n, x_{n-1}\right)\right), \end{aligned} \tag{1}$$

for all $n \geq 1$, where

$$\begin{aligned} &\Delta\left(x_n, x_{n-1}\right) = \\ &= \max\left\{\begin{array}{c} S\left(x_n, x_n, x_{n-1}\right), S\left(x_n, x_n, fx_n\right), S\left(x_{n-1}, x_{n-1}, fx_{n-1}\right), \\ \frac{S(x_n,x_n,fx_n)S(x_{n-1},x_{n-1},fx_{n-1})}{S(x_n,x_n,x_{n-1})}, \\ \frac{S(x_n,x_n,fx_n)S(x_{n-1},x_{n-1},fx_{n-1})}{S(x_n,x_n,x_{n-1})+S(x_n,x_n,fx_{n-1})+S(x_{n-1},x_{n-1},fx_n)}, \\ \frac{S(x_n,x_n,fx_n)S(x_n,x_n,fx_{n-1})+S(x_{n-1},x_{n-1},fx_n)S(x_{n-1},x_{n-1},fx_{n-1})}{S(x_{n-1},x_{n-1},fx_n)+S(x_n,x_n,fx_{n-1})} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} S\left(x_n, x_n, x_{n-1}\right), S\left(x_n, x_n, x_{n+1}\right), S\left(x_{n-1}, x_{n-1}, x_n\right), \\ S\left(x_n, x_n, x_{n+1}\right), S\left(x_n, x_n, x_{n+1}\right), S\left(x_{n-1}, x_{n-1}, x_n\right) \end{array}\right\} \\ &= \max\left\{S\left(x_n, x_n, x_{n-1}\right), S\left(x_n, x_n, x_{n+1}\right)\right\}. \end{aligned}$$

By 1, we get

$$S\left(x_{n+1}, x_{n+1}, x_n\right) \leq \psi\left(\max\left\{S\left(x_n, x_n, x_{n-1}\right), S\left(x_n, x_n, x_{n+1}\right)\right\}\right),$$

for all $n \geq 1$.

Case 1: Let $S\left(x_n, x_n, x_{n-1}\right) \leq S\left(x_n, x_n, x_{n+1}\right)$. Then we get

$$S\left(x_{n+1}, x_{n+1}, x_n\right) \leq \psi\left(S\left(x_n, x_n, x_{n+1}\right)\right) < S\left(x_n, x_n, x_{n+1}\right),$$

a contradiction.

Case 2: Let $S\left(x_n, x_n, x_{n+1}\right) \leq S\left(x_n, x_n, x_{n-1}\right)$. Then we have

$$S\left(x_{n+1}, x_{n+1}, x_n\right) \leq \psi\left(S\left(x_n, x_n, x_{n-1}\right)\right),$$

for all $n \geq 1$. Using the mathematical induction, if we continue this process, we obtain

$$S\left(x_{n+1}, x_{n+1}, x_n\right) \leq \psi^n\left(S\left(x_1, x_1, x_0\right)\right),$$

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for all $n \geq 1$. Then, we get

$$S(x_n, x_n, x_{n+k}) \leq \sum_{p=n}^{n+k-1} 2\psi^p S(x_1, x_1, x_0), \tag{2}$$

for all $k \geq 1$. By 2, if we take $p \to +\infty$, we have

$$S(x_n, x_n, x_{n+k}) \to 0,$$

and so $\{x_n\}$ is Cauchy. Using the completeness hypothesis, there is $a \in X$ such that

$$\lim_{n\to+\infty} x_n = a.$$

By (iii), we have

$$fa = f\left(\lim_{n\to+\infty} x_n\right) = \lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} x_{n+1} = a.$$

Therefore, a is a fixed point of f. □

REMARK 1. Theorem 1 is an existence of a fixed point for the self-mapping f with the continuity hypothesis.

In the following existence theorem, we do not use the continuity hypothesis.

THEOREM 2. *Let (X, S) be a complete S-metric space and $f : X \to X$ be a γ-ψ_S-contractive mapping. If the following conditions are satisfied*:

(*i*) f is γ-admissible,

(*ii*) there exists $x_0 \in X$ such that $\gamma(x_0, x_0, fx_0) \geq 1$, and

(*iii*) If $\{x_n\}$ is a sequence in X such that $\gamma(x_n, x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x_{n_k}, x) \geq 1$ for all k,
then there is $a \in X$ such that $fa = a$.

