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An enhancement of smartphone selection process based on a new 3D Pythagorean fuzzy distance metric via TOPSIS technique

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Abstract

The idea of Pythagorean fuzzy distance metrics (PFDMs) has been used to discuss sundry selection problems. Existing PFDMs often fail to incorporate all three essential parameters of a Pythagorean fuzzy set (PFS), namely; membership degree (MD), non-membership degree (NMD), and hesitation degree (HD), thereby limiting their precision and effectiveness in real-world decision-making situations. To explore this gap, this research presents a novel three-dimensional (3D) weighted distance metric under the Pythagorean fuzzy framework, which integrates MD, NMD, and HD for a more comprehensive representation of imprecision. The proposed PFDM is theoretically validated and it is shown to fulfill the axioms of a distance function. It is then embedded into the technique for order of preference by similarity to ideal solution (TOPSIS) to enhance multi-criteria decision-making (MCDM), particularly in the context of smartphone selection. A comparative analysis against existing PFDMs demonstrates the superior precision and stability of the new approach. Furthermore, a sensitivity analysis of the novel 3D distance model confirms its robustness with respect to changes in criteria weights. This enhanced 3D distance metric provides a more reliable and interpretable tool for decision-makers in decision making fields.

Keywords

Pythagorean fuzzy distance metric, smartphone selection process, TOPSIS technique, decision-making, Pythagorean fuzzy sets

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1 Introduction

In recent years, the proliferation of sundry smartphone models with varying features has made the selection process more complex and uncertain for users.[1] Choosing the most suitable smartphone often involves multiple conflicting criteria such as cost, performance, durability, and brand reliability. These decision contexts are typically characterized with vagueness and imprecision that traditional decision-making models cannot adequately handle. The fuzzy set (FS) theory in[2] and its extensions; intuitionistic fuzzy sets (IFSs)[3] and the Pythagorean fuzzy sets (PFSs)[4] have been widely used to model such uncertainty. PFSs allow for a more flexible representation of these uncertainties by incorporating the three parametric values of PFSs; MD, NMD, and HD, satisfying the property that, the square addition of the MD and NMD is at most one.

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Owing to the flexibility and effectiveness of PFSs in handling complex decision-making problem, it has been used by scholars in sundry fields based on distance metrics to solve varying problems. For instance, a directional correlation coefficient measures for PFSs were applied enhance medical diagnosis and clustering analysis.[5] A Pythagorean fuzzy multi-objective optimization on the basis of ratio analysis (MULTIMOORA) scheme based on the dice and Jaccard distance was developed and applied in evaluating solid-state disk production and energy projects.[6] In,[7] an enhanced Pythagorean fuzzy distance metric was developed and applied in a renewable source energy selection, and two distance metrics were constructed for PFSs inspired by the Hellinger distance metric.[8] Though, several PFDMs have been developed, many existing distance metrics under PFSs framework fail to fully utilize all three parameters, particularly the hesitation component, which limits their descriptive and discriminative efficiency in multi-criteria decision-making (MCDM) applications. Furthermore, some distance models lack a framework for assigning importance weights to different criteria, thereby affecting the reliability of the decision outcomes. These limitations affect some critical decision-making scenarios, such as smartphone selection, medical diagnosis, and supplier evaluation.

1.1 Literature review

The FS theory[2] was able to handle varying decision-making problems, but with a limitation since it could only account for the MD. Due to this drawback, IFSs[3] were introduced, which included the NMD and the HM that exist in real-world decision-making problems, thereby providing an enhanced mathematical framework for handling uncertainties. Due to efficiency of the IFSs' concept, it has been efficient in handling varying decision-making problems in diverse fields; such as in medical diagnostics,[9] social network analysis and supply chain management.[10] Xu and Liao[11] emphasized handling uncertainty in high-stakes decisions with an intuitionistic fuzzy (IF) -TOPSIS approach. Zeng et al.[12] used IF-TOPSIS for investment decision-making, balancing subjective information with decision-makers' attitudes. Chen and Li[13] introduced an IF information fusion model for supplier selection, which has been critical for improving decision-making in uncertain environments. Chen et al.[14] integrated entropy with IF-TOPSIS to enhance precision in supplier selection under uncertain conditions. Kahraman and Kaya[15] proposed a risk assessment approach using IFSs, enhancing decision-making in uncertain industrial risk scenarios. Keshavarz and Amiri[16] discussed interval-valued IF methods for green supplier selection, promoting sustainability in supply chains. Numerous applications of IFSs have been discussed.[17–22]

To further enhance IFSs, Yager[4] presented PFSs, a new framework where the addition of the squares of NMD and MD is at most one ($MD^2 + NMD^2 \leq 1$). This expansion allowed for more flexibility in managing vagueness, leading to better decision-making outcomes, especially in situations with high levels of vagueness.[23] Peng et al.[24] stretched rough set theory to Pythagorean fuzzy (PF) contexts and applied it to classification problems, enhancing uncertainty handling in machine learning models. Peng and Dai[25] applied PFS to image processing tasks, specifically focusing on how PFS can improve information fusion techniques in images with noisy or incomplete data. This innovation has been critical in areas like satellite imagery analysis and biomedical imaging. Lin et al.[5] developed some novel directional correlation coefficients between PFSs by considering four parameters of the PFSs, namely; MD, NMD, direction of commitment and strength of commitment, with two practical applications in disease diagnosis and cluster analysis. Lin et al.[26] stretched the partitioned Bonferroni mean operator and partitioned geometric Bonferroni mean operator to the linguistic PFSs (LPFSs) and used them to develop a new operational law for LPFSs, which took into consideration the connections between the NMD and MD of two LPFSs. Lin et al.[27] developed a correlation coefficient for the relationship measurement between LPFSs with two entropy measures to compute the attribute weight data along with a LPF TOPSIS method to unravel multiple attribute decision-making problems in the selection of firewall productions. Due to the appropriateness of PFSs, many authors have discussed numerous practical problems with the concept.[28–34]

MCDM is the major decision-making approach that determines the finest alternative, taking into account many criteria in the selection process. In Liu et al.,[35] a novel heterogenous MCDM model was developed to expedite the blockchain platforms and assessment process, and a hybrid distance measure is proposed to decide the negative and positive ideal separation matrices of the evaluation criteria. MCDM has been a core area of applications for PFSs and it has various types like ELECTRE, OPA (ordinal priority approach), TOPSIS, WPM (weighted product model), and VIKOR (VIseKriterijumska Optimizacija I Kompromisno Resenje) amongst others. Of these types, the most commonly used, is the TOPSIS method because it is proficient in ranking alternatives. TOPSIS is used for ranking alternatives according to how geometrically far away they are from the optimal solution. The closer an alternative is to the ideal solution, the better it is considered.[36] Zhang and Xu[37] expanded the traditional TOPSIS method to incorporate PFSs, providing more flexibility for decision-making in uncertain environments. TOPSIS has been widely applied in areas like energy resource evaluation,[38,39] financial risk assessment,[40] and agricultural planning,[41] which assist farmers to optimize resource allocation by resource needs, considering crop yield, and environmental factors.

