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CONTRA (m_X, m_Y) -SEMICONINUOUS FUNCTIONS IN m -SPACES

ABSTRACT. In this paper, we introduce the notion of contra (m_X, m_Y) -semicontinuous functions between m -spaces. We obtain many characterizations of these functions and deal with decompositions of the functions and other related functions.

KEY WORDS: contra (m_X, m_Y) -semicontinuity, m_X -semi-closed set, m_X -semi-open set, minimal structure, minimal space.

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1. Introduction

Generalizations of open sets in a topological space: α -sets [8], preopen sets [3], semi-open sets [1] and β -open sets etc are very important for generalizing continuity in topological spaces. Various generalizations of continuity are defined and investigated by many authors. As a generalization of the topology, Maki [2] define the notion of minimal structures. A subfamily m of the power set $P(X)$ on a nonempty set X is called a minimal structure [2] if $\emptyset \in m$ and $X \in m$. The pair (X, m) is called a minimal space. The elements of m are said to be m -open. Recently, several generalizations of m -open sets have been defined and investigated in [4, 5, 6] and [15]. Quite recently, Sengul and Rosas [14] introduced the notion of contra (m_X, m_Y) -continuity between m -spaces.

The purpose of the present paper is to introduce and study the notion of contra (m_X, m_Y) -semicontinuous functions between m -spaces. In Section 3, we obtain many characterizations of contra (m_X, m_Y) -semicontinuity. In Section 4, we deal with decompositions of contra (m_X, m_Y) -semicontinuity and other related spaces. The last section gives some properties of strongly $S - m_X$ -closed spaces.

2. Preliminaries

Definition 1 ([2, 11]). A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a minimal structure (briefly, m -structure) on X if $\emptyset \in m_X$ and $X \in m_X$. The pair (X, m_X) is called a minimal space (briefly, m -space). A member of m_X is said to be m_X -open and the complement of an m_X -open set is said to be m_X -closed.

Definition 2 ([2, 11]). Let (X, m_X) be a minimal space. For a subset A of X , the m_X -closure of A and the m_X -interior of A are defined as follows:

- (1) $m_X - Cl(V) = \bigcap \{F : A \subseteq F, X - F \in m_X\}$.
- (2) $m_X - Int(V) = \bigcup \{U : U \subseteq A, U \in m_X\}$.

Lemma 1 ([2, 11]). Let (X, m_X) be a minimal space and $A, B \subseteq X$. Then the followings hold:

- (1) $m_X - Cl(\emptyset) = \emptyset$, $m_X - Cl(X) = X$.
- (2) $m_X - Int(\emptyset) = \emptyset$, $m_X - Int(X) = X$.
- (3) If $X - A \in m_X$, then $m_X - Cl(A) = A$.
- (4) If $A \in m_X$, then $m_X - Int(A) = A$.
- (5) $A \subseteq m_X - Cl(A)$, $m_X - Int(A) \subseteq A$.
- (6) $m_X - Cl(X - A) = X - (m_X - Int(A))$.
- (7) $m_X - Int(X - A) = X - (m_X - Cl(A))$.
- (8) $m_X - Cl(m_X - Cl(A)) = m_X - Cl(A)$.
- (9) $m_X - Int(m_X - Int(A)) = m_X - Int(A)$.
- (10) If $A \subseteq B$, then $m_X - Cl(A) \subseteq m_X - Cl(B)$.
- (11) If $A \subseteq B$, then $m_X - Int(A) \subseteq m_X - Int(B)$.

Definition 3 ([2]). Let (X, m_X) be a minimal space. The m -structure m_X is said to have property \mathcal{B} if the union of any family of subsets belonging to m_X belongs to m_X .

Lemma 2 ([11]). Let (X, m_X) be a minimal space and m_X satisfy property of \mathcal{B} . For $A \subseteq X$, the followings hold:

- (1) $A \in m_X$ if and only if $m_X - Int(A) = A$.
- (2) A is m_X -closed if and only if $m_X - Cl(A) = A$.
- (3) $m_X - Int(A) \in m_X$.
- (4) $m_X - Cl(A)$ is m_X -closed.