Proof. Using the proof of Theorem 1, we can say that the sequence $\{x_n\}$ defined by

$$fx_n = x_{n+1},$$

for all $n \geq 0$, converges for some $a \in X$. By (iii), there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k}, x_{n_k}, a) \geq 1,$$

for all k. Using the γ-ψ_S-contractive mapping hypothesis, we obtain

$$\begin{aligned} S(x_{n_k+1}, x_{n_k+1}, fa) &= S(fx_{n_k}, fx_{n_k}, fa) \\ &\leq \gamma(x_{n_k}, x_{n_k}, a) S(fx_{n_k}, fx_{n_k}, fa) \\ &\leq \psi(\Delta(x_{n_k}, a)), \end{aligned} \tag{3}$$

where

$$\Delta(x_{n_k}, a) = \max \left\{ \begin{array}{c} S(x_{n_k}, x_{n_k}, a), S(x_{n_k}, x_{n_k}, x_{n_k+1}), S(a, a, fa), \\ \frac{S(x_{n_k}, x_{n_k}, x_{n_k+1}) S(a,a,fa)}{S(x_{n_k}, x_{n_k}, a)}, \\ \frac{S(x_{n_k}, x_{n_k}, x_{n_k+1}) S(a,a,fa)}{S(x_{n_k}, x_{n_k}, a) + S(x_{n_k}, x_{n_k}, fa) + S(a, a, x_{n_k+1})}, \\ \frac{S(x_{n_k}, x_{n_k}, x_{n_k+1}) S(x_{n_k}, x_{n_k}, fa) + S(a,a,x_{n_k+1}) S(a,a,fa)}{S(a,a,x_{n_k+1}) + S(x_{n_k}, x_{n_k}, fa)} \end{array} \right\}.$$

Let us take $k \to \infty$, we have

$$\lim_{k \to +\infty} \Delta(x_{n_k}, a) = S(a, a, fa).$$

Now, we assume that $S(a, a, fa) > 0$. For large enough k, we have

$$\Delta(x_{n_k}, a) > 0$$

and

$$\psi(\Delta(x_{n_k}, a)) < \Delta(x_{n_k}, a).$$

By 1, we obtain

$$S(x_{n_{k+1}}, x_{n_{k+1}}, fa) < \Delta(x_{n_k}, a).$$

Let us take $k \to +\infty$, we get

$$S(a, a, fa) < S(a, a, fa),$$

a contradiction. It should be $S(a, a, fa) = 0$, that is, $fa = a$. Therefore, a is a fixed point of f. □

EXAMPLE 4. Let $X = \{(1,0), (0,1)\} \subset \mathbb{R}^2$ and define the S-metric as

$$S((x,y),(u,v),(a,b)) = |u-a| + |u+a-2x| + |v-b| + |u+b-2y|,$$

for all $(x,y),(u,v),(a,b) \in X$. Then, (X,S) is a complete S-metric space and the S-metric is not generated by any metric. Let us define the self-mapping $f : X \to X$ as

$$f(x,y) = (x,y),$$

for all $(x,y) \in X$. It is clear that f is continuous. Also, the following inequality is satisfied for any $\psi \in \Psi$,

$$\gamma((x,y),(x,y),(u,v))\, S(f(x,y), f(x,y), f(u,v)) \leq \psi(\Delta((x,y),(u,v))),$$

for all $(x,y),(u,v) \in X$, where

$$\gamma((x,y),(u,v),(a,b)) = \begin{cases} 1, & (x,y) = (u,v) = (a,b) \\ 0, & \text{otherwise} \end{cases}.$$

Then, f is a γ-ψ_S-contractive mapping. Also, f is γ-admissible. Indeed, for all $(x,y),(u,v),(a,b) \in X$, we get

$$\begin{aligned} \gamma((x,y),(u,v),(a,b)) &\geq 1 \\ &\Longrightarrow (x,y) = (u,v) = (a,b) \\ &\Longrightarrow f(x,y) = f(u,v) = f(a,b) \\ &\Longrightarrow \gamma(f(x,y), f(u,v), f(a,b)) \geq 1. \end{aligned}$$

Also, for all $(x,y) \in X$, we have

$$\gamma((x,y),(u,v),(a,b)) \geq 1.$$

Hence, the conditions of Theorem 1 are satisfied. On the other hand, if $\{(x_n, y_n)\}$ is a sequence in X that converges to some point $(a,b) \in X$ with

$$\gamma((x_n,y_n),(x_n,y_n),(a,b)) \geq 1,$$

for all n, then by the definition of γ, we get

$$(x_n, y_n) = (a,b),$$

for all n, which implies that

$$\gamma((x_n,y_n),(x_n,y_n),(a,b)) = 1.$$

So, the conditions of Theorem 2 are satisfied. Consequently, f has two fixed points in X, that is, $Fix(f) = X$.

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REMARK 2. From Example 4, it can be said that the fixed points of a given self-mapping which satisfies the conditions Theorem 1 (or Theorem 2) cannot be unique. For this reason, it is important to investigate the uniqueness conditions for the existence theorems. For this purpose, we consider the following condition:

(iv) There is $z \in X$ such that

$$\gamma(x, x, z) \geq 1 \text{ and } \gamma(y, y, z) \geq 1,$$

for all $x, y \in Fix(f)$.