Garg[42] introduced aggregation methods for PFSs based on Einstein operations, which significantly improved decision-making in MCDM scenarios. Wei[43] explored various MCDM techniques within the PFS framework to address real-world problems such as risk evaluation and supplier selection, illustrating the advantage of PFS in tackling uncertainties typical in logistics and supply chain decisions. Additional discussion on MCDM approaches under PF domain were presented in.[44–48] In recent time, blockchain platform selection was carried out using with heterogeneous MCDM approach,[49] and the decision-making applications of PFSs and LPFSs were presented using correlation coefficient,[50] decision science via MCDM, distance metric, and operator of Bonferroni mean aggregation.[51]

The integration of PFSs with the TOPSIS along with recent advancements in 3D distance metrics, provides an improved framework for decision-making under fuzziness. The development of 3D distance metrics within the PFS framework is a significant advancement aimed at enhancing decision-making processes. These metrics simultaneously consider the MD, NMD, and HD, making them a notable contribution to PFS research. Liang and Xu[52] developed a hesitant PF distance measure by incorporating hesitant FSs into PFSs using TOPSIS method to analyze energy selection projects. Xu and Ren[53] introduced distance and similarity measures to compare PFSs, which are critical for decision-making applications in fields like recognition of pattern and medical diagnostics. Zeng et al.[54] created distance and similarity measures for PFSs by considering five parameters, stating that their method was more efficient than others.

Hussain and Yang[55] introduced distance measures for PFSs using the Hausdorff metric, generalizing Hamming and Euclidean distances, which were applied to MCDM applications. Similarly, Xiao and Ding[56] introduced a distance model for PFSs derived from the Jensen-Shannon divergence. They explored the theoretical foundations of their proposed measure and demonstrate its application in medical diagnosis. In,[57] a weighted PFDM was explored, and importance of incorporating weights was discussed in in recognition of pattern and disease diagnosis, demonstrating how the proposed method improves classification outcomes and decision-making efficiency. Mahanta and Panda[58] proposed a distance method for distinguishing uncertain PFSs, with applications in COVID-19 preventive measures and school selection in the private sector. In,[59] a 3D PF distance operator was presented which aimed at improving performance in disease diagnosis and recognition of pattern. Singh et al.[60] presented a generalized 3D metric under PFS, which proved decision-making accuracy in supply chain management. Garg and Jain[61] examined various decision-making approaches based on PFSs in the management of supply chain. The authors analyze different methods and their applications, highlighting the benefits of using fuzzy approaches to address uncertainty and imprecision in supply chain decisions.

Adak and Kumar[62] proposed a spherical distance measurement method under PFS to discuss the TOPSIS method, which is widely used for ranking options based on their closeness to an ideal solution. Liu et al.[63] demonstrated the utility of 3D PFDM for improving smartphone selection. Zhang et al.[64] explored the application PFDMs in group decision-making scenarios, emphasizing their effectiveness in collaborative environments. Ali and Hossain[65] highlighted the use of PFDM in supply chain management, demonstrating their practical implications. Li and Wang[66] examined the application of PFDM in medical decision-making scenarios, emphasizing their role in improving diagnostic accuracy. Furthermore, recent studies by Thakur et al.[67] and Kumar et al.[68] have explored spherical distance measures under PFSs, to enhance applications in medical diagnostics. Palvinder et al.[69] introduced another PFDM, enhancing the computational simplicity and reliability of decision models. Hussain et al.[1] introduced a PF distance measure and integrated them into the TOPSIS method, particularly for smartphone selection. Their study demonstrated the practical use of these metrics in consumer decision-making but also revealed gaps in the inclusion of HD and weights. Therefore, this research proposes a new weighted 3D distance metric under the PF setting with applications in smartphone selection using the TOPSIS technique to enhance decision-making reliability.

1.2 Motivation and contributions

Owing to the vital role PFDMs play in solving real-world decision-making problems in complex fields, this work is motivated by the need to overcome key gaps in existing PFDMs, which include:

i. Exclusion of the HD in most 2D frameworks, which leads to incomplete representation of uncertainty and error of exclusion.
ii. Lack of distance precision and reliability in terms of applications.
iii. Lack of weight sensitive distance metrics that can reflect the varying importance of the decision criteria.
iv. The inability of the distance metric to satisfy all the standard axioms of a distance metric.

To address these challenges, this study offers the following contributions:

i. Development of a new 3D weighted PFDM that incorporates the three parametric values of PFSs (MD, NMD and HD), along with a flexible mathematical framework for computing the weights.
ii. Theoretical validation of the proposed PFDM, showing it fulfills all the standard axioms of a distance function.
iii. Application of the proposed 3D PFDM to a practical case study of smartphone selection, using expert evaluation based on linguistic variables to simulate real-world decision-making via the TOPSIS technique.
iv. Comparative analysis showing the effectiveness of the novel 3D PFDM over the obtainable PFDMs based on the TOPSIS technique.

Besides the introductory section, Section 2 contains the preliminaries of the work by stating some of the existing PFDMs. Section 3 presents the new 3D PFDM by describing it numerical and theoretical validations, and comparative investigation. Section 4 dwells on the selection process of smartphone using the TOPSIS method. The section also presents the method's sensitivity analysis and the comparative examination of the new 3D PFDM with the existing PFDMs to express the edge of the novel method. Section 5 concludes the work, stating limitations of the extant PFDMs and future work's direction.

2 Preliminaries

Here, we first present some of the concepts that relates to PFSs and distance measures for PFSs. All through this article, let $\mathfrak{B} = \{b_1, b_2, \ldots, b_k\}$ be nonempty finite set, where k is the cardinality of $\mathfrak{B}$ and $PFS(\mathfrak{B})$ is the collection of the family of PFSs in $\mathfrak{B}$.

Definition 1 2. A fuzzy set $\mathfrak{Y}$ in $\mathfrak{B}$ is of the form:

$$\mathfrak{Y} = \{(b_i, \mathfrak{Y}_m(b_i))|b_i \in \mathfrak{B}\},$$

where $i = 1, 2, \ldots, k$ and $\mathfrak{Y}_m : \mathfrak{B} \to [0, 1]$ is the function representing the MD.

Definition 2 3. An IFS $\bar{\mathfrak{Y}}$ in $\mathfrak{B}$ is of the form:

$$\bar{\mathfrak{Y}} = \{(\mathfrak{b}_i,\ \bar{\mathfrak{Y}}_m(\mathfrak{b}_i),\ \bar{\mathfrak{Y}}_n(\mathfrak{b}_i)) \mid \mathfrak{b}_i \in \mathfrak{B}\},$$

where $i = 1, 2, \ldots, k$, the functions, $\bar{\mathfrak{Y}}_m, \bar{\mathfrak{Y}}_n : \mathfrak{B} \to [0, 1]$ are the MD and NMD respectively, and for each $\mathfrak{b}_i \in \mathfrak{B}$, $0 \le \bar{\mathfrak{Y}}_m(\mathfrak{b}_i) + \bar{\mathfrak{Y}}_n(\mathfrak{b}_i) \le 1$. It also has a hesitancy part, $\bar{\mathfrak{Y}}_h(\mathfrak{b}_i) = 1 - (\bar{\mathfrak{Y}}_m(\mathfrak{b}_i) + \bar{\mathfrak{Y}}_n(\mathfrak{b}_i))$ called the hesitation degree (HD) for every $\mathfrak{b}_i \in \mathfrak{B}$ for $\bar{\mathfrak{Y}}$ and $0 \le \bar{\mathfrak{Y}}_h(\mathfrak{b}_i) \le 1$.