Lemma 3 ([11]). Let (X, m_X) be a minimal space and $A \subseteq X$. Then $x \in m_X - Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ such that $x \in U$.

Definition 4. Let (X, m_X) be a minimal space. A subset A of X is said to be m_X -clopen if it is m_X -open and m_X -closed.

Definition 5. Let (X, m_X) be a minimal space. A subset A of X is called

- (1) an αm_X - open set [6] if $A \subseteq m_X - \text{Int}(m_X - \text{Cl}(m_X - \text{Int}(A)))$.
- (2) an m_X -preopen set [4, 13] if $A \subseteq m_X - \text{Int}(m_X - \text{Cl}(A))$.
- (3) a β - m_X -open set [7, 15] if $A \subseteq m_X - \text{Cl}(m_X - \text{Int}(m_X - \text{Cl}(A)))$.

Definition 6 ([5]). Let (X, m_X) be a minimal space. A subset A of X is called an m_X -semiopen set if $A \subseteq m_X - \text{Cl}(m_X - \text{Int}(A))$. The complement of an m_X -semiopen set is called an m_X -semiclosed set. The family of all m_X -semiopen sets in X is denoted by $MSO(X)$.

Lemma 4 ([5]). Let (X, m_X) be a minimal space and $A \subseteq X$. Then

- (1) A is an m_X -semiclosed set if and only if $m_X - \text{Int}(m_X - \text{Cl}(A)) \subseteq A$.
- (2) $MSO(X)$ is a minimal structure with property \mathcal{B} .

Definition 7 ([5]). Let (X, m_X) be a minimal space and $A \subseteq X$. The m_X -semi-closure of A and the m_X -semi-interior of A are defined as follows:

- (1) $m_X s \text{Cl}(A) = \bigcap \{F : A \subseteq F, F \text{ is } m_X\text{-semiclosed in } X\}$.
- (2) $m_X s \text{Int}(A) = \bigcup \{U : U \subseteq A, U \text{ is } m_X\text{-semiopen in } X\}$.

Lemma 5. Let (X, m_X) be a minimal space. For a subset of A of X , the following hold:

- (1) A is m_X -semiopen if and only if $m_X s \text{Int}(A) = A$.
- (2) A is m_X -semiclosed if and only if $m_X s \text{Cl}(A) = A$.

Proof. This follows easily from Lemmas 2 and 4. ■

Definition 8 ([13]). Let (X, m_X) be a minimal space. Then a subset A of X is said to be m_X -gs-closed if $m_X s \text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m_X$.

Definition 9 ([13]). Let (X, m_X) be a minimal space. Then $A \subseteq X$ is called an m_X -regular open set if $A = m_X - \text{Int}(m_X - \text{Cl}(A))$. Also $A \subseteq X$ is called an m_X -regular closed set if $X - A$ is m_X -regular open.

If A is m_X -closed, then $m_X - \text{cl}(A) = A$ but the converse is not always true. Therefore, m_X -regular open (resp. m_X - regular closed) is not always m_X -open (resp. m_X -closed).

Definition 10 ([12]). A subset U of a nonempty set X with a minimal structure m_X is said to be m_X -compact relative to (X, m_X) if any cover of U by m_X -open sets has a finite subcover.

Definition 11 ([14]). Let (X, m_X) and (Y, m_Y) be two minimal spaces. Then a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be contra (m_X, m_Y) -continuous if $f^{-1}(V) = m_X - \text{Cl}(f^{-1}(V))$ for every m_Y -open set V of Y .

3. Contra (m_X, m_Y) -semi continuous functions

In this section, we introduce the concept of a contra (m_X, m_Y) -semi continuous function between m -spaces and investigate some characterizations of this continuity.

Definition 12. Let (X, m_X) and (Y, m_Y) be two minimal spaces. Then a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be contra (m_X, m_Y) -semi continuous if $f^{-1}(V)$ is m_X -semiclosed in X for every m_Y -open set V of Y .