THEOREM 3. *If we add the condition (iv) to the hypothesis of Theorem 1 (resp. Theorem 2), then a is a unique fixed point of f.*

Proof. Let b be another fixed point of f with $a \neq b$. By (iv), there exists $z \in X$ such that

$$\gamma(a, a, z) \geq 1 \text{ and } \gamma(b, b, z) \geq 1.$$

From the α-admissibility of f, we get

$$\gamma(a, a, f^n z) \geq 1 \text{ and } \gamma(b, b, f^n z) \geq 1,$$

for all n. Let us define the sequence $\{z_n\}$ in X by

$$fz_n = z_{n+1},$$

for all $n \geq 0$ and $z_0 = z$. Then, we have

$$\begin{aligned} S(a, a, z_{n+1}) &= S(fa, fa, fz_n) \\ &\leq \gamma(a, a, z_n) S(fa, fa, fz_n) \\ &\leq \psi(\Delta(a, z_n)), \end{aligned}$$

where

$$\Delta(a, z_n) = \max\left\{ \begin{array}{c} S(a, a, z_n), S(a, a, a), S(z_n, z_n, z_{n+1}), \\ \frac{S(a,a,a)S(z_n,z_n,z_{n+1})}{S(a,a,z_n)}, \\ \frac{S(a,a,a)S(z_n,z_n,z_{n+1})}{S(a,a,z_n)+S(a,a,z_{n+1})+S(z_n,z_n,a)}, \\ \frac{S(a,a,a)S(a,a,z_{n+1})+S(z_n,z_n,a)S(z_n,z_n,z_{n+1})}{S(z_n,z_n,a)+S(a,a,z_{n+1})} \end{array} \right\}$$

$$\leq \quad \max\left\{S\left(a,a,z_n\right), S\left(a,a,z_{n+1}\right)\right\}.$$

Hence, using the monotone property of ψ, we obtain

$$S\left(a,a,z_{n+1}\right) \leq \psi\left(\max\left\{S\left(a,a,z_n\right), S\left(a,a,z_{n+1}\right)\right\}\right),$$

for all n. Without the generality, we assume that

$$S\left(a,a,z_n\right) > 0,$$

for all n.

Case 1: Let $\max\left\{S\left(a,a,z_n\right), S\left(a,a,z_{n+1}\right)\right\} = S\left(a,a,z_{n+1}\right)$. Then we have

$$S\left(a,a,z_{n+1}\right) \leq \psi\left(S\left(a,a,z_{n+1}\right)\right) < S\left(a,a,z_{n+1}\right),$$

a contradiction.

Case 2: $\max\left\{S\left(a,a,z_n\right), S\left(a,a,z_{n+1}\right)\right\} = S\left(a,a,z_n\right)$. Hence we get

$$S\left(a,a,z_n+1\right) \leq \psi\left(S\left(a,a,z_n\right)\right),$$

for all n. If we continue this process, then we have

$$S\left(a,a,z_n\right) \leq \psi^n\left(S\left(a,a,z_0\right)\right),$$

for all $n \geq 1$. Let us take $n \to +\infty$, we obtain

$$\lim_{n \to +\infty} S\left(a,a,z_n\right) = 0$$

and similarly,

$$\lim_{n \to +\infty} S\left(b,b,z_n\right) = 0$$

From the uniqueness of the limit point, we have

$$a = b.$$

Then a is the unique fixed point of f. □

EXAMPLE 5. Let $X=[0,1]$ and the S-metric be defined as

$$S(x,y,z)=|x-z|+|x+z-2y|,$$

for all $x,y,z\in X$ (Özgür & Taş, 2017). Then, the pair (X,S) is a complete S-metric space. Let us define the self-mapping $f:X\to X$ by

$$fx=\begin{cases}\frac{1}{8}, & x\in[0,1)\\ 0, & x=1\end{cases},$$

for all $x\in[0,1]$. It is clear that f is not a continuous function at the point $x_0=1$. Now we define the mapping $\gamma:X\times X\times X\to[0,+\infty)$ by

$$\gamma(x,y,z)=\begin{cases}1, & (x,y,z)\in\left(\left[0,\frac{1}{8}\right]\times\left[0,\frac{1}{8}\right]\times\left[\frac{1}{8},1\right]\right)\\ & \quad\cup\left(\left[\frac{1}{8},1\right]\times\left[\frac{1}{8},1\right]\times\left[0,\frac{1}{8}\right]\right)\\ 0, & \text{otherwise}\end{cases}$$

We show the validity of this example under the cases:

Case 1: f is a γ-ψ_S-contractive mapping with

$$\psi(t)=\frac{t}{4},$$

for all $t\geq 0$. If $x\in\left[0,\frac{1}{8}\right]$ and $z=1$, we have

$$\begin{aligned}\gamma(x,x,y)S(fx,fx,fy) &= S(fx,fx,fy)\\ &= 2|fx-fy|\\ &= 2\left|\frac{1}{8}-0\right|\\ &= \frac{1}{4}\\ &= \frac{1}{8}S(y,y,fy)\\ &\leq \psi(\Delta(x,y)).\end{aligned}$$

If $x=1$ and $y\in\left[0,\frac{1}{8}\right]$, we have

$$\begin{aligned}\gamma(x,x,y)S(fx,fx,fy) &= S(fx,fx,fy)\\ &= 2|fx-fy|\\ &= 2\left|0-\frac{1}{8}\right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4} \\
&= \frac{1}{8} S(x, x, fx) \\
&\leq \psi(\Delta(x, y)).
\end{aligned}
$$

The other cases are clear.

