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Polynomial Approximation in Smirnov-Orlicz Classes

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Abstract. We prove a direct theorem for polynomial approximation of functions in certain subclasses of Smirnov-Orlicz classes.

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1. Introduction and Main Result

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. This curve separates the plane into two domains $G := \operatorname{Int}\Gamma$ and $G^- := \operatorname{Ext}\Gamma$. Without loss of generality we may assume $0 \in G$. In particular, if $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$ denotes the unit circle, then $\mathbb{D} := \operatorname{Int}\mathbb{T}$ is the open unit disk and $\mathbb{D}^- := \operatorname{Ext}\mathbb{T}$ is the complement of the closed unit disk. Let $w = \varphi(z)$ be the conformal mapping of G^- onto $\mathbb{D}^-$ normalized by

$$\varphi(\infty) = \infty, \qquad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0,$$

and let ψ denote the inverse of φ.

By $L_p(\Gamma)$, resp. $E_p(G)$, $1 \le p < \infty$, we denote the Lebesgue space of measurable complex valued functions on Γ, resp. the Smirnov class of analytic functions in G. Since Γ is rectifiable, we have $\varphi' \in E_1(G^-)$ and $\psi' \in E_1(\mathbb{D}^-)$. Hence, the functions φ', resp. ψ', admit nontangential limits a.e. on Γ, resp. on $\mathbb{T}$ and these functions belong to $L_1(\Gamma)$, resp. $L_1(\mathbb{T})$ (cf. [8, p. 419]).

For $z \in \Gamma$ and $\varepsilon > 0$ let $\Gamma(z, \varepsilon)$ denote the portion of Γ which is inside the open disk of radius ε centered at z, i.e. $\Gamma(z, \varepsilon) := \{t \in \Gamma : |t - z| < \varepsilon\}$. Further, let $|\Gamma(z, \varepsilon)|$ denote the length of $\Gamma(z, \varepsilon)$.

Definition 1. A rectifiable Jordan curve Γ is called a *Carleson curve* if

$$\sup_{\varepsilon>0} \sup_{z\in\Gamma} \frac{1}{\varepsilon} |\Gamma(z, \varepsilon)| < \infty.$$

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A convex and continuous function $M\colon [0,\infty) \to [0,\infty)$ which satisfies the four conditions

$$\begin{aligned} M(0) &= 0, \\ M(x) &> 0 \qquad \text{for } x > 0, \\ \lim_{x\to 0} \frac{M(x)}{x} &= 0, \qquad \text{and} \\ \lim_{x\to\infty} \frac{M(x)}{x} &= \infty \end{aligned}$$

is called an *N-function.* The *complementary N-function* to M is defined by

$$N(y) := \max_{x\geq 0}(xy - M(x)) \qquad \text{for } y \geq 0.$$

Let M be an N-function and N be its complementary function. By $L_M(\Gamma)$ we denote the linear space of Lebesgue measurable functions $f\colon \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M(\alpha|f(z)|)\,|dz| < \infty$$

for some $\alpha > 0$. Equipped with the norm

$$\|f\|_{L_M(\Gamma)} := \sup\left\{\int_\Gamma |f(z)g(z)|\,|dz| \,:\, g \in L_N(\Gamma),\, \rho(g,N) \leq 1\right\}, \tag{1}$$

where

$$\rho(g,N) := \int_\Gamma N(|g(z)|)\,|dz|,$$

the space $L_M(\Gamma)$ becomes a Banach space, cf. [16, pp. 52–68]. The norm $\|\cdot\|_{L_M(\Gamma)}$ is called *Orlicz norm* and the Banach space $L_M(\Gamma)$ is called *Orlicz space.* It is known, cf. [16, p. 50], that every function in $L_M(\Gamma)$ is integrable on Γ, i.e.

$$L_M(\Gamma) \subset L_1(\Gamma).$$

A N-function M satisfies the Δ_2*-condition* if

$$\limsup_{x\to\infty} \frac{M(2x)}{M(x)} < \infty.$$

The Orlicz space $L_M(\Gamma)$ is reflexive if and only if the N-function M and its complementary function N both satisfy the Δ_2-condition (cf. [16, p. 113]). Further information about Orlicz spaces may be found in [15] and [16].