Lemma 6. Every contra (m_X, m_Y) -continuous function is contra (m_X, m_Y) -semi continuous.

Proof. Let $f : (X, m_X) \rightarrow (Y, m_Y)$ be a contra (m_X, m_Y) -continuous function and V be any m_Y -open set of Y . Then $m_X - Cl(f^{-1}(V)) = f^{-1}(V)$ and $m_X - Int(m_X - Cl(f^{-1}(V))) = m_X - Int f^{-1}(V) \subseteq f^{-1}(V)$. Therefore, Lemma 4, $f^{-1}(V)$ is m_X -semiclosed and f is contra (m_X, m_Y) -semi continuous. ■

Remark 1. The converse of Lemma 6 is not always true as the following example shows.

Example 1. Let $X = \{a, b, c\}$ and m_{X_1}, m_{X_2} be two minimal structures on X as follows:

$$\begin{aligned} m_{X_1} &= \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}, \\ m_{X_2} &= \{\emptyset, X, \{c\}\}. \end{aligned}$$

Define a function $f : (X, m_{X_1}) \rightarrow (X, m_{X_2})$ as follows:

$$f(a) = b, \quad f(b) = c, \quad f(c) = a.$$

Then f is contra (m_X, m_Y) -semi continuous, but it is not contra (m_X, m_Y) -continuous.

Theorem 1. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is contra (m_X, m_Y) -semi continuous if and only if $f : (X, MSO(X)) \rightarrow (Y, m_Y)$ is contra (m_X, m_Y) -continuous.

Proof. *Necessity.* Let $f : (X, m_X) \rightarrow (Y, m_Y)$ be contra (m_X, m_Y) -semi continuous and V be any m_Y -open set of Y . Then, by hypothesis $f^{-1}(V)$ is m_X -semiclosed in X and, by Lemma 5, $f^{-1}(V) = m_X sCl(f^{-1}(V))$. Therefore, $f : (X, MSO(X)) \rightarrow (Y, m_Y)$ is contra (m_X, m_Y) -continuous.

Sufficiency. Let V be any m_Y -open set of Y . By hypothesis, $f^{-1}(V) = m_X sCl(f^{-1}(V))$ and, by Lemma 5, $f^{-1}(V)$ is m_X -semi-closed. Therefore, $f : (X, m_X) \rightarrow (Y, m_Y)$ is contra (m_X, m_Y) -semi continuous. ■

Definition 13. Let (X, m_X) and (Y, m_Y) be two minimal spaces. Then a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be contra (m_X, m_Y) -semicontinuous at $x \in X$ if for each m_Y -closed V of Y containing $f(x)$, there exists an m_X -semiopen set U of X containing x such that $f(U) \subseteq V$.

Theorem 2. Let $(X, m_X), (Y, m_Y)$ be two minimal spaces. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is contra (m_X, m_Y) -semi continuous if and only if f is contra (m_X, m_Y) -semicontinuous at each point $x \in X$.

Proof. *Necessity.* Let $x \in X$ and V be any m_Y -closed set of Y containing $f(x)$. Then $Y - V$ is m_Y -open. By hypothesis, $f^{-1}(Y - V)$ is an m_X -semiclosed subset of X . Thus $f^{-1}(V)$ is m_Y -semiopen. Put $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$. This shows that f is contra (m_X, m_Y) -semicontinuous at each point $x \in X$.

Sufficiency. Let V be any m_Y -open set of Y and $x \in f^{-1}(Y - V)$. Then $f(x) \in Y - V$ and $Y - V$ is m_Y -closed. By hypothesis, there exists an m_X -semiopen set U_x containing x such that $f(U_x) \subseteq Y - V$; hence $x \in U_x \subseteq f^{-1}(Y - V)$. Therefore, we have $\cup\{U_x : x \in f^{-1}(Y - V)\} = f^{-1}(Y - V)$. Since $MSO(X)$ satisfies property \mathcal{B} , $f^{-1}(Y - V)$ is m_X -semiopen and $f^{-1}(V)$ is m_X -semiclosed in X . This shows that f contra (m_X, m_Y) -semi continuous. ■