Definition 3 4. A PFS $\hat{\mathfrak{Y}}$ in $\mathfrak{B}$ is described as follows:

$$\hat{\mathfrak{Y}} = \{(\mathfrak{b}_i,\ \hat{\mathfrak{Y}}_m(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}_n(\mathfrak{b}_i)) \mid \mathfrak{b}_i \in \mathfrak{B}\}$$

where $\hat{\mathfrak{Y}}_m, \hat{\mathfrak{Y}}_n : \mathfrak{B} \to [0, 1]$, such that $0 \le \hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i) + \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i) \le 1$ for all $\mathfrak{b}_i \in \mathfrak{B}$, where $\hat{\mathfrak{Y}}_m(\mathfrak{b}_i)$ is the MD and $\hat{\mathfrak{Y}}_n(\mathfrak{b}_i)$ is the NMD of $\mathfrak{b}_i \in \mathfrak{B}$. The HD denoted as $\hat{\mathfrak{Y}}_h(\mathfrak{b}_i)$ defined by

$$\hat{\mathfrak{Y}}_h(\mathfrak{b}_i) = (1 - (\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i) + \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i)))^{1/2}$$

is the degree of non-determinacy of $\mathfrak{b}_i \in \mathfrak{B}$ to $\hat{\mathfrak{Y}}$ and $\hat{\mathfrak{Y}}_h(\mathfrak{b}_i) \in [0, 1]$.

Definition 4 4. Let $\hat{\mathfrak{Y}}, \tilde{\mathfrak{Y}} \in PFS(\mathfrak{B})$, then the following are some set-operations on the PFSs:

i. complement; $\hat{\mathfrak{Y}}^c = \{(\mathfrak{b}_i,\ \hat{\mathfrak{Y}}_n(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}_m(\mathfrak{b}_i)) \mid \mathfrak{b}_i \in \mathfrak{B}\}$,
ii. intersection; $\tilde{\mathfrak{Y}} \cap \hat{\mathfrak{Y}} = \{(\mathfrak{b}_i, \min\{\tilde{\mathfrak{Y}}_m(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}_m(\mathfrak{b}_i)\}, \max\{\tilde{\mathfrak{Y}}_n(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}_n(\mathfrak{b}_i)\})|\mathfrak{b}_i \in \mathfrak{B}\}$,
iii. union; $\tilde{\mathfrak{Y}} \cup \hat{\mathfrak{Y}} = \{(\mathfrak{b}_i, \max\{\tilde{\mathfrak{Y}}_m(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}_m(\mathfrak{b}_i)\}, \min\{\tilde{\mathfrak{Y}}_n(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}_n(\mathfrak{b}_i)\})|\mathfrak{b}_i \in \mathfrak{B}\}$,
iv. inclusion relation; $\tilde{\mathfrak{Y}} \subseteq \hat{\mathfrak{Y}} \Leftrightarrow \tilde{\mathfrak{Y}}_m(\mathfrak{b}_i) \le \hat{\mathfrak{Y}}_m(\mathfrak{b}_i)$ and $\tilde{\mathfrak{Y}}_n(\mathfrak{b}_i) \ge \hat{\mathfrak{Y}}_n(\mathfrak{b}_i)$ for all $\mathfrak{b} \in \mathfrak{B}$,
v. equality; $\tilde{\mathfrak{Y}} = \hat{\mathfrak{Y}} \Leftrightarrow \tilde{\mathfrak{Y}}_m(\mathfrak{b}_i) = \hat{\mathfrak{Y}}_m(\mathfrak{b}_i)$ and $\tilde{\mathfrak{Y}}_n(\mathfrak{b}_i) = \hat{\mathfrak{Y}}_n(\mathfrak{b}_i)$ for all $\mathfrak{b} \in \mathfrak{B}$.

Next, we describe the PF distance metrics. The PF distance metrics are mathematical operators used to compute the distance between two PFSs, enabling the assessment of uncertainty in complex systems, thereby facilitating more accurate decision-making, pattern recognition and risk assessment in various fields. Zhang and Xu[37] defined the PF distance metric as follows:

Definition 5 37. Let d be a function, $d : PFS(\mathfrak{V}) \times PFS(\mathfrak{V}) \to [0,1]$, and let $\tilde{\mathfrak{Y}}$, $\hat{\mathfrak{Y}}$ and $\acute{\mathfrak{Y}}$ be three PFSs in $\mathfrak{V}$, then $d(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}})$ is called a distance metric between $\tilde{\mathfrak{Y}}$ and $\hat{\mathfrak{Y}}$ if it fulfills the following axioms:

i. $0 \le d(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) \le 1$,
ii. $d(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = 0$ if and only if $\tilde{\mathfrak{Y}} = \hat{\mathfrak{Y}}$,
iii. $d(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = d(\hat{\mathfrak{Y}}, \tilde{\mathfrak{Y}})$,
iv. $d(\tilde{\mathfrak{Y}}, \tilde{\mathfrak{Y}}) = 1$, if and only if $\tilde{\mathfrak{Y}}$ is a crisp set,
v. If $\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}, \acute{\mathfrak{Y}} \in PFS(\mathfrak{V})$ and $\tilde{\mathfrak{Y}} \subseteq \hat{\mathfrak{Y}} \subseteq \acute{\mathfrak{Y}}$, then $d\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \ge d(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}})$ and $d\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \ge d\left(\hat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$.

2.1 Existing methods of Pythagorean fuzzy distance metrics

In this subdivision, some extant methods of PF distance metrics are outlined. For any two arbitrary PFSs,

$$\hat{\mathfrak{Y}} = \{(\mathfrak{b}_i, \hat{\mathfrak{Y}}_m(\mathfrak{b}_i), \hat{\mathfrak{Y}}_n(\mathfrak{b}_i)) \mid \mathfrak{b}_i \in \mathfrak{V}\} \quad and \quad \tilde{\mathfrak{Y}} = \{(\mathfrak{b}_i, \tilde{\mathfrak{Y}}_m(\mathfrak{b}_i), \tilde{\mathfrak{Y}}_n(\mathfrak{b}_i)) \mid \mathfrak{b}_i \in \mathfrak{V}\}$$

in $\mathfrak{V}$. Then, some existing distance functions for PFSs are presented as enumerated:

Zhang and Xu[37] developed the first distance metric between PFS as presented in (1):

$$d_{ZX}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{1}{2}\sum_{i=1}^{k}\left[|\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)| + |\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)| + |\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)|\right]. \tag{1}$$

In (1), the cardinality of $\mathfrak{V}$ is not included, which may affect the result of the computation.

Hussain and Yang[55] created a distance metric based on the Hausdorff metric by a generalization of existing distance scheme as presented in (2):

$$d_{HY}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{1}{k}\sum_{i=1}^{k}(max\{|\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)|, |\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)|\}). \tag{2}$$

In (2), the HD is not built-in the computation which may result to error of exclusion.

Xiao and Ding[56] constructed a divergence PFDM using the Jensen-Shannon divergence without including the parametric difference as presented by (3).

$$d_{XD}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \sqrt{\frac{1}{2k}\sum_{i=1}^{k}\left((\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i))^2 + (\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i))^2 + (\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i))^2\right)}. \tag{3}$$

In (3), the number of the parametric differences is not included in the scheme.

Ejegwa[70] modified distance metric in[37] by including the cardinality of $\mathfrak{V}$ as shown in (4):

$$d_{E}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{1}{2k}\sum_{i=1}^{k}\left[|\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)| + |\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)| + |\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)|\right]. \tag{4}$$

In (4), the complete number of the parametric differences is not built-in the scheme.

Mahanta and Panda[58] introduced a distance metric between PFSs which is able to make a distinction between highly uncertain PFSs uniquely, as presented in (5):

$$d_{MP}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{\sum_{i=1}^{k}(|\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)| + |\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)|)}{k\left(\sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)) + \sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i))\right)}. \tag{5}$$

In (5), the HD is not built-in the computation which may result to error of omission.