Case2: f is γ-admissible. To show this, we assume that $(x, y, z) \in X \times X \times X$ such that

$$\gamma(x, y, z) \geq 1.$$

If $(x, y, z) \in \left[0, \frac{1}{8}\right] \times \left[0, \frac{1}{8}\right] \times \left[\frac{1}{8}, 1\right]$, then $(fx, fy, fz) \in \left[\frac{1}{8}, 1\right] \times \left[\frac{1}{8}, 1\right] \times \left[0, \frac{1}{8}\right]$ which implies

$$\gamma(fx, fy, fz) = 1.$$

If $(x, y, z) \in \left[\frac{1}{8}, 1\right] \times \left[\frac{1}{8}, 1\right] \times \left[0, \frac{1}{8}\right]$, then $(fx, fy, fz) \in \left[0, \frac{1}{8}\right] \times \left[0, \frac{1}{8}\right] \times \left[\frac{1}{8}, 1\right]$ which implies

$$\gamma(fx, fy, fz) = 1.$$

Consequently, f is γ-admissible.

Case 3: There is $x_0 \in X$ such that

$$\gamma(x_0, x_0, fx_0) \geq 1.$$

If we take $x_0 = 0$, then we have

$$\gamma(x_0, x_0, fx_0) = \gamma\left(0, 0, \frac{1}{8}\right) = 1.$$

Case 4: If $\{x_n\}$ is a sequence in X such that $\gamma(x_n, x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to +\infty$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k}, x_{n_k}, x) \geq 1,$$

for all k. By the definition of γ, we get

$$(x_n, x_n, x_{n+1}) \in \left(\left[0, \frac{1}{8}\right] \times \left[0, \frac{1}{8}\right] \times \left[\frac{1}{8}, 1\right]\right) \cup \left(\left[\frac{1}{8}, 1\right] \times \left[\frac{1}{8}, 1\right] \times \left[0, \frac{1}{8}\right]\right)$$

for all n.

Case 5: The uniqueness condition (iv) is satisfied. Let $(a, b) \in [0, 1] \times [0, 1]$. For $z = \frac{1}{8}$, we get

$$\gamma(a, a, z) = 1, \ \gamma(b, b, z) = 1.$$

Consequently, f has a unique fixed point $a = \frac{1}{8}$.

REMARK 3. If the S-metric is generated by any metric, then the notion of a γ-ψ_S-contractive mapping coincides with the notion of a generated α-ψ-contractive type mapping introduced in (Raji et al, 2024). Also, under this case, Theorem 1 (resp. Theorem 2, Theorem 3) coincides with Theorem 3.2 (resp. Theorem 3.3, Theorem 3.4) proven in (Raji et al, 2024).

If the b-metric d^S is generated by the S-metric, then we obtain the following definitions and corollaries.

DEFINITION 6. *Let (X, d^S) be a b-metric space and $f : X \to X$ be a self-mapping. f is called an α-ψ_b-contractive mapping if there are two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)\, d^S(fx, fy) \leq \psi(\Delta_b(x, y)),$$

for all $x, y \in X$, where

$$\Delta_b(x, y) = \max\left\{\begin{array}{c} d^S(x, y), d^S(x, fx), d^S(y, fy), \\ \frac{d^S(x,fx)d^S(y,fy)}{d^S(x,y)}, \\ \frac{d^S(x,fx)d^S(y,fy)}{d^S(x,y)+d^S(x,fy)+d^S(y,fx)}, \\ \frac{d^S(x,fx)d^S(x,fy)+d^S(y,fx)d^S(y,fy)}{d^S(y,fx)+d^S(x,fy)} \end{array}\right\}.$$

THEOREM 4. *Let (X, d^S) be a complete b-metric space and $f : X \to X$ be an α-ψ_b-contractive mapping. If the following conditions are satisfied:*

(i) *f is α-admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1, and$*
(iii) *f is continuous,*
then there exists $a \in X$ such that $fa = a$.

THEOREM 5. *Let (X, d^S) be a complete b-metric space and $f : X \to X$ be an α-ψ_b-contractive mapping. If the following conditions are satisfied:*
(i) *f is α-admissible,*
(ii) *there exists $x_0 \in$ X such that $\alpha(x_0, fx_0) \geq 1$, and*
(iii) *if $\{x_n\}$ is a sequence in Xsuch that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in$ X as n $\to \infty$,then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k, then there is a $\in$ X such that fa=a.*
(iv) *There is z $\in$ X such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, for all x,y $\in$ Fix(f).*

THEOREM 6. *If we add the condition* (iv) *to the hypothesis of Theorem 5 (resp. Theorem 6), then* a *is a unique fixed point of* f.