Theorem 3. Let (X, m_X) and (Y, m_Y) be two minimal spaces. For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, the following statements are equivalent:

- (1) f is contra (m_X, m_Y) -semi continuous;
- (2) $f^{-1}(V)$ is m_X -semiopen in X for every m_Y -closed subset V of Y ;
- (3) $m_X - Int(m_X - Cl(f^{-1}(V))) = m_X - Int(f^{-1}(V))$ for every m_Y -open subset V of Y ;
- (4) $m_X - Cl(m_X - Int(f^{-1}(V))) = m_X - Cl(f^{-1}(V))$ for every m_Y -closed subset V of Y .

Proof. (1) \Rightarrow (2). Let V be any m_Y -closed set of Y . Then $Y - V$ is m_Y -open. Using the hypothesis, $f^{-1}(Y - V) = X - f^{-1}(V)$ is m_X -semiclosed in X . As a consequence, $f^{-1}(V)$ is m_X -semiopen in X .

(2) \Rightarrow (3). Let V be any m_Y -open set of Y . Then $Y - V$ is m_Y -closed. By (2), $f^{-1}(Y - V)$ is m_X - semiopen and $f^{-1}(V)$ is m_X - semiclosed in X . By Lemma 4, $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq f^{-1}(V)$ and hence by Lemma 1 $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq m_X - Int(f^{-1}(V)) \subseteq m_X - Int(m_X - Cl(f^{-1}(V)))$. Therefore, we obtain (3).

(3) \Rightarrow (4). It is clear from the complement of (3).

(4) \Rightarrow (1). Let V be any m_Y -open subset of Y . Then $Y - V$ is m_Y -closed. By hypothesis,

$$m_X - Cl(m_X - Int(f^{-1}(Y - V))) = m_X - Cl(f^{-1}(Y - V)).$$

Then we obtain that

$$m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(V))) = m_X - \text{Int}(f^{-1}(V)) \subseteq f^{-1}(V).$$

By Lemma 4, $f^{-1}(V)$ is m_X -semiclosed in X . ■

Theorem 4. *Let (X, m_X) , (Y, m_Y) be two minimal spaces and m_Y satisfy property \mathcal{B} . For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, the following statements are equivalent:*

- (1) f is contra (m_X, m_Y) -semi continuous;
- (2) $f^{-1}(B)$ is m_X -semiopen in X for every m_Y -closed set B in Y ;
- (3) $f^{-1}(B) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(B))))$ for every subset B in Y ;
- (4) $m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(m_Y - \text{Int}(B)))) \subseteq f^{-1}(B)$ for every subset B in Y ;
- (5) $A \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(f(A))))$ for every subset A in X .

Proof. (1) \Leftrightarrow (2). It is obvious from Theorem 3.

(2) \Rightarrow (3). Let $B \subseteq Y$. Then $m_Y - \text{Cl}(B)$ is an m_Y -closed set in Y since m_Y satisfies property \mathcal{B} . By (2), $f^{-1}(m_Y - \text{Cl}(B))$ is m_X -semiopen in X . Therefore, $f^{-1}(m_Y - \text{Cl}(B)) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(B))))$. As a consequence, $f^{-1}(B) \subseteq f^{-1}(m_Y - \text{Cl}(B)) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(B))))$.

(3) \Leftrightarrow (4). It is clear from the complement.

(4) \Rightarrow (5). Let $A \subseteq X$. Then $f(A) \subseteq Y$. By (3), $A \subseteq f^{-1}(f(A)) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(f(A))))$.

