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COMPLEX FACTORIZATIONS OF THE k-FIBONACCI AND k-LUCAS NUMBERS

BY

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Abstract. In this paper we obtain several complex factorizations of the k-Fibonacci and k-Lucas numbers.

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1. Introduction

Let $k \geq 1$ be any integer. In this paper we consider k-Fibonacci and k-Lucas numbers. The k-th Fibonacci sequence $\{F_{k,n}\}$ is defined recurrently by $F_{k,0} = 0$, $F_{k,1} = 1$ and $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$.

As particular cases, if $k = 1$ we obtain the classical Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, ...\}$ and if k=2 the Pell sequence appears $\{0, 1, 2, 5, 12, 29, ...\}$.

The k-th Lucas sequence $\{L_{k,n}\}$ is defined as follows:

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1} \text{ for } n \geq 1$$

with initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$. As particular cases, if $k = 1$ we obtain the classical Lucas sequence $\{2, 1, 3, 4, 7, 11, 18, ...\}$ and if $k = 2$ the Pell-Lucas sequence appears $\{2, 2, 6, 14, 34, 82, ...\}$ (see fore more details [4], [3] and the references therein).

Complex factorizations of the Fibonacci and Lucas numbers are given by Cahill, D'Errico and Spence in [2]. These complex factorizations are given of the forms $F_n = \prod_{j=1}^{n-1}(1 - 2i\cos\frac{\pi j}{n}),\ n \geq 2$ and $L_n = \prod_{j=1}^{n}(1 - 2i\cos\frac{\pi(j-\frac{1}{2})}{n}), n \geq 1$.

In this paper we obtain complex factorizations of the k-Fibonacci and k-Lucas numbers. We use the following lemma in our investigations.

Lemma 1.1 (see [2], Lemma 1). *Let* $\{H(n), n = 1, 2, ...\}$ *be a sequence of tridiagonal matrices of the form*

$$\begin{pmatrix} h_{1,1} & h_{1,2} & & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & & \\ & h_{3,2} & h_{3,3} & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & h_{n-1,n} \\ & & & & h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

Then the successive determinants of $H(n)$ are given by the recursive formula $|H(1)| = h_{1,1}, |H(2)| = h_{1,1}h_{2,2} - h_{1,2}h_{2,1}, |H(n)| = h_{n,n}\,|H(n-1)| - h_{n-1,n}h_{n,n-1}\,|H(n-2)|$.

2. Complex factorization of the k-Fibonacci numbers

Let $\{C(n), n = 1, 2, 3, ...\}$ be the sequences of matrices where $C(n)$ is the nxn tridiagonal matrix with initial conditions $c_{j,j} = k, 1 \leq j \leq n$ and $c_{j-1,j} = c_{j,j-1} = i$, $2 \leq j \leq n$. In other words,

$$C(n) = \begin{pmatrix} k & i & & & & \\ i & k & i & & & \\ & i & k & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & i \\ & & & & i & k \end{pmatrix}.$$

According to Lemma 1.1 successive determinants of $C(n)$ are given by the following formula $|C(1)| = k, |C(2)| = k^2 + 1, |C(n)| = k\,|C(n-1)| + |C(n-2)|$.

It is obvious that this is the k-Fibonacci sequence starting with $F_{k,2}$ and we have

$$F_{k,n} = |C(n-1)|\,,\ n \geq 2. \tag{2.1}$$

It is known that the determinant of a matrix is the product of its eigenvalues. For this reason, we should determine eigenvalues of $C(n)$.

Let $\{G(n), n = 1, 2, ...\}$ be a matrix sequence satisfying $g_{j,j} = 0, 1 \leq j \leq n$ and $g_{j-1,j} = g_{j,j-1} = 1, 2 \leq j \leq n$, so $G(n)$ be in the following form:

$$G(n) = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ & & & & 1 & 0 \end{pmatrix}.$$

In this case we have $C(n) = iG(n) + kI$. Let λ_j be the eigenvalue of $G(n)$ regarding the eigenvector x_j for $j = 1, 2, ..., n$. For each j we get $C(n)x_j = [kI + iG(n)]x_j = kIx_j + iG(n)x_j = kx_j + i\lambda_j x_j = (k + i\lambda_j)x_j$.