Let Γ_r be the image of the circle $\{w \in \mathbb{C} : |w| = r,\, 0 < r < 1\}$ under some conformal mapping of $\mathbb{D}$ onto G and let M be an N-function. By $E_M(G)$ we denote the class of analytic functions f in G which satisfy

$$\int_{\Gamma_r} M(|f(z)|)\,|dz| < \infty$$

uniformly in r.

Definition 2 ([13]). The class $E_M(G)$ is called the *Smirnov-Orlicz class.*

If $M(x) = M(x,p) := x^p$, $1 < p < \infty$, then the Smirnov−Orlicz class $E_M(G)$ coincides with the usual Smirnov class $E_p(G)$. As observed in [13], every function in the class $E_M(G)$ has nontangential boundary values a.e. and the boundary function belongs to $L_M(\Gamma)$.

For $\zeta \in \Gamma$ we define the point $\zeta_h \in \Gamma$ by

$$\zeta_h := \psi(\varphi(\zeta)e^{ih}), \qquad h \in [0, 2\pi].$$

For $f \in L_M(\Gamma)$ the *shift* $T_h f$ is defined by

$$T_h f(\zeta) := \frac{f(\zeta_h)}{\varphi'(\zeta_h)}\varphi'(\zeta), \qquad \zeta \in \Gamma. \tag{2}$$

In particular, if $\Gamma = \mathbb{T}$, then $T_h f(w) = f(we^{ih})$ and hence $T_h f \in L_M(\mathbb{T})$ as soon as $f \in L_M(\mathbb{T})$. Moreover, if

$$0 < c_1 \leq |\varphi'(z)| \leq c_2 < \infty \qquad \text{for all } z \in \Gamma$$

with some constants c_1 and c_2 independent of z, then it is easy to verify that $L_M(\Gamma)$ is invariant under the shift $T_h f$. Starting from this we define the function $\omega_M^*(\cdot, f)$ by

$$\omega_M^*(\delta, f) := \sup_{|h|\leq\delta} \|f - T_h f\|_{L_M(\Gamma)}, \qquad \delta \geq 0.$$

Let $\omega(\delta)$ be a nonnegative, continuous, nondecreasing real function which satisfies $\omega(0) = 0$, $\omega(\delta) > 0$ for $\delta > 0$, and $\omega(n\delta) \leq c_3 n\omega(\delta)$ for every $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and for some constant $c_3 > 0$. The class of functions $f \in E_M(G)$ which satisfy the condition

$$\omega_M^*(\delta, f) \leq c_4\omega(\delta), \qquad \delta > 0,$$

with some constant c_4 independent of f and δ will be denoted by $H_\Gamma^\omega E_M(G)$.

Obviously $T_h f \in L_M(\Gamma)$ for every $f \in H_\Gamma^\omega E_M(G)$. Moreover, if $f, g \in H_\Gamma^\omega E_M(G)$, then

$$\begin{aligned}
\omega_M^*(0, f) &= 0, \\
\omega_M^*(\delta, f) &\geq 0 \qquad \text{for } \delta \geq 0, \\
\lim_{\delta\to 0} \omega_M^*(\delta, f) &= 0, \\
\omega_M^*(\delta, f + g) &\leq \omega_M^*(\delta, f) + \omega_M^*(\delta, g).
\end{aligned}$$

In this paper we discuss approximation problems in the class $H_\Gamma^\omega E_M(G)$. Our main result, Theorem 1 below, is a direct theorem for approximation by polynomials in these classes. In the proof, the approximating polynomials are constructed by using the Dzjadyk sums of the boundary function of f (cf. [2, pp. 71–96]).

Theorem 1. *Let Γ be a Carleson curve, let $L_M(\Gamma)$ be a reflexive Orlicz space on Γ, and let $f \in H_\Gamma^\omega E_M(G)$. Then for any $n \in \mathbb{N}$ there exists an algebraic polynomial $P_n(z,f)$ of degree at most n such that*

$$\|f - P_n(\cdot, f)\|_{L_M(\Gamma)} \le c\,\omega(1/n) \tag{3}$$

with some constant c independent of n.