(5) \Rightarrow (2). Let B be any m_Y -closed set in Y . Then $f^{-1}(B) \subseteq X$. By (5), $f^{-1}(B) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(f(f^{-1}(B)))))) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(B))))$. Then we obtain

$$f^{-1}(B) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(B)))$$

since B is m_Y -closed in Y . As a consequence, $f^{-1}(B)$ is m_X -semiopen in X . ■

Theorem 5. *Let (X, m_X) , (Y, m_Y) be two minimal spaces and m_X, m_Y satisfy property \mathcal{B} . For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, the following statements are equivalent:*

- (1) f is contra (m_X, m_Y) -semi continuous;
- (2) $f^{-1}(V)$ is m_X -semiopen in X for every m_Y -closed subset V of Y ;
- (3) There exists an m_X -semiclosed set U such that $x \notin U$ and $f^{-1}(V) \subseteq U$ for each $x \in X$ and each m_Y -open V with $f(x) \notin V$;

- (4) $f^{-1}(F) \subseteq m_X sInt(f^{-1}(F))$ for any m_Y -closed set F in Y ;
- (5) $m_X sCl(f^{-1}(F)) \subseteq f^{-1}(F)$ for any m_Y -open set F in Y ;
- (6) $m_X sCl(f^{-1}(m_Y - Int(F))) \subseteq f^{-1}(m_Y - Int(F))$ for any subset $F \subseteq Y$;
- (7) $f^{-1}(m_Y - Cl(F)) \subseteq m_X sInt(f^{-1}(m_Y - Cl(F)))$ for any subset $F \subseteq Y$.

Proof. (1) \Leftrightarrow (2) is already shown in Theorem 3.

(1) \Rightarrow (3). Let $x \in X$ and V be any m_Y -open subset of Y with $f(x) \notin V$. Then $f^{-1}(V)$ is m_X -semiclosed. Put $U = f^{-1}(V)$. Then $f^{-1}(V) \subseteq U$ and $x \notin U$.

(3) \Rightarrow (1). Let V be any m_Y -open subset of Y . For each $x \in f^{-1}(Y - V)$, $f(x) \notin V$. By hypothesis, there exists an m_X -semiclosed set U_x such that $x \notin U_x$ and $f^{-1}(V) \subseteq U_x$. Then $x \in X - U_x \subseteq X - f^{-1}(V) = f^{-1}(Y - V)$. We obtain

$$\bigcup_{x \in f^{-1}(Y - V)} \{x\} \subseteq \bigcup_{x \in f^{-1}(Y - V)} (X - U_x) \subseteq f^{-1}(Y - V).$$

Hence $f^{-1}(Y - V) = \bigcup_{x \in f^{-1}(Y - V)} (X - U_x)$ is m_X -semiopen. Thus $f^{-1}(V)$ is m_X -semiclosed. As a consequence, f is contra (m_X, m_Y) -semi continuous.

(1) \Rightarrow (4). Let F be any m_Y -closed subset of Y . For each $x \in f^{-1}(F)$, $f(x) \in F$. By Theorem 2, there exists an m_X -semiopen set U such that $x \in U$ and $f(U) \subseteq F$. Since $x \in U \subseteq f^{-1}(F)$, we obtain $x \in m_X sInt(f^{-1}(F))$. As a consequence, $f^{-1}(F) \subseteq m_X sInt(f^{-1}(F))$.

(4) \Rightarrow (5). It is obvious from taking the complement of (4).

(5) \Rightarrow (6). Let F be any subset of Y . Since m_Y satisfies property \mathcal{B} , $m_Y - Int(F)$ is an m_Y -open subset of Y and by (5), we obtain

$$m_X sCl(f^{-1}(m_Y - Int(F))) \subseteq f^{-1}(m_Y - Int(F)).$$

(6) \Rightarrow (7). It is clear from the complement of (6).

(7) \Rightarrow (1). Let V be any m_Y -open subset of Y . Then $Y - V$ is m_Y -closed. By (7), $X - f^{-1}(V) = f^{-1}(Y - V) = f^{-1}(m_Y - Cl(Y - V)) \subseteq m_X sInt(f^{-1}(m_Y - Cl(Y - V))) = m_X sInt(f^{-1}(Y - V)) = X - m_X sCl(f^{-1}(V))$. Therefore, $m_X - sCl(f^{-1}(V)) \subseteq f^{-1}(V)$ and hence $m_X - sCl(f^{-1}(V)) = f^{-1}(V)$. Since m_X satisfies property \mathcal{B} , $f^{-1}(V)$ is m_X -semiclosed in X . As a consequence, f is contra (m_X, m_Y) -semi continuous. ■

4. Decompositions of contra (m_X, m_Y) -semicontinuity

In this section, we obtain decompositions of contra (m_X, m_Y) -semicontinuous functions and other related functions.