Some integral type fixed-point results on S-metric spaces

DEFINITION 7. *Let* (X, S) *be an* S*-metric space and* $f : X \to X$ *be a self-mapping.* f *is called an integral type* γ-ψ_S*-contractive mapping if there exist* $\gamma : X \times X \to [0, +\infty)$ *and* $\psi \in \Psi$ *such that*

$$\int_0^{\gamma(x,x,y)S(fx,fx,fy)} \varphi(t)\, dt \leq \int_0^{\psi(\Delta(x,y))} \varphi(t)\, dt,$$

for all $x, y \in X$, *where* $\varphi : [0, +\infty) \to [0, +\infty)$ *is a Lebesque integrable mapping, which is summable, that is, with finite integral on each compact subset of* $[0, +\infty)$, *nonnegative and such that*

$$\int_0^{\varepsilon} \varphi(t)\, dt > 0,$$

for each $\varepsilon > 0$.

REMARK 4. If we take $\varphi(t) = 1$ in Definition 7, then the notions of a γ-ψ_S-contractive mapping and an integral type γ-ψ_S-contractive mapping coincide.

THEOREM 7. *Let* (X, S) *be a complete* S*-metric space and* $f : X \to X$ *be an integral type* γ-ψ_S*-contractive mapping. If the following conditions are satisfied*:

(a) f *is* γ*-admissible,*

(b) *there exists* $x_0 \in X$ *such that* $\gamma(x_0, x_0, fx_0) \geq 1, and$

(c) f *is continuous,*

then there exists $a \in X$ *such that* $fa = a$.

Proof. By the similar arguments used in the proof of Theorem 1, this can be easily proved. □

REMARK 5. If we take $\varphi(t) = 1$ in Theorem 7, then Theorem 7 and Theorem 1 coincide.

THEOREM 8. *Let (X, S) be a complete S-metric space and $f : X \to X$ be an integral type γ-ψ_S-contractive mapping. If the following conditions are satisfied*:

(a) *f is γ-admissible,*
(b) *there exists $x_0 \in X$ such that $\gamma(x_0, x_0, fx_0) \geq 1, and$*
(c) *if $\{x_n\}$ is a sequence in X such that $\gamma(x_n, x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x_{n_k}, x) \geq 1$ for all k,*

then there is $a \in X$ such that $fa = a$.

Proof. By the similar arguments used in the proof of Theorem 2, this can be easily proved. □

REMARK 6. If we take $\varphi(t) = 1$ in Theorem 8, Theorem 8 and Theorem 2 coincide.

THEOREM 9. (d) *There is $z \in X$ such that*

$$\gamma(x, x, z) \geq 1 \text{ and } \gamma(y, y, z) \geq 1,$$

for all $x, y \in Fix(f)$.

THEOREM 10. *If we add the condition (d) to the conditions (a)-(b)-(c) given in Theorem 8 (resp. Theorem 7), then a is a unique fixed point of f.*

REMARK 7.
(1) If we take $\varphi(t) = 1$ in Theorem 10, then Theorem 3 and Theorem 10 coincide.
(2) Theorem 7, Theorem 8 and Theorem 10 are integral type fixed point results and generalize Theorem 3.2, Theorem 3.3 and Theorem 3.5 proved in (Raji et al, 2024).
(3) The two examples given earlier provide integral type fixed-point theorems with $\varphi(t) = 1$.
(4) On b-metric spaces, the notion of an integral type α-ψ_b-contractive mapping can be defined and new integral type fixed point results can be proved as seen in the previous theorems.
(5) The results obtained in this article are new generalized fixed-point results for both S-metric spaces and b-metric spaces.

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Definition 8. *Let (X,S) be an S-metric space and $f:X\to X$ be a self-mapping. f is called a γ-ψ_S-f_{x_0}-contractive mapping if there exists $x_0\in X$, $\gamma : X\times X\times X\to[1,+\infty)$ and $\psi\in\Psi$ such that*

$$S(fx,fx,x)>0\Rightarrow\gamma(x,x,x_0)S(fx,fx,x)\le\psi(\Delta^*(x,x_0)),$$

for all $x\in X$, where

$$\Delta^*(x,y)=\max\left\{\begin{array}{c} S(x,x,y),S(x,x,fx),S(y,y,fy), \\ \frac{S(x,x,fx)S(y,y,fy)}{S(x,x,y)+S(x,x,fy)+S(y,y,fx)}, \\ \frac{S(x,x,fx)S(x,x,fy)+S(y,y,fx)S(y,y,fy)}{S(y,y,fx)+S(x,x,fy)}\end{array}\right\}.$$

Proposition 1. *If f is a γ-ψ_S-f_{x_0}-contractive mapping with $x_0\in X$, then we have*

$$fx_0=x_0.$$

Proof. On the contrary, we assume $fx_0\neq x_0$, that is,

$$S(fx_0,fx_0,x_0)>0.$$

Using the contraction hypothesis, we get

$$\gamma(x_0,x_0,x_0)S(fx_0,fx_0,x_0)\le\psi(\Delta^*(x_0,x_0))$$

and

$$\begin{aligned}\Delta^*(x_0,x_0) &= \left\{\begin{array}{c} S(x_0,x_0,x_0),S(x_0,x_0,fx_0),S(x_0,x_0,fx_0), \\ \frac{S(x_0,x_0,fx_0)S(x_0,x_0,fx_0)}{S(x_0,x_0,x_0)+S(x_0,x_0,fx_0)+S(x_0,x_0,fx_0)}, \\ \frac{S(x_0,x_0,fx_0)S(x_0,x_0,fx_0)+S(x_0,x_0,fx_0)S(x_0,x_0,fx_0)}{S(x_0,x_0,fx_0)+S(x_0,x_0,fx_0)}\end{array}\right\} \\ &= S(x_0,x_0,fx_0)\end{aligned}$$