Wu et al.[71] developed certain PF distance techniques with improved precision, and they are as presented in (6) and (7):

$$d_{W_1}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{1}{k}\sum_{i=1}^{k}\left(Avg\left\{\begin{array}{c}|\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)|, |\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)|,\\ |\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)|\end{array}\right\}\right), \tag{6}$$

where *Avg* stands for Average.

$$d_{W_2}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{1}{2k^2}\sum_{i=1}^{k}\left(|\tilde{\mathfrak{Y}}_\mathrm{m}^2(\mathfrak{b}_\mathrm{i}) - \hat{\mathfrak{Y}}_\mathrm{m}^2(\mathfrak{b}_\mathrm{i})| + |\tilde{\mathfrak{Y}}_\mathrm{n}^2(\mathfrak{b}_\mathrm{i}) - \hat{\mathfrak{Y}}_\mathrm{n}^2(\mathfrak{b}_\mathrm{i})| + |\tilde{\mathfrak{Y}}_\mathrm{h}^2(\mathfrak{b}_\mathrm{i}) - \hat{\mathfrak{Y}}_\mathrm{h}^2(\mathfrak{b}_\mathrm{i})|\right). \tag{7}$$

In (6) and (7), the complete number of the parametric differences are not built-in the computation.

Ejegwa and Onyeke[57] presented an improved distance scheme as presented in (8).

$$d_{EO}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{1}{3k}\sum_{i=1}^{k}\left[|\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)| + |\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)| + |\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)|\right]. \tag{8}$$

The technique in (8), has no mathematical limitation except in the aspect of precision.

Hussain et al.[1] presented a distinct distance scheme based on 2D as presented in (9):

$$d_{He}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{1}{k}\sum_{j=1}^{k}\sqrt{\left(\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)}{2} - \frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)}{2}\right)^2 + \frac{1}{3}\left(\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)}{2} - \frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)}{2}\right)^2}. \tag{9}$$

In (9), the HD is not built-in the computation which may perhaps lead to inaccuracy of result.

Thakur et al.[67] created a 2D distance metric based on Euler's number to enhance accuracy as presented in (10):

$$d_{Te}(\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}) = \frac{1}{k}\sum_{j=1}^{k}\left[\left(\sqrt{e} - 1\right)|\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)| + \left(\sqrt{e} - 1\right)|\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)|\right], \tag{10}$$

where e is a mathematical constant in the region of 2.7182. On the contrary, (10) does not built-in the HD which may perhaps lead to inaccuracy of result.

3 New Pythagorean fuzzy distance method

In this section, the new method of PF distance metric is presented with illustrative examples and a comparative exploration to show the efficiency of the new distance scheme. Sundry PF distance metrics have been constructed by some authors as previously presented, each with varying applications in decision-making and pattern recognition amongst other relevant fields of human endeavors. It is worthy to note that, majority of the distance schemes excluded the HD and only a fraction of the schemes satisfy the distance metric's axioms. Recently, Hussain et al.[1] constructed a PFDM without minding the significance of the HD and the relevance of feature space's weights. The exclusion of the hesitation compartment from its computation may possibly affect the outcome of distance outcomes. To correct this setback, the distance scheme in[1] is modified based on the 3D approach, which integrates the HD to MD and NMD to boost the trustworthiness of the computed outputs as well as the truthfulness of the decisions made from the computed distance values.

Definition 6. Given two PFSs $\tilde{\mathfrak{Y}}$ and $\hat{\mathfrak{Y}}$ in $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_n\}$, where

$$\tilde{\mathfrak{Y}} = \{(\mathfrak{b}_i, \tilde{\mathfrak{Y}}_m(\mathfrak{b}_i), \tilde{\mathfrak{Y}}_n(\mathfrak{b}_i)) \mid \mathfrak{b}_i \in \mathfrak{B}\} \quad and \quad \hat{\mathfrak{Y}} = \{(\mathfrak{b}_i, \hat{\mathfrak{Y}}_m(\mathfrak{b}_i), \hat{\mathfrak{Y}}_n(\mathfrak{b}_i)) \mid \mathfrak{b}_i \in \mathfrak{B}\}$$

with the hesitations,

$$\tilde{\mathfrak{Y}}_h(\mathfrak{b}_i) = (1 - (\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)))^{1/2} \quad and \quad \hat{\mathfrak{Y}}_h(\mathfrak{b}_i) = (1 - (\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)))^{1/2}$$

respectively. Then, the propose distance metric between $\tilde{\mathfrak{Y}}$ and $\hat{\mathfrak{Y}}$ is presented in (11).

$$d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = \frac{1}{k}\sum_{i=1}^{k}\left(\begin{array}{c}\left(\frac{\tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}-\frac{\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i)\ -\hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i)\ -\hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}\right)^2\\ +\frac{1}{3}\left(\frac{\tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i)\ +\tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)\ +\tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}-\frac{\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i)\ +\hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i)\ +\hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}\right)^2\end{array}\right)^{1/2} \tag{11}$$

Because of the significance of the feature space-weight of the PFSs in distance metric as witness in different methods, the approach (11) is modified as (12):

$$d^w_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = \sum_{i=1}^{k} w_i\left(\begin{array}{c}\left(\frac{\tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}-\frac{\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i)\ -\hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i)\ -\hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}\right)^2\\ +\frac{1}{3}\left(\frac{\tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i)\ +\tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)\ +\tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}-\frac{\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i)\ +\hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i)\ +\hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}\right)^2\end{array}\right)^{\frac{1}{2}} \tag{12}$$

where

$$w_i = \frac{\frac{3\tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i)+3\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i)+\hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i)+\hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}}{\sum_{i=1}^{n}\left(\frac{3\tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i)+3\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i)+\hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i)+\hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i)}{3}\right)} \tag{13}$$

such that $w_i \in\ [0,1]$ and $\sum_{i=1}^{k} w_i = 1$.

3.1 Theoretical and numerical validations

Now, we show that the new PF distance metric fulfill all the axioms of a distance metric with the aid of theorems and validate its efficiency in distinguishing between closely related PFSs with the aid of a numerical example.

Theorem 1. Given three PFSs $\tilde{\mathfrak{Y}}$, $\hat{\mathfrak{Y}}$ and $\acute{\mathfrak{Y}}$ defined in $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_n\ \}$. Then, the new PF distance metric fulfills the following:

i. $0 \le d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})\ \le 1$,
ii. $d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})\ = 0$ if and only if $\tilde{\mathfrak{Y}} = \hat{\mathfrak{Y}}$,
iii. $d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = d_*(\hat{\mathfrak{Y}}, \tilde{\mathfrak{Y}})$,
iv. $d_*(\tilde{\mathfrak{Y}},\ \tilde{\mathfrak{Y}}) = 1$, if and only if $\tilde{\mathfrak{Y}}$ is a crisp set .
v. If $\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\ \in PFS(\mathfrak{b}_i)$ and $\tilde{\mathfrak{Y}} \subseteq \hat{\mathfrak{Y}} \subseteq \acute{\mathfrak{Y}}$, then $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \ge d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})$ and $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \ge d_*\left(\hat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$,
vi. $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \ge \min\left\{d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}), d_*\left(\hat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)\right\}$,
vii. $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \ge \max\left\{d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}), d_*\left(\hat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)\right\}$.