Definition 14. Let (X, m_X) be a minimal space. A subset A of X is called

- (1) an m_X -semi-regular set if A is both m_X -semiopen and m_X -semiclosed.
- (2) an m_X - B -set if $A = U \cap V$, where $U \in m_X$ and V is m_X -semiclosed.

Lemma 7. Let (X, m_X) be a minimal space and $A \subseteq X$. Then the following conditions are equivalent:

- (1) A is m_X -semi-regular;
- (2) A is both βm_X -open and m_X -semiclosed.

Proof. It is obvious by Lemma 4. ■

Remark 2. A βm_X -open set and an m_X -semiclosed set are independent of each other as the following examples show.

Example 2. Let $X = \{a, b, c\}$ and $m_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $A = \{a, b\}$ is an m_X -open set and hence βm_X -open, but it is not an m_X -semiclosed set.

Example 3. Let $X = \{a, b, c\}$ and $m_X = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}\}$. Then $A = \{a, b\}$ is an m_X -closed set and hence m_X -semiclosed set, but it is not a βm_X -open set.

Lemma 8. Let (X, m_X) be a minimal space and m_X satisfy property \mathcal{B} . Then for a subset A of X , the following conditions are equivalent:

- (1) A is both m_X -open and m_X -semiclosed;
- (2) A is both αm_X -open and m_X -semiclosed;
- (3) A is both m_X -preopen and m_X -semiclosed.

Proof. It is clear. ■

Remark 3. An m_X -preopen set and an m_X -semiclosed set are independent of each other as the following example shows.

Example 4. Consider Example 2, then the set $A = \{a, b\}$ is an m_X -preopen set, but it is not m_X -semiclosed. Also in Example 3, the set A is an m_X -semiclosed set, but it is not an m_X -preopen set.

Lemma 9. Let (X, m_X) be a minimal space and $A \subseteq X$. If A is both βm_X -open and m_X -closed, then it is m_X -regular closed.

Proof. It is an immediate result. ■

Remark 4. A βm_X -open set and an m_X -closed set are independent of each other as the following example shows.

Example 5. Consider Example 2, then the set $A = \{a, b\}$ is a βm_X -open set, but it is not an m_X -closed set. Also in Example 3, the set A is an m_X -closed set, but it is not a βm_X -open set.

Definition 15. Let (X, m_X) and (Y, m_Y) be two minimal spaces. Then a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be

- (1) (m_X, m_Y) -perfectly continuous if $f^{-1}(V)$ is m_X -clopen in X for every m_Y -open set V of Y ,
- (2) (m_X, m_Y) -completely continuous if $f^{-1}(V)$ is m_X -regular open in X for every m_Y -open set V of Y ,
- (3) (m_X, m_Y) -semi-regular continuous (briefly, (m_X, m_Y) - SR-continuous) if $f^{-1}(V)$ is m_X -semi-regular open in X for every m_Y -open set V of Y ,
- (4) (m_X, m_Y) -regular closed continuous (briefly, (m_X, m_Y) - RC-continuous) if $f^{-1}(V)$ is m_X -regular closed in X for every m_Y -open set V of Y ,
- (5) (m_X, m_Y) - B-continuous if $f^{-1}(V)$ is an m_X - B-set in X for every m_Y -open set V of Y .

Definition 16 ([7]). Let m_X, m_Y be two minimal structures. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be $M - \beta$ -continuous if $f^{-1}(V)$ is βm_X -open in X for every m_Y -open set V of Y .