So, $\mu_j = k + i\lambda_j$ is an eigenvalue of $C(n)$ for $j = 1, 2, ..., n$ and therefore we find

$$|C(n)| = \prod_{j=1}^{n}(k + i\lambda_j),\ n \geq 1. \tag{2.2}$$

Since λ_j is a zero of $p_n(\lambda) = |G(n) - \lambda I|$ and $G(n) - \lambda I$ tridiagonal matrix, we find $p_1(\lambda) = -\lambda, p_2(\lambda) = \lambda^2 - 1, p_n(\lambda) = -\lambda p_{n-1}(\lambda) - p_{n-2}(\lambda)$. If we substitute $\lambda = -2x$, then we have $U_1(x) = 2x, U_2(x) = 4x^2 - 1, U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, where $\{U_n(x), n \geq 1\}$ is the set of Chebyshev polynomials of the second kind to be written as (see [6])

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}.$$

Then, the roots of $U_n(x) = 0$ are $x_j = \cos\frac{\pi j}{n+1}$ $(1 \leq j \leq n)$ and we find

$$\lambda_j = -2\cos\frac{\pi j}{n+1},\quad j = 1, 2, ..., n. \tag{2.3}$$

Combining (2.1), (2.2), (2.3), we have the following complex factorization of the k-Fibonacci numbers:

$$F_{k,n+1} = \prod_{j=1}^{n}(k + i\lambda_j) = \prod_{j=1}^{n}(k - 2i\cos\frac{\pi j}{n+1}),\ n \geq 1. \tag{2.4}$$

3. Complex factorization of the k-Lucas numbers

Let $\{D(n), n = 1, 2, 3, ...\}$ be a sequence of matrices where $D(n)$ is the nxn tridiagonal matrix with initial conditions $d_{1,1} = \frac{k}{2}, d_{j,j} = k, 2 \leq j \leq n$ and $d_{j-1,j} = d_{j,j-1} = i, 2 \leq j \leq n$. In other words,

$$D(n) = \begin{pmatrix} \frac{k}{2} & i & & & & \\ i & k & i & & & \\ & i & k & . & & \\ & & . & . & . & \\ & & & . & . & i \\ & & & & i & k \end{pmatrix}.$$

According to Lemma 1.1 successive determinants of $D(n)$ are given by the following formula $|D(1)| = \frac{k}{2}, |D(2)| = \frac{k^2}{2} + 1, |D(n)| = k\,|D(n-1)| + |D(n-2)|$. Each number in the above sequence is the half of the n-th k-Lucas number. So we have

$$L_{k,n} = 2\,|D(n)|\,,\ n \geq 1. \tag{3.1}$$

After some calculations, we see that $L_{k,n} = 2\,|D(n)| = \left|kI + i(G(n) + e_1e_2^T)\right|$.

Let γ_j be an eigenvalue of $D(n)$ corresponding to the eigenvector y_j $(j = 1, 2, ..., n)$. For each j we have $L_{k,n}y_j = (kI + i(G(n) + e_1e_2^T))y_j = ky_j + i(G(n) + e_1e_2^T)y_j = ky_j + i\gamma_jy_j = (k + i\gamma_j)y_j$. Then, $\varsigma_j = k + i\gamma_j$, $j = 1, 2, ..., n$ are the eigenvalues of $2D(n)$, therefore we find

$$2\,|D(n)| = \prod_{j=1}^{n}(k + i\gamma_j). \tag{3.2}$$

Since γ_j is a zero of $q_n(\gamma) = \left|G(n) + e_1e_2^T - \gamma I\right|$ and $\left|I - \frac{1}{2}e_1e_1^T\right| = \frac{1}{2}$, we have $q_n(\gamma) = 2\left|(I - \frac{1}{2}e_1e_1^T)(G(n) + e_1e_2^T - \gamma I)\right|$ and $q_1(\gamma) = -\frac{\gamma}{2}, q_2(\lambda) = \frac{\gamma^2}{2} - 1, q_n(\lambda) = -\gamma q_{n-1}(\gamma) - q_{n-2}(\gamma)$.