Proof. First, we prove (i), i.e., $0\ \le\ d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})\ \le\ 1$ for two PFSs $\tilde{\mathfrak{Y}}$ and $\hat{\mathfrak{Y}}$ defined in $\mathfrak{B}$. From Definition 3, we get $0 \le \tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i) \le 1$ and $0 \le \hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i) + \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i) \le 1$ such that, $\tilde{\mathfrak{Y}}_h(\mathfrak{b}_i)\ = (1 - (\tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)))^{1/2}$ and $\hat{\mathfrak{Y}}_h(\mathfrak{b}_i)\ = (1 - (\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i) + \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i)))^{1/2}$. Since $\tilde{\mathfrak{Y}}_m(\mathfrak{b}_i),\ \ \tilde{\mathfrak{Y}}_n(\mathfrak{b}_i), \tilde{\mathfrak{Y}}_h(\mathfrak{b}_i)\ \in\ [0,1]$ and $\hat{\mathfrak{Y}}_m(\mathfrak{b}_i),\ \ \hat{\mathfrak{Y}}_n(\mathfrak{b}_i), \hat{\mathfrak{Y}}_h(\mathfrak{b}_i)\ \in\ [0,1]$, the maximum possible difference between the corresponding components is 1. Hence from (11), we get:

$$\left(\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}-\frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}\right)^2 \geq 0 \quad and$$
$$\left(\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}-\frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}\right)^2 \geq 0.$$

Thus, $d_*\left(\hat{\mathfrak{Y}},\hat{\mathfrak{Y}}\right) \geq 0$, for all $i \in \{1,2,3,\ldots,k\}$. Since $d_*\left(\hat{\mathfrak{Y}},\hat{\mathfrak{Y}}\right) \geq 0$ has been proved, we need to show that $d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) \leq 1$.

From (11), we have

$$d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = \frac{1}{k}\sum_{i=1}^{k}\left(\left(\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}\right)^2 + \frac{1}{3}\left(\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)\ +\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)\ +\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}\right)^2\right)^{1/2}$$
$$= \frac{1}{k}\left[\frac{\sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i))^2}{9} + \frac{\sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i))^2}{27}\right]^{\frac{1}{2}}$$
$$= \frac{1}{9k}\left[\left(\sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i))-\sum_{i=1}^{k}(\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i))\right)^2 + \frac{\left(\sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i))-\sum_{i=1}^{k}(\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i))\right)^2}{3}\right]^{\frac{1}{2}}.$$

Hence,

$$d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = \frac{1}{9k}\left[\left(\sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i))-\sum_{i=1}^{k}(\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i))\right)^2 + \frac{\left(\sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i))-\sum_{i=1}^{k}(\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i))\right)^2}{3}\right]^{\frac{1}{2}}.$$

To enhance simplification, we assume that

$$\sum_{i=1}^{k}\left(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)\right) = \alpha,$$
$$\sum_{i=1}^{k}(\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)) = \beta,$$
$$\sum_{i=1}^{k}(\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)) = \bar{\alpha},$$
$$\sum_{i=1}^{k}(\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)) = \bar{\beta}.$$

Then,

$$d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = \frac{1}{9k}\left[(\alpha-\beta)^2 + \frac{(\bar{\alpha}-\bar{\beta})^2}{3}\right]^{\frac{1}{2}}$$

$$\Rightarrow d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})^2 = \frac{1}{81k^2}\left[(\alpha-\beta)^2 + \frac{(\bar{\alpha}-\bar{\beta})^2}{3}\right]$$

$$= \frac{1}{81k^2}\left[\frac{3(\alpha-\beta)^2 + (\bar{\alpha}-\bar{\beta})^2}{3}\right] = \frac{3(\alpha-\beta)^2 + (\bar{\alpha}-\bar{\beta})^2}{243k^2}.$$

By subtracting -1 from both sides we have,

$$d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})^2 - 1 = \frac{3(\alpha-\beta)^2 + (\bar{\alpha}-\bar{\beta})^2}{243k^2} - 1$$

$$= \frac{3(\alpha-\beta)^2 + (\bar{\alpha}-\bar{\beta})^2 - 243k^2}{243k^2}$$

$$= -\frac{(243k^2 - 3(\alpha-\beta)^2 - (\bar{\alpha}-\bar{\beta})^2)}{243k^2} \leq 0,$$

which means $d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})^2 - 1 \leq 0$, and so $d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})^2 \leq 1$. Thus, $d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) \leq 1$. Hence, the axiom (i) is proved.

Next, we prove (ii). Suppose $d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = 0$, we can get

$$\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} - \frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} = 0 \quad \textit{and}$$

$$\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} - \frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} = 0,$$

for all $i \in \{1, 2, 3, \ldots, n\}$.

Hence,

$$\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} = \frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} \textit{ and}$$

$$\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} = \frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}.$$

Thus,

$$\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) = \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i),$$

$$\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) = \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i),$$

which imply that

$$\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) = \hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i), \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) = \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) \textit{ and } \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) = \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i).$$

Hence, $\tilde{\mathfrak{Y}} = \hat{\mathfrak{Y}}$.

Conversely, assume $\tilde{\mathfrak{Y}} = \hat{\mathfrak{Y}}$, then we obtain

$$\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} - \frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) - \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} = 0 \quad \textit{and}$$

$$\frac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) + \tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} - \frac{\hat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) + \hat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3} = 0,$$

which imply that $d_*(\tilde{\mathfrak{Y}},\ \widehat{\mathfrak{Y}})\ = 0$. Hence (ii) is proved.

For the proof of (iii), we have

$$\begin{aligned}
d_*(\tilde{\mathfrak{Y}},\ \widehat{\mathfrak{Y}}) &= \frac{1}{k}\sum_{i=1}^{k}\left(\begin{array}{l}\left(\dfrac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)\ -\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)-\widehat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\widehat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\widehat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}\right)^2\\ \qquad +\dfrac{1}{3}\left(\dfrac{\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)\ +\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)\ +\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)-\widehat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\widehat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\widehat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}\right)^2\end{array}\right)^{1/2}\\
&= \frac{1}{k}\sum_{i=1}^{k}\left(\begin{array}{l}\left(\dfrac{\widehat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\widehat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\widehat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)-\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}\right)^2\\ \qquad +\dfrac{1}{3}\left(\dfrac{\widehat{\mathfrak{Y}}_m^2(\mathfrak{b}_i)+\widehat{\mathfrak{Y}}_n^2(\mathfrak{b}_i)+\widehat{\mathfrak{Y}}_h^2(\mathfrak{b}_i)-\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i)-\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i)-\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i)}{3}\right)^2\end{array}\right)^{1/2}\\
&= d_*(\ \widehat{\mathfrak{Y}},\tilde{\mathfrak{Y}}).
\end{aligned}$$

Thus, property (iii) is satisfied.

For (iv), let $\tilde{\mathfrak{Y}}$ and $\widehat{\mathfrak{Y}}$ be two PFSs in $\mathfrak{V}$, if $\tilde{\mathfrak{Y}}$ is crisp then either $\tilde{\mathfrak{Y}} = 0$ or $\tilde{\mathfrak{Y}} = 1$, in both cases $d_*(\tilde{\mathfrak{Y}},\ \tilde{\mathfrak{Y}}^c\ ) = 1$. Thus, (iv) is proved.