Theorem 6. For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, the following statements are equivalent:

- (1) f is (m_X, m_Y) - SR-continuous;
- (2) f is $M - \beta$ -continuous and contra (m_X, m_Y) -semi continuous.

Proof. It is an immediate result of Lemma 7. ■

Definition 17 ([4]). Let m_X, m_Y be two minimal structures. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be M -pre continuous if $f^{-1}(V)$ is m_X -preopen in X for every m_Y -open set V of Y .

Theorem 7. If a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is M -pre continuous and contra (m_X, m_Y) -semi continuous, it is (m_X, m_Y) -completely continuous.

Proof. It is clear from the fact that every m_X -preopen and m_X -semiclosed set is m_X -regular open. ■

Theorem 8. If a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is $M - \beta$ -continuous and contra (m_X, m_Y) -continuous, it is (m_X, m_Y) - RC-continuous.

Proof. It is obvious from Lemma 9. ■

Definition 18. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be contra (m_X, m_Y) - gs -continuous if $f^{-1}(V)$ is m_X - gs -closed in X for every m_Y -open set V of Y .

Theorem 9. For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, the following statements are equivalent:

- (1) f is contra (m_X, m_Y) -semi continuous;
- (2) f is (m_X, m_Y) - B -continuous and contra (m_X, m_Y) - gs -continuous.

Proof. (1) \Rightarrow (2). It is clear.

(2) \Rightarrow (1). Let V be any m_Y -open set of Y . Since f is (m_X, m_Y) - B -continuous, $f^{-1}(V) = U \cap F$, where $U \in m_X$ and F is m_X -semiclosed in X . Then $f^{-1}(V) \subseteq U$ and $U \in m_X$. $f^{-1}(V)$ is m_X - gs -closed and since f is contra (m_X, m_Y) - gs -continuous, $m_X sCl(f^{-1}(V)) \subseteq U$. Since $MSO(X)$ satisfies property \mathcal{B} , $m_X sCl(f^{-1}(V))$ is m_X - semiclosed and by Lemma 4 $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq m_X - Int(m_X - Cl(m_X sCl(f^{-1}(V)))) \subseteq m_X sCl(f^{-1}(V)) \subseteq U$. On the other hand, F is m_X - semiclosed and by Lemma 4 $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq m_X - Int(m_X - Cl(F)) \subseteq F$. Therefore, we obtain $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq U \cap F = f^{-1}(V)$. As a consequence, $f^{-1}(V)$ is m_X -semiclosed. \blacksquare

Remark 5. The notions of (m_X, m_Y) - B -continuity and contra (m_X, m_Y) - gs -continuity are independent of each other as shown by the following example.

Example 6. Let $X = \{1, 2\}$, $Y = \{a, b\}$, $m_X = \{\emptyset, X, \{2\}\}$ and $m_Y = \{\emptyset, Y\}$. Let $f : (X, m_X) \rightarrow (X, m_X)$ be the identity function. Then f is (m_X, m_Y) - B -continuous but it is not contra (m_X, m_Y) - gs -continuous. Also, let $g : (Y, m_Y) \rightarrow (X, m_X)$ be a function defined as follows:

$$g(a) = 1, \quad g(b) = 2.$$

Then g is contra (m_X, m_Y) - gs -continuous, but it is not (m_X, m_Y) - B -continuous.

Corollary 1. For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, the following statements are equivalent:

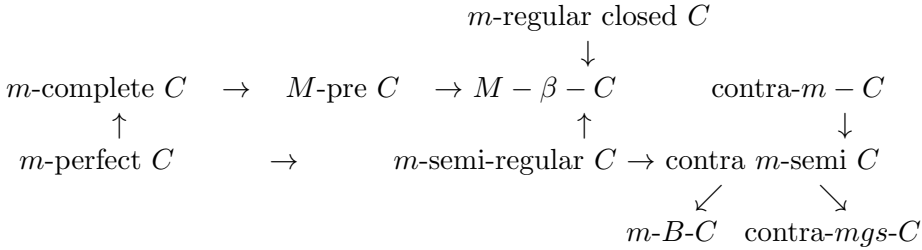
- (1) f is (m_X, m_Y) - SR -continuous;
- (2) f is M - β -continuous, (m_X, m_Y) - B -continuous and contra (m_X, m_Y) - gs -continuous.