Theorem 1 has not been known before, even not for the spaces $L_p(\Gamma)$, $1 < p < \infty$.

Some inverse problems of approximation theory in Smirnov-Orlicz classes have been investigated by Kokilashvili [13] in the case that Γ is a smooth Jordan curve and that $\theta(s)$, the angle between the tangent and the positive real axis defined as a function of the arclength s, has a modulus of continuity $\Omega(s,\theta)$ satisfying the Dini-smooth condition

$$\int_0^\delta \frac{\Omega(s,\theta)}{s}\,ds < \infty \qquad \text{for } \delta > 0.$$

Similar problems for the spaces $L_p(\Gamma)$ and $E_p(G)$, $1 \le p < \infty$, have been studied in [1], [3], [9], [14], [7], [5], [10], [11]. All this has been done under different restrictive conditions on $\Gamma = \partial G$

We use $c, c_1, c_2, \ldots$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest.

2. Auxiliary results

Let Γ be a rectifiable Jordan curve and consider a function $f \in L_1(\Gamma)$. Then the functions f^+ and f^-, defined by

$$f^+(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \qquad \text{for } z \in G, \tag{4}$$

$$f^-(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \qquad \text{for } z \in G^-, \tag{5}$$

are analytic in G, resp. G^-, and we have $f^-(\infty) = 0$. The *Cauchy singular integral* of f at a point $z \in \Gamma$ is defined by

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i}\int_{\Gamma \setminus \Gamma(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

if the limit exists. If one of the functions f^+ or f^- has nontangential limits a.e. on Γ, then $S_\Gamma(f)(z)$ exists a.e. on Γ and also the other one has nontangential limits a.e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a.e. on Γ, then both functions f^+ and f^- have nontangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z), \qquad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z) \tag{6}$$

and hence $f = f^+ - f^-$ hold a.e. on Γ (cf. [8, p. 431]).

To f we associate the function $S_\Gamma(f)$ by taking the value $S_\Gamma(f)(z)$ a.e. on Γ. The linear operator S_Γ defined in this way is called the *Cauchy singular operator.*

The following theorem, proved in [12], characterizes the curves for which the singular operator S_Γ is bounded in a reflexive Orlicz space $L_M(\Gamma)$.

Theorem 2. *Let Γ be a rectifiable Jordan curve and let $L_M(\Gamma)$ be a reflexive Orlicz space on Γ. Then the singular operator S_Γ is bounded on $L_M(\Gamma)$, i.e.*

$$\|S_\Gamma(f)\|_{L_M(\Gamma)} \leq c\|f\|_{L_M(\Gamma)} \qquad \text{for all } f \in L_M(\Gamma) \tag{7}$$

for some constant $c > 0$, if and only if Γ is a Carleson curve.

Let k be a nonnegative integer. Then the function $\varphi^k(z)\varphi'(z)$ has a pole of order k at ∞. Hence there exist a polynomial $B_k(z)$ of degree k and a function $E_k(z)$ analytic in G^- such that $E_k(\infty) = 0$ and

$$\varphi^k(z)\varphi'(z) := B_k(z) + E_k(z) \qquad \text{for all } z \in G^-.$$

The polynomials $B_k(z)$ are called the *Faber polynomials of second kind* of $\overline{G}$. It is known (cf. [17, p. 95]) that

$$\frac{1}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{B_k(z)}{w^{k+1}} \qquad \text{for } z \in G,\ w \in \mathbb{D}^-.$$

3. Proof of Theorem 1

Let $f \in L_M(\Gamma)$. Then $f \in L_1(\Gamma)$ and the function

$$f_0(w) := f(\psi(w))\psi'(w)$$

is integrable on $\mathbb{T}$. For $w \in \mathbb{T}$ we can associate to f_0 a series of the form

$$f_0(w) \sim \sum_{k=0}^{\infty} a_k w^k + \sum_{k=1}^{\infty} \frac{b_k}{w^k}.$$

Let

$$K_n(\theta) = \sum_{m=-n}^{n} \lambda_m^{(n)} e^{im\theta}$$

be an even, nonnegative trigonometric polynomial satisfying the conditions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\theta)\, d\theta = 1, \tag{8}$$

$$\int_0^{\pi} \theta K_n(\theta)\, d\theta \leq \frac{c}{n} \tag{9}$$

for $n \in \mathbb{N}$ with some constant $c > 0$. In particular, the *Jackson kernel*

$$J_n(\theta) := \frac{3(\sin\frac{n\theta}{2})^4}{n(2n^2+1)(\sin\frac{\theta}{2})^4}$$

satisfies these conditions (cf. [6, pp. 203–204]).