Now we give the proof of (v). For the given condition $\tilde{\mathfrak{Y}} \subseteq \widehat{\mathfrak{Y}} \subseteq \acute{\mathfrak{Y}}$, we have the following inequalities:

$$\begin{aligned}
\tilde{\mathfrak{Y}}_m^2(\mathfrak{b}_i) \le \widehat{\mathfrak{Y}}_m^2(\mathfrak{b}_i) \le \acute{\mathfrak{Y}}_m^2(\mathfrak{b}_i),\\
\tilde{\mathfrak{Y}}_n^2(\mathfrak{b}_i) \ge \widehat{\mathfrak{Y}}_n^2(\mathfrak{b}_i) \ge \acute{\mathfrak{Y}}_n^2(\mathfrak{b}_i),\\
\tilde{\mathfrak{Y}}_h^2(\mathfrak{b}_i) \ge \widehat{\mathfrak{Y}}_h^2(\mathfrak{b}_i) \ge \acute{\mathfrak{Y}}_h^2(\mathfrak{b}_i).
\end{aligned}$$

From (11), we derive a function $\Delta(\ell, m, \mathfrak{y})$, which is defined as:

$$\Delta(\ell,\ m, \mathfrak{y}) = ((\ell - m - \mathfrak{y}) - (p - q - r))^2 + \frac{1}{3}((\ell + m + \mathfrak{y}) - (p + q + r))^2,$$

where the three variables ℓ, m, and $\mathfrak{y}$ represent the Pythagorean fuzzy parameters and ℓ, m, $\mathfrak{y}$, p, q, $r \in [0, 1]$.

Now, we find the partial derivatives of $\Delta(\ell, m, \mathfrak{y})$ with respect to ℓ, m, and $\mathfrak{y}$ are:

$$\begin{aligned}
\frac{\partial_\Delta}{\partial_\ell} &= 2((\ell - m - \mathfrak{y}) - (p - q - r)) + \frac{2}{3}((\ell + m + \mathfrak{y}) - (p + q + r))\\
&= 2(\ell - m - \mathfrak{y} - p + q + r) + \frac{2}{3}(\ell + m + \mathfrak{y} - p - q - r)\\
&= \frac{8}{3}\ell - \frac{4}{3}m - \frac{4}{3}\mathfrak{y} - \frac{8}{3}p + \frac{4}{3}q + \frac{4}{3}r\\
&= \frac{8}{3}(\ell - p) - \frac{4}{3}(m + \mathfrak{y} - q - r)\\
&= \frac{4}{3}(2(\ell - p) - (m + \mathfrak{y} - q - r)).\\
\frac{\partial_\Delta}{\partial_m} &= -2((\ell - m - \mathfrak{y}) - (p - q - r)) + \frac{2}{3}((\ell + m + \mathfrak{y}) - (p + q + r))\\
&= -2(\ell - m - \mathfrak{y} - p + q + r) + \frac{2}{3}(\ell + m + \mathfrak{y} - p - q - r)\\
&= -\frac{4}{3}\ell + \frac{8}{3}m + \frac{8}{3}\mathfrak{y} + \frac{4}{3}p - \frac{8}{3}q - \frac{8}{3}r
\end{aligned}$$

$$= -\frac{4}{3}(\ell - p) + \frac{8}{3}(m + \mathfrak{y} - q - r)$$
$$= \frac{4}{3}(2(m + \mathfrak{y} - q - r) - (\ell - p)).$$

Similarly, we have

$$\frac{\partial_\Delta}{\partial_\mathfrak{y}} = \frac{4}{3}(2(m + \mathfrak{y} - q - r) - (\ell - p)).$$

Thus,

$$\frac{\partial_\Delta}{\partial_\ell} = \frac{4}{3}(2(\ell - p) - (m + \mathfrak{y} - q - r)),$$
$$\frac{\partial_\Delta}{\partial_m} = \frac{4}{3}(2(m + \mathfrak{y} - q - r) - (\ell - p)),$$
$$\frac{\partial_\Delta}{\partial_\mathfrak{y}} = \frac{4}{3}(2(m + \mathfrak{y} - q - r) - (\ell - p)).$$

To conclude the verification, the following cases are considered:

(i) Suppose $0 \le p \le \ell \le 1$, $0 \le m \le q \le 1$, and $0 \le \mathfrak{y} \le r \le 1$. Then, we have

$$\frac{\partial_\Delta}{\partial_\ell} \ge 0, \quad \frac{\partial_\Delta}{\partial_m} \le 0 \quad and \quad \frac{\partial_\Delta}{\partial_\mathfrak{y}} \le 0.$$

Thus $\Delta(\ell, m, \mathfrak{y})$ is an increasing and decreasing function for ℓ, m, and, $\mathfrak{y}$, respectively. Let $p = \tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i)$, $q = \tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)$ and $r = \tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i)$, then $p = \tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i) \le \hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i) \le \acute{\mathfrak{Y}}^2_m(\mathfrak{b}_i)$, $q = \tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i) \ge \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i) \ge \acute{\mathfrak{Y}}^2_n(\mathfrak{b}_i)$ and $r = \tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i) \ge \hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i) \ge \acute{\mathfrak{Y}}^2_h(\mathfrak{b}_i)$ for $i = 1, 2, 3, \ldots, n$. From the monotonicity of $\Delta(\ell, m, \mathfrak{y})$, we see that $\Delta(\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i)) \le \left(\acute{\mathfrak{Y}}^2_m(\mathfrak{b}_i), \acute{\mathfrak{Y}}^2_n(\mathfrak{b}_i), \acute{\mathfrak{Y}}^2_h(\mathfrak{b}_i)\right)$ for all $i = 1, 2, 3, \ldots, n$. Using $p = \tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i)$, $q = \tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i)$ and $r = \tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i)$ for all $i = 1, 2, 3, \ldots, n$, the specified statements fulfilled:

$$d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = \frac{1}{2n}\sum_{i=1}^{n}\sqrt{\Delta(\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i))} \quad and$$
$$d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) = \frac{1}{2n}\sum_{i=1}^{n}\sqrt{\Delta\left(\acute{\mathfrak{Y}}^2_m(\mathfrak{b}_i), \acute{\mathfrak{Y}}^2_n(\mathfrak{b}_i), \acute{\mathfrak{Y}}^2_h(\mathfrak{b}_i)\right)}.$$

Hence, $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \ge d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})$.

(ii) Suppose $0 \le p \le \ell \le 1$, $0 \le q \le m \le 1$, and $0 \le \mathfrak{y} \le r \le 1$. Then, we have

$$\frac{\partial_\Delta}{\partial_\ell} \le 0, \quad \frac{\partial_\Delta}{\partial_m} \ge 0 \quad and \quad \frac{\partial_\Delta}{\partial_\mathfrak{y}} \ge 0.$$

So $\Delta(\ell, m, \mathfrak{y})$ is an increasing and decreasing function for variables ℓ, m, and $\mathfrak{y}$. If $p = \acute{\mathfrak{Y}}^2_m(\mathfrak{b}_i)$, $q = \acute{\mathfrak{Y}}^2_n(\mathfrak{b}_i)$, $r = \acute{\mathfrak{Y}}^2_h(\mathfrak{b}_i)$ then $p = \tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i) \le \hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i) \le \acute{\mathfrak{Y}}^2_m(\mathfrak{b}_i)$, $q = \tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i) \ge \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i) \ge \acute{\mathfrak{Y}}^2_n(\mathfrak{b}_i)$ and $q = \tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i) \ge \hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i) \ge \acute{\mathfrak{Y}}^2_h(\mathfrak{b}_i)$. From the monotonicity of $\Delta(\ell, m, \mathfrak{y})$, we have $\Delta(\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i), \hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i)) \le \Delta\left(\acute{\mathfrak{Y}}^2_m(\mathfrak{b}_i), \acute{\mathfrak{Y}}^2_n(\mathfrak{b}_i), \acute{\mathfrak{Y}}^2_h(\mathfrak{b}_i)\right)$ for all $i = 1, 2, 3, \ldots, n$. For