If we substitute $\gamma = -2x$, then we find $T_1(x) = x, T_2(x) = 2x^2 - 1, T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, where $\{T_n(x), n \geq 1\}$ is the set of Chebyshev polynomials of the first kind. Defining $x = \cos\theta$ allows the Chebyshev polynomials of the first kind to be written as (see [6]) $T_n(x) = \cos n\theta$. Then, the roots of $T_n(x) = 0$ are $x_j = \cos\frac{\pi(j-\frac{1}{2})}{n}$ $(j = 1, 2, ..., n)$ and we find

$$\gamma_j = -2\cos\frac{\pi(j-\frac{1}{2})}{n},\ j = 1, 2, ..., n. \tag{3.3}$$

Finally, combining (3.1), (3.2), (3.3), we have the following complex factorization of the k-Lucas numbers:

$$L_{k,n} = \prod_{j=1}^{n}(k + i\gamma_j) = \prod_{j=1}^{n}(k - 2i\cos\frac{\pi(j-\frac{1}{2})}{n}),\ n \geq 1. \tag{3.4}$$

4. Another factorization of the k-Fibonacci numbers

Using Lemma 1.1, we can generalize the family of tridiagonal matrices to a subsequence of k-Fibonacci numbers which is a family of tridiagonal matrices whose successive determinants are given by that sequence. To do this, we give the following lemma.

Lemma 4.1. *For $n \geq 1$ we have $F_{k,m+n} = L_{k,n}F_{k,m} + (-1)^{n+1}F_{k,m-n}$.*

Proof. We use the principle of mathematical induction. Let n=1, then the lemma yields $F_{k,m+1}$=$L_{k,1}F_{k,m}$+$(-1)^2F_{k,m-1}$=$kF_{k,m}$+$F_{k,m-1}$, which defines the $k-$Fibonacci sequence.

We assume that $F_{k,m+n} = L_{k,n}F_{k,m} + (-1)^{n+1}F_{k,m-n}$ for $n \geq 1$. Then, we have $F_{k,m+n+1} = kF_{k,m+n} + F_{k,m+n-1} = k(L_{k,n}F_{k,m} + (-1)^{n+1}F_{k,m-n}) + (L_{k,n-1}F_{k,m} + (-1)^nF_{k,m-n+1}) = (kL_{k,n} + L_{k,n-1})F_{k,m} + (-1)^{n+2}(F_{k,m-n+1} - kF_{k,m-n}) = L_{k,n+1}F_{k,m} + (-1)^{n+2}F_{k,m-(n+1)}$. □

For $k = 1$, another proof of this equation can be found in [1] (see Lemma 1 on page 218).

Theorem 4.1. *Let $M_{k,\alpha\beta}(n)$ be the family of symmetric tridiagonal matrices $n = 1, 2, ...$ whose elements are satisfy following conditions:*

$$m_{1,1} = F_{k,\alpha+\beta}, m_{2,2}=\left\lceil\frac{F_{k,2\alpha+\beta}}{F_{k,\alpha+\beta}}\right\rceil, m_{1,2}=m_{2,1}=\sqrt{m_{2,2}F_{k,\alpha+\beta}-F_{k,2\alpha+\beta}},$$

$$m_{j,j+1} = m_{j+1,j} = \sqrt{(-1)^\alpha},\ 2 \leq j \leq 3,\ \ m_{j,j} = L_{k,\alpha,\beta},\ 3 \leq j \leq k,$$

with $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{N}$. The successive determinants of this family of matrices is $|M_{k,\alpha,\beta}(n)| = F_{k,\alpha n+\beta}$.

Proof. We use the principle of mathematical induction. We have

$$|M_{k,\alpha,\beta}(1)| = \det F_{k,\alpha+\beta} = F_{k,\alpha+\beta},$$

$$|M_{k,\alpha,\beta}(2)| = \det\begin{pmatrix} F_{k,\alpha+\beta} & \sqrt{m_{2,2F_{k,\alpha+\beta}-F_{k,2\alpha+\beta}}} \\ \sqrt{m_{2,2F_{k,\alpha+\beta}-F_{k,2\alpha+\beta}}} & \left\lceil\frac{F_{k,2\alpha+\beta}}{F_{k,\alpha+\beta}}\right\rceil \end{pmatrix} = F_{k,2\alpha+\beta}.$$