Let us consider the integral

$$I(\theta, z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta_{-\theta})}{\varphi'(\zeta_{-\theta})} \frac{\varphi'(\zeta)}{\zeta - z} \, d\zeta, \qquad z \in G.$$

By a change of variables we obtain

$$I(\theta, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_0(e^{i(t-\theta)}) \frac{e^{it}}{\psi(e^{it}) - z} \, dt.$$

Since

$$f_0(e^{it}) \sim \sum_{k=0}^{\infty} a_k e^{ikt} + \sum_{k=1}^{\infty} \frac{b_k}{e^{ikt}}, \qquad \frac{e^{it}}{\psi(e^{it}) - z} \sim \sum_{k=0}^{\infty} \frac{B_k(z)}{e^{ikt}}$$

we can, by [4, pp. 74–75], associate to $I(\theta, z)$ the series

$$I(\theta, z) \sim \sum_{k=0}^{\infty} a_k B_k(z) e^{-ik\theta}.$$

The function $K_n(\theta)$ is of bounded variation and $I(\cdot, z) \in L_1([-\pi, \pi])$. Therefore, by applying the generalized Parseval identity [4, pp. 225-228], we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) I(\theta, z) \, d\theta = \sum_{k=0}^{n} \lambda_k^{(n)} a_k B_k(z).$$

This implies

$$\frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} K_n(\theta) \, d\theta \int_\Gamma \frac{f(\zeta_{-\theta})}{\varphi'(\zeta_{-\theta})} \frac{\varphi'(\zeta)}{\zeta - z} \, d\zeta = \sum_{k=0}^{n} \lambda_k^{(n)} a_k B_k(z) \qquad \text{for } z \in G.$$

Hence,

$$P_n(z, f) := \frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} K_n(\theta) \, d\theta \int_\Gamma \frac{f(\zeta_{-\theta})}{\varphi'(\zeta_{-\theta})} \frac{\varphi'(\zeta)}{\zeta - z} \, d\zeta, \qquad z \in G,$$

is a polynomial of degree n. Since the kernel $K_n(\theta)$ is an even function we get

$$P_n(z, f) = \frac{1}{4\pi^2 i} \int_0^{\pi} K_n(\theta) \, d\theta \int_\Gamma \left[\frac{f(\zeta_\theta)}{\varphi'(\zeta_\theta)} + \frac{f(\zeta_{-\theta})}{\varphi'(\zeta_{-\theta})} \right] \frac{\varphi'(\zeta)}{\zeta - z} \, d\zeta \qquad \text{for } z \in G.$$

Now, by using (2), we obtain

$$P_n(z, f) = \frac{1}{4\pi^2 i} \int_0^{\pi} K_n(\theta) \, d\theta \int_\Gamma \left[T_\theta f(\zeta) + T_{-\theta} f(\zeta) \right] \frac{d\zeta}{\zeta - z}, \qquad z \in G,$$

and finally, taking into account (4),

$$(10) \qquad P_n(z, f) = \frac{1}{2\pi} \int_0^{\pi} K_n(\theta) \left[(T_\theta f)^+(z) + (T_{-\theta} f)^+(z) \right] \, d\theta, \qquad z \in G.$$

We are now ready to work on the main part of the proof of Theorem 1. Let $f \in H_\Gamma^\omega E_M(G)$ and $z' \in G$. Then, by (8),

$$f(z') = f^+(z') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^+(z')K_n(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{\pi} 2f^+(z')K_n(\theta)\,d\theta,$$

which, together with (10), implies

$$f(z') - P_n(z',f) = \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left\{2f^+(z') - \left[(T_\theta f)^+(z') + (T_{-\theta}f)^+(z')\right]\right\}\,d\theta.$$