Hence, using the symmetry property and the property of ψ, we have

$$\begin{aligned}\gamma(x_0,x_0,x_0)S(fx_0,fx_0,x_0) &\le \psi(S(fx_0,fx_0,x_0)) \\ &< S(fx_0,fx_0,x_0),\end{aligned}$$

a contradiction. So, it should be

$$fx_0=x_0.$$

□

THEOREM 11. *Let f be a γ-ψ_S-f_{x_0}-contractive mapping with $x_0 \in X$ and the number*

$$r = \inf \{S(fx, fx, x) : x \in X, x \neq fx\}. \tag{4}$$

If $fx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$, then f fixes the circle $C^S_{x_0,r}$.

Proof. Let us consider the following cases:

Case 1: Let $r = 0$. Then, we get

$$C^S_{x_0,r} = \{x_0\}$$

and so by Proposition 1, we say that f fixes the circle $C^S_{x_0,r}$.

Case 2: Let $r > 0$ and $x \in C^S_{x_0,r}$ be any point such that

$$S(fx, fx, x) > 0.$$

Using the contraction hypothesis and $fx \in C^S_{x_0,r}$, we get

$$\gamma(x, x, x_0) S(fx, fx, x) \leq \psi(\Delta^*(x, x_0))$$

and

$$\begin{aligned} \Delta^*(x, x_0) &= \max \left\{ \begin{array}{c} S(x, x, x_0), S(x, x, fx), S(x_0, x_0, fx_0), \\ \frac{S(x,x,fx)S(x_0,x_0,fx_0)}{S(x,x,x_0)+S(x,x,fx_0)+S(x_0,x_0,fx)}, \\ \frac{S(x,x,fx)S(x,x,fx_0)+S(x_0,x_0,fx)S(x_0,x_0,fx_0)}{S(x_0,x_0,fx)+S(x,x,fx_0)} \end{array} \right\} \\ &= S(fx, fx, x). \end{aligned}$$

So, we obtain

$$\gamma(x, x, x_0) S(fx, fx, x) \leq \psi(S(fx, fx, x)) < S(fx, fx, x),$$

a contradiction and thus, it should be

$$S(fx, fx, x) = 0 \Longrightarrow fx = x.$$

Consequently, f fixes the circle $C^S_{x_0,r}$. □

COROLLARY 1. *Let f be a γ-ψ_S-f_{x_0}-contractive mapping with $x_0 \in X$ and r be defined as in 4. If $fx \in D^S_{x_0,r}$ for each $x \in D^S_{x_0,r}$, then f fixes the disc $D^S_{x_0,r}$.*

EXAMPLE 6. Let $X=\mathbb{R}$ be the S-metric space with the S-metric defined as

$$S(x,y,z)=|x-z|+|x+z-2y|,$$

for all $x,y,z\in\mathbb{R}$ (Özgür & Taş, 2017). This S-metric is not generated by any metric. Let us define the mappings $f:X\to X$, $\gamma:X\times X\times X\to[1,+\infty)$ and $\psi:[0,+\infty)\to[0,+\infty)$ as, respectively,

$$fx=\begin{cases}41, & x=40\\ 0, & x\in\mathbb{R}-\{0\}\end{cases},$$

$$\gamma(x,y,z)=1$$

and

$$\psi(t)=\frac{t}{2}.$$

If we take $x_0=0$, then for $x=40$, we get

$$\begin{aligned}\Delta^*(40,0) &= \max\left\{\begin{array}{c} S(40,40,0),S(40,40,f40),S(0,0,f0),\\ \frac{S(40,40,f40)S(0,0,f0)}{S(40,40,0)+S(40,40,f40)+S(0,0,f40)},\\ \frac{S(40,40,f40)S(40,40,f0)+S(0,0,f40)S(0,0,f0)}{S(0,0,f40)+S(40,40,f0)}\end{array}\right\}\\ &= \max\left\{80,2,0,0,\frac{160}{102}\right\}\\ &= 80\end{aligned}$$

and

$$S(40,40,f40)=S(40,40,41)=2.$$

Then we have

$$\gamma(40,40,f40)\,S(f40,f40,40)=2\leq 40=\frac{80}{2}=\psi(\Delta^*(40,0)).$$

Consequently, f is a γ-ψ_S-f_{x_0}-contractive mapping with $x_0=0$. Also we have

$$r=2$$

and so f fixes the circle $C_{0,2}^S=\{-1,1\}$ and the disc $D_{0,2}^S=[-1,1]$.