Proof. It is obvious from Theorems 6 and 9. \blacksquare

Remark 6. The function $f : (X, m_X) \rightarrow (X, m_X)$ in Example 6 is (m_X, m_Y) -pre continuous, but it is not contra (m_X, m_Y) - gs -continuous. Also, the function $g : (Y, m_Y) \rightarrow (X, m_X)$ in Example 6 is (m_X, m_Y) -pre continuous, but it is not (m_X, m_Y) - B -continuous.

Remark 7. We obtain the following diagram which shows the relationships between contra (m_X, m_Y) -semicontinuous functions and other related functions.

DIAGRAM



In the diagram, C denotes continuity and m means (m_X, m_Y) .

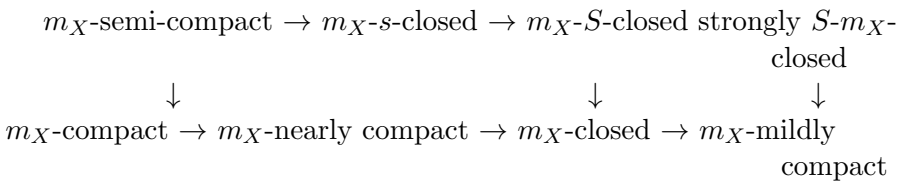
5. Strongly $S - m_X$ -closed spaces

Definition 19. A minimal space (X, m_X) is said to be

- (1) m_X -semi-compact if there exists a finite subset J of I such that $X = \cup\{U_i : i \in J\}$ for every m_X -semiopen cover $\{U_i : i \in I\}$ of X ,
- (2) m_X -s-closed if there exists a finite subset J of I such that $X = \cup\{m_X sCl(U_i) : i \in J\}$ for every m_X -semiopen cover $\{U_i : i \in I\}$ of X ,
- (3) $m_X - S$ -closed if there exists a finite subset J of I such that $X = \{m_X - Cl(U_i) : i \in J\}$ for every m_X -semiopen cover $\{U_i : i \in I\}$ of X ,
- (4) [14] m_X -nearly compact if there exists a finite subset J of I such that $X = \cup\{m_X - Int(m_X - Cl(U_i)) : i \in J\}$ for every m_X -open cover $\{U_i : i \in I\}$ of X ,
- (5) [9] m_X -closed if there exists a finite subset J of I such that $X = \cup\{m_X - Cl(U_i) : i \in J\}$ for every m_X -open cover $\{U_i : i \in I\}$ of X ,
- (6) [10] strongly S - m_X -closed if every m_X -closed cover of X has a finite subcover,
- (7) m_X -mildly compact if every m_X -clopen cover of X has a finite subcover.

We obtain the following diagram:

DIAGRAM



Theorem 10. Let (X, m_X) , (Y, m_Y) be two minimal spaces and a function $f : (X, m_X) \rightarrow (Y, m_Y)$ be a surjection. If one of the following statements holds, then (Y, m_Y) is strongly S - m_Y -closed.

- (1) f is contra (m_X, m_Y) -semi continuous and (X, m_X) is m_X -semi-compact,
- (2) f is (m_X, m_Y) -perfectly continuous and (X, m_X) is m_X -mildly compact.

Proof. Suppose (2) holds: Let $\{U_i : i \in I\}$ be an m_Y -closed cover of Y . $\{f^{-1}(U_i) : i \in I\}$ is an m_X -clopen cover of X since f is (m_X, m_Y) -perfectly continuous. Then there exists a finite $J \subseteq I$ such that $X = \bigcup_{i \in J} f^{-1}(U_i)$ as (X, m_X) is m_X -mildly compact. Hence $Y = \bigcup_{i \in J} U_i$. As a consequence, (Y, m_Y) is strongly S - m_Y -closed. ■

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