Taking the limit $z' \to z \in \Gamma$ along all nontangential paths inside Γ we obtain, by using (6),

$$\begin{aligned} f(z) - P_n(z,f) &= \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left[S_\Gamma(f - T_\theta f)(z) + S_\Gamma(f - T_{-\theta}f)(z)\right]\,d\theta \\ &\quad + \frac{1}{4\pi}\int_0^{\pi} K_n(\theta)\left[(f - T_\theta f)(z) + (f - T_{-\theta}f)(z)\right]\,d\theta \end{aligned}$$

for almost all $z \in \Gamma$. Now, we use (1), apply Fubini's Theorem, and take the supremum into the integral to obtain

$$\begin{aligned} &\|f - P_n(\cdot,f)\|_{L_M(\Gamma)} \\ &= \sup \int_\Gamma |f(z) - P_n(z,f)|\,|g(z)|\,|dz| \\ &\le \sup \int_\Gamma \left|\frac{1}{2\pi}\int_0^{\pi} K_n(\theta)[S_\Gamma(f - T_\theta f)(z) + S_\Gamma(f - T_{(-\theta)}f)(z)]d\theta\right| |g(z)|\,|dz| \\ &\quad + \sup \int_\Gamma \left|\frac{1}{4\pi}\int_0^{\pi} K_n(\theta)[(f - T_\theta f)(z) + (f - T_{(-\theta)}f)(z)]d\theta\right| |g(z)|\,|dz| \\ &\le \sup \int_\Gamma \left\{\frac{1}{2\pi}\int_0^{\pi} K_n(\theta)[|S_\Gamma(f - T_\theta f)(z)| + |S_\Gamma(f - T_{-\theta}f)(z)|]d\theta\right\} |g(z)|\,|dz| \\ &\quad + \sup \int_\Gamma \left\{\frac{1}{4\pi}\int_0^{\pi} K_n(\theta)[|(f - T_\theta f)(z)| + |(f - T_{-\theta}f)(z)|]d\theta\right\} |g(z)|\,|dz| \\ &\le \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)\left\{\sup \int_\Gamma [|S_\Gamma(f - T_\theta f)(z)| + |S_\Gamma(f - T_{-\theta}f)(z)|]|g(z)|\,|dz|\right\} d\theta \\ &\quad + \frac{1}{4\pi}\int_0^{\pi} K_n(\theta)\left\{\sup \int_\Gamma [|(f - T_\theta f)(z)| + |(f - T_{-\theta}f)(z)|]|g(z)|\,|dz|\right\} d\theta \\ &\le \frac{1}{2\pi}\int_0^{\pi} K_n(\theta)[\|S_\Gamma(f - T_\theta f)\|_{L_M(\Gamma)} + \|S_\Gamma(f - T_{-\theta}f)\|_{L_M(\Gamma)}]d\theta \\ &\quad + \frac{1}{4\pi}\int_0^{\pi} K_n(\theta)[\|f - T_\theta f\|_{L_M(\Gamma)} + \|f - T_{-\theta}f\|_{L_M(\Gamma)}]d\theta, \end{aligned}$$

where all the suprema above are taken over all functions $g \in L_N(\Gamma)$ with $\rho(g, N) \leq 1$. By virtue of (7) this implies

$$\|f - P_n(\cdot, f)\|_{L_M(\Gamma)} \leq c_5 \int_0^\pi K_n(\theta) \left[\|f - T_\theta f\|_{L_M(\Gamma)} + \|f - T_{-\theta} f\|_{L_M(\Gamma)}\right] d\theta.$$

Finally, we recall the definitions of $\omega_M^*(\delta, f)$ and the class $H_\Gamma^\omega E_M(G)$ to get

$$\begin{aligned} \|f - P_n(\cdot, f)\|_{L_M(\Gamma)} &\leq c_6 \int_0^\pi K_n(\theta) \omega_M^*(\theta, f)\, d\theta \\ &\leq c_7 \int_0^\pi K_n(\theta) \omega(\theta)\, d\theta \\ &\leq c_8 \omega(1/n) \int_0^\pi K_n(\theta)(n\theta + 1)\, d\theta. \end{aligned}$$

According to (9) this implies (3) and completes the proof of Theorem 1.

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