Since every S-metric generates a b-metric, then we get the following:

DEFINITION 9. *Let* $\left(X, d^S\right)$ *be a* b*-metric space and* $f : X \to X$ *be a self-mapping.* f *is called an* α*-*ψ_b*-*f_{x_0}*-contractive mapping if there exist* $x_0 \in X$, $\alpha : X \times X \to [1, +\infty)$ *and* $\psi \in \Psi$ *such that*

$$d^S\left(fx, x\right) > 0 \Longrightarrow \alpha\left(x, x_0\right) d^S\left(fx, x\right) \leq \psi\left(\Delta_b^*\left(x, x_0\right)\right),$$

for all $x \in X$*, where*

$$\Delta_b^*\left(x, y\right) = \max\left\{\begin{array}{c} d^S\left(x, y\right), d^S\left(x, fx\right), d^S\left(y, fy\right), \\ \frac{d^S(x,fx)d^S(y,fy)}{d^S(x,y)+d^S(x,fy)+d^S(y,fx)}, \\ \frac{d^S(x,fx)d^S(x,fy)+d^S(y,fx)d^S(y,fy)}{d^S(y,fx)+d^S(x,fy)} \end{array}\right\}.$$

PROPOSITION 2. *If* f *is an* α*-*ψ_b*-*f_{x_0}*-contractive mapping* $x_0 \in X$*, then we have* $fx_0 = x_0$.

THEOREM 12. *Let* f *be an* α*-*ψ_b*-*f_{x_0}*-contractive mapping with* $x_0 \in X$ *and the number*

$$r^* = \inf\left\{d^S\left(fx, x\right) : x \in X, x \neq fx\right\}. \tag{5}$$

If $fx \in C^b_{x_{0,r^*}}$ *for each* $x \in C^b_{x_{0,r^*}}$*, then* f *fixes the circle* $C^b_{x_{0,r^*}}$.

COROLLARY 2. *Let* f *be an* α*-*ψ_b*-*f_{x_0}*-contractive mapping with* $x_0 \in X$ *and* r^* *be defined as in 5. If* $fx \in D^b_{x_{0,r^*}}$ *for each* $x \in D^b_{x_{0,r^*}}$*, then* f *fixes the disc* $D^b_{x_{0,r^*}}$.

Using the integral type technique, we give the following.

DEFINITION 10. *Let* (X, S) *be an* S*-metric space and* $f : X \to X$ *be a self-mapping.* f *is called an integral type* α*-*ψ_b*-*f_{x_0}*-contractive mapping if there exist* $x_0 \in X, \gamma : X \times X \times X \to [1, +\infty)$ *and* $\psi \in \Psi$ *such that*

$$S\left(fx, fx, x\right) > 0 \Longrightarrow \int_0^{\gamma(x,x,x_0)S(fx,fx,x)} \varphi\left(t\right) dt \leq \int_0^{\psi(\Delta^*(x,x_0))} \varphi\left(t\right) dt,$$

for all $x \in X$.

REMARK 8. If we take $\varphi\left(t\right) = 1$ in Definition 10, then the notions of a γ-ψ_S-f_{x_0}-contractive mapping and an integral type γ-ψ_S-f_{x_0}-contractive mapping coincide.

PROPOSITION 3. *If* f *is an integral type* γ*-*ψ_S*-*f_{x_0}*-contractive mapping* $x_0 \in X$*, then we have* $fx_0 = x_0$.

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Proof. By the similar arguments used in the proof of Proposition 1, this can be easily seen. □

THEOREM 13. *Let f be an integral type γ-ψ_S-f_{x_0}-contractive mapping $x_0 \in X$ and the number r be defined as in 4. If $fx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$, then f fixes the circle $C^S_{x_0,r}$.*

Proof. By the similar arguments used in the proof of Theorem 11, this can be easily proved. □

COROLLARY 3. *Let f be an integral type γ-ψ_S-f_{x_0}-contractive mapping $x_0 \in X$ and the number r be defined as in 4. If $fx \in D^S_{x_0,r}$ for each $x \in D^S_{x_0,r}$, then f fixes the disc $D^S_{x_0,r}$.*

REMARK 9.

1. If we take $\varphi(t) = 1,$ then Proposition 1 and Proposition 3 coincide.
2. If we take $\varphi(t) = 1,$ then Theorem 11 and Theorem 13 coincide.
3. If we take $\varphi(t) = 1,$ then Corollary 1 and Corollary 3 coincide.
4. On b-metric spaces, the notion of an integral type α-ψ_b-f_{x_0}-contractive mapping can be defined and some integral type fixed-circle results can be proved.

Conclusion

In this article, three different techniques used in fixed-point theory are used together. For this purpose, fixed-point theorems are obtained by using multiple new generalized contractive conditions on S-metric spaces, which are an example of generalized metric spaces. As an application of the obtained theorems, new results are obtained for the fixed-circle problem, which is a geometric generalization approach. On the other hand, another point that makes this study important is to further investigate the applications of the obtained fixed-circle results to activation functions. For example, we consider the Rectified linear unit (ReLU) activation function, (for more details, see (Nair & Hinton, 2010)), defined as

$$ReLU(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$$

Let us take $X = \{-10\} \cup [0, +\infty)$ with the S-metric defined as in Example 3 and define the mappings $\gamma : X \times X \times X \to [1, +\infty)$ and $\psi : [0, +\infty) \to$