Now we assume that $|M_{k,\alpha,\beta}(n)| = F_{k,\alpha n+\beta}$ for $1 \leq k \leq n$. Then by Lemma 1.1 and Lemma 4.1 we have $M_{k,\alpha,\beta}(n+1) = m_{n,n}\,|M_{k,\alpha,\beta}(n)| - m_{n,n-1}m_{n-1,n}\,|M_{k,\alpha,\beta}(n-1)| = L_{k,\alpha}\,|M_{k,\alpha,\beta}(n)| - (-1)^{\alpha}\,|M_{k,\alpha,\beta}(n-1)| = L_{k,\alpha}F_{k,\alpha n+\beta} + (-1)^{\alpha+1}F_{k,\alpha(n+1)+\beta} = F_{k,\alpha+\alpha n+\beta} = F_{k,\alpha(n+1)+\beta}$. □

Due to Theorem 4.1, we establish a family of tridiagonal matrices whose successive determinants generate any subsequence of the k-Fibonacci numbers. For example,

$$F_{k,4n-2}=\begin{pmatrix} k & 0 & 0 & & & \\ 0 & k^4+4k^2+3 & 1 & & & \\ 0 & 1 & k^4+4k^2+2 & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ & & & & 1 & k^4+4k^2+2 \end{pmatrix}.$$

Now, we give another factorization of the k-Fibonacci numbers. We consider the symmetric tridiagonal matrices:

$$B_m(n) = \begin{pmatrix} L_{k,2m}F_{k,2m} & \sqrt{F_{k,2m}} & & & & \\ \sqrt{F_{k,2m}} & L_{k,2m} & 1 & & & \\ & 1 & L_{k,2m} & 1 & & \\ & & . & . & & \\ & & & . & . & 1 \\ & & & & 1 & L_{k,2m} \end{pmatrix}.$$

By Lemma 4.1, we get $F_{k,4m} = L_{k,2m}F_{k,2m}$, $\lceil F_{k,6m}/F_{k,4m} \rceil = L_{k,2m}$ and $\sqrt{\lceil F_{k,6m}/F_{k,4m} \rceil F_{k,4m} - F_{k,6m}} = \sqrt{F_{k,2m}}$. So we obtain $M_{k,2m,2m} = B_m(n)$. Therefore, by Theorem 4.1 we obtain $|B_m(n)| = F_{k,2m(n+1)}$.

Let e_j be the j^{th} column of nxn identity matrix I and $C_m(n) = (I + (\frac{1}{F_{k,2m}} - 1)e_1e_1^T)B_m(n)$. And so, we have $|B_m(n)| = F_{2m}\,|C_m(n)|$. We know that the determinant is the product of the eigenvalues. For this reason, let λ_j be an eigenvalue of $C_m(n)$ related to the eigenvector x_j for $j = 1, 2, ..., n$. We get $|C_m(n)| = \prod_{j=1}^{n} \lambda_j$. Letting $G_m(n) = C_m(n) - L_{k,2m}I$, we obtain that $G_m(n)x_j = (C_m(n) - L_{k,2m}I)x_j = C_m(n)x_j - L_{k,2m}Ix_j = (\lambda_j - L_{k,2m})x_j$. Then $\gamma_j = \lambda_j - L_{k,2m}$ is an eigenvalue of $G_m(n)$. Hence, γ is a root of the characteristic polynomial $|G_m(n) - \gamma I| = 0$. This polynomial is transformed Cheybshev polynomial of the second kind which has root $\gamma_j = -2\cos\frac{\pi j}{n+1}$. Finally, we found $F_{k,2m(n+1)} = |B_m(n)| = F_{k,2m}\,|C_m(n)| =$

$F_{k,2m}\prod_{j=1}^{n}\lambda_j$ and so

$$F_{k,2m(n+1)} = F_{k,2m}\prod_{j=1}^{n}\left(L_{k,2m} - 2\cos\frac{\pi j}{n+1}\right). \tag{4.1}$$

5. k-Lucas subsequences

Using Lemma 1.1, we can generalize the family of tridiagonal matrices to a subsequence of k-Lucas numbers which is a family of tridiagonal matrices whose successive determinants are given by that sequence. To do this, we give the following lemma.