$p = \acute{\mathfrak{Y}}^2_m(\mathfrak{b}_i),\ q = \acute{\mathfrak{Y}}^2_n(\mathfrak{b}_i),\ r = \acute{\mathfrak{Y}}^2_h(\mathfrak{b}_i)$, for all $i = 1, 2, 3, \ldots, n$, the specified statements are fulfilled:

$$d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) = \frac{1}{2n}\sum_{i=1}^{n}\sqrt{\Delta(\tilde{\mathfrak{Y}}^2_m(\mathfrak{b}_i),\ \tilde{\mathfrak{Y}}^2_n(\mathfrak{b}_i),\ \tilde{\mathfrak{Y}}^2_h(\mathfrak{b}_i))} \text{ and}$$

$$d_*\left(\hat{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) = \frac{1}{2n}\sum_{i=1}^{n}\sqrt{\Delta(\hat{\mathfrak{Y}}^2_m(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}^2_n(\mathfrak{b}_i),\ \hat{\mathfrak{Y}}^2_h(\mathfrak{b}_i))}.$$

Thus, we obtain $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \geq d_*\left(\hat{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right)$.

From (I) and (II), we get, $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \geq d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})$ and $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \geq d_*\left(\hat{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right)$. Hence, (v) is fulfilled.

Since $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \geq d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}})$ and $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \geq d_*\left(\hat{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right)$, then $d_*\left(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}\right) \geq \min\left\{d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}), d_*\left(\hat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)\right\}$ and $d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) \geq \max\left\{d_*(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}), d_*\left(\hat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)\right\}$. Hence, (vi) and (vii) are established. □

Theorem 2. Given three PFSs $\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}$ and $\acute{\mathfrak{Y}}$ defined in $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_n\}$. Then, the new weighted PF distance metric satisfies the distance conditions.

Proof. The verification is akin to that of Theorem 1. □

Example 1. Given a set $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2\}$ and assume there are three similar PFSs $\tilde{\mathfrak{Y}}, \hat{\mathfrak{Y}}$, and $\acute{\mathfrak{Y}}$ of $\mathfrak{B}$ as:

$$\tilde{\mathfrak{Y}} = \{\langle\mathfrak{b}_1,\ 0.6,\ 0.5\rangle, \langle\mathfrak{b}_2,\ 0.7, 0.3\rangle\},$$
$$\hat{\mathfrak{Y}} = \{\langle\mathfrak{b}_1,\ 0.7,\ 0.5\rangle, \langle\mathfrak{b}_2,\ 0.6, 0.3\rangle\}$$
$$\acute{\mathfrak{Y}} = \{\langle\mathfrak{b}_1,\ 0.8,\ 0.3\rangle, \langle\mathfrak{b}_2,\ 0.6, 0.3\rangle\}.$$

We use the new method to determine which of the given PFSs are more similar, and get the following results:

$$d_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = 0.0613, \quad d_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) = 0.1029, \quad d_*\left(\hat{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) = 0.0500.$$

Similarly, for the case of feature space-weight, if $w = \{(w_1\ 0.5245), (w_2\ 0.4755)\}$, we get

$$d^w_*(\tilde{\mathfrak{Y}},\ \hat{\mathfrak{Y}}) = 0.0867, \quad d^w_*\left(\tilde{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) = 0.1478, \quad d^w_*\left(\hat{\mathfrak{Y}},\ \acute{\mathfrak{Y}}\right) = 0.0724.$$

From both cases, $\hat{\mathfrak{Y}}$ and $\acute{\mathfrak{Y}}$ are closer to each other because their distance is the shortest.

3.2 Numerical comparison of the schemes

Here, we perform a comparative analysis with the aid of numerical examples, to illustrate the efficiency and effectiveness of the new scheme over the already existing schemes.

Example 2. Given a finite set $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2\}$. Assume there are two PFSs $\tilde{\mathfrak{Y}}$ and $\hat{\mathfrak{Y}}$ in $\mathfrak{B}$ as:

$$\tilde{\mathfrak{Y}} = \left\{\langle\mathfrak{b}_1,\ 0.60,\ 0.65\rangle, \langle\mathfrak{b}_2,\ 0.54,\ 0.68\rangle\right\},$$
$$\hat{\mathfrak{Y}} = \left\{\langle\mathfrak{b}_1,\ 0.65,\ 0.54\rangle, \langle\mathfrak{b}_2,\ 0.73, 0.40\rangle\right\}.$$

Suppose a PFS $\hat{\mathfrak{Y}} = \left\{\langle\mathfrak{b}_1,\ 0.55,\ 0.56\rangle, \langle\mathfrak{b}_2,\ 0.70, 0.35\rangle\right\}$ is given which is very similar but not exactly as $\tilde{\mathfrak{Y}}$ and $\hat{\mathfrak{Y}}$, we use the new and the existing PF distance metrics to determine which of the given sets is more similar to $\acute{\mathfrak{Y}}$. The computed distance outputs between the PFSs are presented in Table 1, with a graphical representation of the distance outputs shown in Figure 1.

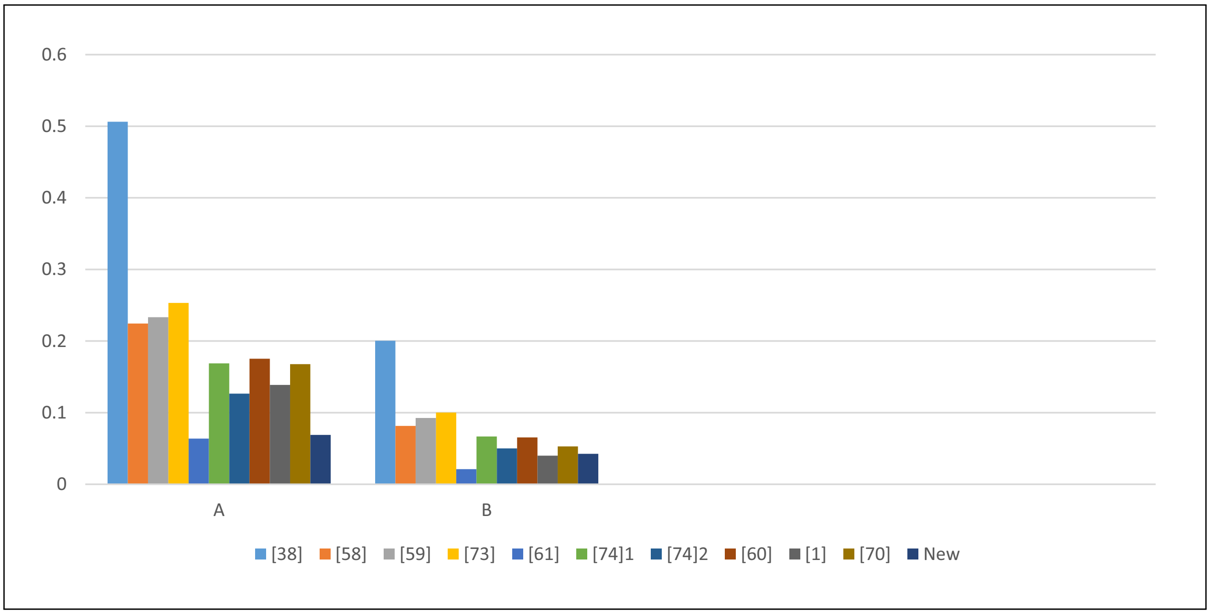

Figure 1. Graphic representation of distance values for Example 2.