$[0, +\infty)$ as, respectively,

$$\gamma(x, y, z) = 1$$

and

$$\psi(t) = \frac{t}{2}.$$

If we take $x_0 = 20$, then for $x = -10$, we get

$$\Delta^*(-10, 20) = 60$$

and

$$S(-10, -10, ReLU(-10)) = 20.$$

Then we have

$$\begin{aligned} \gamma(-10, -10, 20)\, S(ReLU(-10), ReLU(-10), -10) &= 20 \leq 30 \\ &= \frac{60}{2} \\ &= \psi(\Delta^*(-10, 20)). \end{aligned}$$

Hence, ReLU is a γ-ψ_S-f_{x_0}-contractive mapping $x_0 = 20$. Also, we have

$$r = 20$$

and so ReLU fixes the circle $C^S_{20,20} = \{10, 30\}$ and the disc $D^S_{20,20} = [10, 30]$.

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Algunos teoremas de punto fijo mediante contracciones γ-ψ_S-en espacios S-métricos

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TIPO DE ARTÍCULO: artículo científico original

Resumen

Introducción: Este artículo se centra en la extensión de la teoría de puntos fijos en espacios S-métricos mediante la introducción de nuevas condiciones contractivas generalizadas. Estos desarrollos buscan enriquecer las herramientas analíticas disponibles para el estudio de dichos espacios

Métodos: Se establecen diversos teoremas de punto fijo mediante la aplicación de las condiciones contractivas recientemente definidas. La metodología incluye aplicaciones contractivas tanto estándar como integrales. Además, se utiliza un enfoque geométrico para obtener nuevos teoremas de círculo fijo dentro del marco de la métrica S.

Resultados: Se demuestran varios teoremas de punto fijo y círculo fijo en las condiciones propuestas. Se proporcionan ejemplos ilustrativos para validar los hallazgos teóricos y demostrar su aplicabilidad.

Conclusión: Los hallazgos de este estudio no solo amplían el alcance de la teoría de punto fijo en espacios S-métricos, sino que también ofrecen posibles implicaciones para aplicaciones prácticas. En particular, los resultados podrían contribuir al desarrollo de las matemáticas computacionales y al diseño de funciones de activación de redes neuronales.

Palabras claves: espacio S-métrico, teorema del punto fijo, contracción generalizada, contracción de tipo integral, círculo fijo, enfoque geométrico, función de activación, redes neuronales.

Некоторые теоремы о неподвижной точке γ-ψ_S-сжатия на S-метрическом пространстве

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ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение: Данная статья посвящена расширению теории неподвижных точек в S-метрических пространствах путем введения новых обобщенных условий сжатия. Исследование направлено на улучшение аналитических инструментов, доступных в изучении подобных пространств

Методы: Установлен ряд теорем о неподвижных точках с применением условий. Методология включает как стандартные, так и инте- гральные принципы сжимающих отображений. Помимо того, используется геометрический подход для получения новых теорем о неподвижных точках в S метрическом пространстве.

Результаты: Доказаны некоторые теоремы о непо- движных точках и неподвижных окружностях. предложен- ных условиях. Приведены наглядные примеры для подтверждения теоретических выводов и демонстрации применимости результатов.

Заключение: Результаты данного исследования не только расширяют область применения теории неподвижной точки в S-метрических пространствах, но и могут быть использованы в прикладных приложениях. В частности, результаты могут способствовать развитию вычислительной математики и разработке функций активации нейронных сетей. В частности, они могут способствовать развитию вычислительной математики и проектирова- нию функций активации в нейронных сетях.

Ключевые слова: S-метрическое пространство, теорема о неподвижной точке, обобщенное сжатиеённая , сжатие интегрального типа, неподвижная окружность, геометрический подход, функция активации, нейронные сети.

Неке теореме о непокретној тачки преко γ-ψ_S контракција на С-метричким просторима

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ОБЛАСТ: математика

КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод: Тежиште овог рада је на проширењ теорије непокретних тачака у С-метричке просторе увођењем новеих генерализованеих контрактивнеих условеа. Циљ овог проширивања јесте да се обогате аналитички алати доступни за проучавање оваквих простора.

Методе: Различите теореме о непокретној тачки успостављене су коришћењем новодефинисаних контрактивних услова. Методологија обухвата стандардна пресликавања, као и пресликавања интегралног типа контракције Такође, користи се и геометријски приступ дакако би се добиле нове теореме о непокретној тачцки унутар С-метричког простора.

Резултати: Доказано је неколико теорема о непокретним тачкама и непокретним круговима под предложеним условима. Наведени примери илуструју теоријске налазе и показују применљивост резултата.

Закључак: Резултати овог рада не само да проширују домен теорије непокретних тачака на С-метричке просторе већ, и нуде потенцијалне импликације за примену у стварности. Такође, могу да допринесу развоју у областима рачунарске математике и пројектовања активационих функција неуронских мрежа.

Кључне речи: С-метрички простор, теорема о непокретној тачки, генерализована контракција, контракција интегралног типа, непокретни круг, геометријски приступ, активациона функција, неуронске мреже.

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