Lemma 5.1. *We have $L_{k,m+n} = L_{k,n}L_{k,m} + (-1)^{n+1}L_{k,m-n}$ for $n \geq 1$.*

Proof. We use the principle of mathematical induction. Let $n = 1$, then the lemma yields $L_{k,m+1} = L_{k,1}L_{k,m} + (-1)^2 FL_{k,m-1} = kL_{k,m} + L_{k,m-1}$, which defines the k-Lucas sequence.

We assume that $L_{k,m+n}{=}L_{k,n}L_{k,m}{+}(-1)^{n+1}L_{k,m-n}$ for $n \geq 1$. Then, $L_{k,m+n+1}{=}kL_{k,m+n}{+}L_{k,m+n-1}{=}k(L_{k,n}L_{k,m}{+}(-1)^{n+1}L_{k,m-n}){+}(L_{k,n-1}L_{k,m}$ $+(-1)^n L_{k,m-n+1}) = (kL_{k,n}+L_{k,n-1})L_{k,m}+(-1)^{n+2}(L_{k,m-n+1}-L_{k,m-n}) =$ $L_{k,n+1}L_{k,m} + (-1)^{n+2}L_{k,m-(n+1)}$. □

Theorem 5.1. *Let $T_{k,\alpha\beta}(n)$, $n = 1, 2, \ldots$ be the family of symmetric tridiagonal matrices whose elements are satisfy the following conditions:*

$$\begin{aligned} t_{1,1} &= L_{k,\alpha+\beta},\ t_{2,2} = \left\lceil \frac{L_{k,2\alpha+\beta}}{L_{k,\alpha+\beta}} \right\rceil, \\ t_{1,2} &= t_{2,1} = \sqrt{t_{2,2}L_{k,\alpha+\beta} - L_{k,2\alpha+\beta}}, \\ t_{j,j+1} &= t_{j+1,j} = \sqrt{(-1)^{\alpha}},\ 2 \leq j \leq 3, \\ t_{j,j} &= L_{k,\alpha},\ 3 \leq j \leq k, \end{aligned}$$

with $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{N}$. The successive determinants of this family of matrices is $|T_{k,\alpha\beta}(n)| = L_{k,\alpha n+\beta}$.

Proof. We use the principle of mathematical induction and we have $|T_{k,\alpha,\beta}(1)| = \det L_{k,\alpha+\beta} = L_{k,\alpha+\beta}$ and

$$\begin{aligned} |T_{k,\alpha,\beta}(2)| &= \det\begin{pmatrix} L_{k,\alpha+\beta} & \sqrt{m_{2,2}L_{k,\alpha+\beta} - L_{k,2\alpha+\beta}} \\ \sqrt{m_{2,2}L_{k,\alpha+\beta} - L_{k,2\alpha+\beta}} & \left\lceil \frac{L_{k,2\alpha+\beta}}{L_{k,\alpha+\beta}} \right\rceil \end{pmatrix} \\ &= L_{k,2\alpha+\beta}. \end{aligned}$$

Now we assume that $|T_{\alpha,\beta}(n)|{=}L_{k,\alpha n+\beta}$ for $1{\leq}k{\leq}n$. Then by Lemma 1.1 and Lemma 5.1 we find $T_{k,\alpha,\beta}(n{+}1){=}t_{n,n}|T_{k,\alpha,\beta}(n)|{-}t_{n,n-1}t_{n-1,n}|T_{k,\alpha,\beta}(n{-}1)|$ $=L_{k,\alpha}|T_{k,\alpha,\beta}(n)|-(-1)^{\alpha}|T_{k,\alpha,\beta}(n-1)| = L_{k,\alpha}L_{k,\alpha n+\beta}+(-1)^{\alpha+1}L_{k,\alpha(n-1)+\beta}$ $= L_{k,\alpha+\alpha n+\beta} = L_{k,\alpha(n+1)+\beta}$. $\square$

Due to Theorem 5.1, we establish a family of tridiagonal matrices whose successive determinants generate any subsequence of the k-Lucas numbers. For example, we obtain

$$L_{k,4n-2} = \begin{pmatrix} k^2+2 & 0 & 0 & & & \\ 0 & k^4+4k^2+1 & 1 & & & \\ 0 & 1 & k^4+4k^2+2 & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ & & & & 1 & k^4+4k^2+2 \end{pmatrix}.$$

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