Table 1. Distance outputs for Example 2.

Methods	$d\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$d\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	Ordering	Most Similar Pair
d_{ZX}[37]	0.5063	0.2004	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{HY}[55]	0.2244	0.0815	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{XD}[56]	0.2333	0.0925	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{E}[70]	0.2532	0.1002	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{MP}[58]	0.0637	0.0211	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{W1}[71]	0.1688	0.0668	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{W2}[71]	0.1266	0.0501	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{EO}[57]	0.1754	0.0654	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{He}[1]	0.1388	0.0400	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{Te}[67]	0.1677	0.0529	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$
d_{*}	0.0689	0.0425	$\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) \succ \left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$	$\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right)$

The results in Table 1 as shown in Figure 1 indicate that the PFSs $\widehat{\mathfrak{Y}}$ and $\acute{\mathfrak{Y}}$ are the closest. In addition, the new distance scheme yields the most accurate outcomes. It is worthy to note that, both the new and the existing methods in[1,37,57–60,69,70,71] give the same ordering. Although the distance scheme d_{MP}[58] seems to be accurate, its results cannot be reliable because the method excludes the hesitation parameter, which is vital in decision-making.

$$d\left(\tilde{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) = A, \quad d\left(\widehat{\mathfrak{Y}}, \acute{\mathfrak{Y}}\right) = B$$

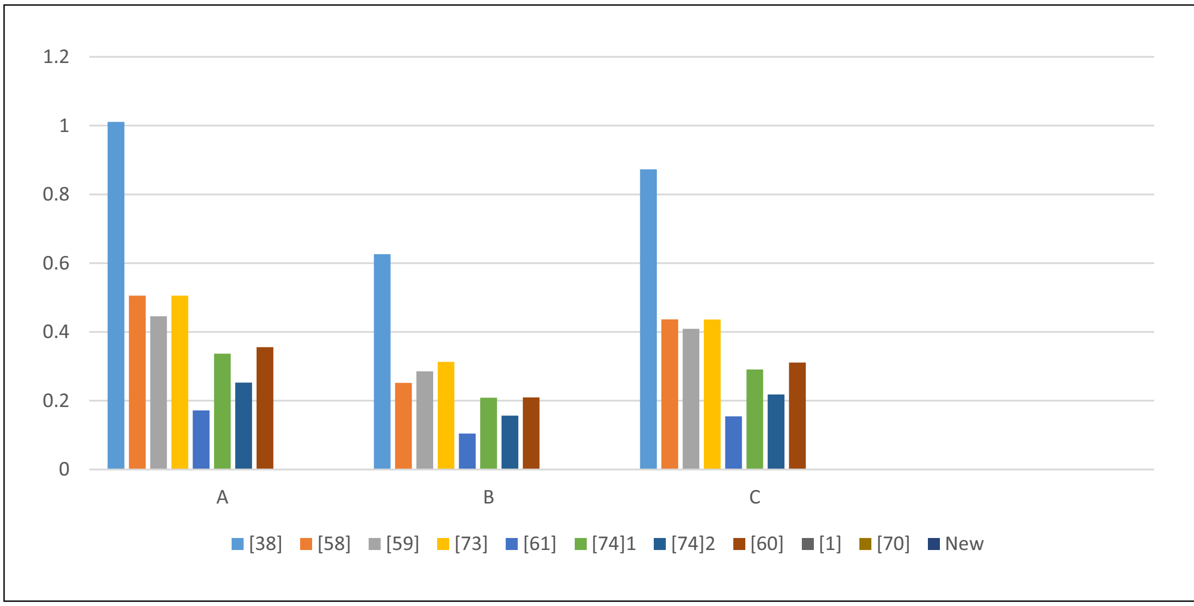

Figure 2. Graphic representation of distance values of Example 3.

Table 2. Distance outputs for Example 3.

Methods	$d\left(\tilde{\mathfrak{Y}}, \mathfrak{M}\right)$	$d\left(\widehat{\mathfrak{Y}}, \mathfrak{M}\right)$	$d\left(\acute{\mathfrak{Y}}, \mathfrak{M}\right)$	Ordering	Most Similar Pair
d_{ZX} [37]	1.0107	0.6264	0.8727	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{HY} [55]	0.5054	0.2518	0.4364	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{XD} [56]	0.4458	0.2856	0.4091	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{E} [70]	0.5054	0.3132	0.4363	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{MP} [58]	0.1717	0.1047	0.1547	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{W1} [71]	0.3369	0.2088	0.2909	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{W2} [71]	0.2527	0.1566	0.2182	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{EO} [57]	0.3558	0.2098	0.3109	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{He} [1]	0.2953	0.1721	0.2420	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{Te} [67]	0.3932	0.2147	0.3161	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$
d_{*}	0.2382	0.1412	0.2081	$(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ (\acute{\mathfrak{Y}}, \mathfrak{M}) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$	$(\widehat{\mathfrak{Y}}, \mathfrak{M})$

Example 3. For a finite set $\mathfrak{V} = \{\mathfrak{b}_1, \mathfrak{b}_2\}$, given three known models represented as PFSs $\tilde{\mathfrak{Y}}$, $\widehat{\mathfrak{Y}}$, and $\acute{\mathfrak{Y}}$ on $\mathfrak{V}$ as:

$$\tilde{\mathfrak{Y}} = \left\{\langle \mathfrak{b}_1,\ 0.45,\ 0.56\rangle, \langle \mathfrak{b}_2,\ 0.74, 0.30\rangle\right\}, \quad \widehat{\mathfrak{Y}} = \left\{\langle \mathfrak{b}_1,\ 0.54,\ 0.48\rangle, \langle \mathfrak{b}_2,\ 0.36, 0.50\rangle\right\},$$
$$\acute{\mathfrak{Y}} = \left\{\langle \mathfrak{b}_1,\ 0.45,\ 0.40\rangle, \langle \mathfrak{b}_2,\ 0.64, 0.35\rangle\right\}.$$

An unknown model to be classified is also given in terms of PFS represented by

$$\mathfrak{M} = \left\{\langle \mathfrak{b}_1,\ 0.84,\ 0.21\rangle, \langle \mathfrak{b}_2,\ 0.20, 0.68\rangle\right\}.$$

We classify the unknown model $\mathfrak{M}$ with the known models using both the new and the existing schemes for a comparative analysis. The computed outputs and classification are presented in Table 2 and Figure 2, respectively.

Table 2 and Figure 2 show that both the new and existing schemes give the same ordering, when the known models are classified with the unknown model. The ordering of the distance values is: $(\tilde{\mathfrak{Y}}, \mathfrak{M}) \succ \left(\acute{\mathfrak{Y}}, \mathfrak{M}\right) \succ (\widehat{\mathfrak{Y}}, \mathfrak{M})$, thereby classifying the unknown model $\mathfrak{M}$ with the known model $\widehat{\mathfrak{Y}}$. It is perceived that, the newly developed scheme gives the most reliable distance outputs as compared to the methods in.[1,37,55,56–58,69,70,71]

4 Application in smartphones selection

The African's smartphone markets have become increasingly saturated with various brands, models and features, making the selection process an overwhelming task for consumers. The fuzziness involved in this process arises from the numerous criteria that need to be considered, some of which include; price, durability, camera quality, battery life, and operating system, among others. Here, we apply the new PF distance metric to enhance the selection process via TOPSIS technique. MCDM is a structured approach for assessing and ranking multiple conflicting criteria in decision-making process. Among the various MCDM methods, the TOPSIS technique is widely used to rank alternatives in either ascending or descending order. It finds extensive application across fields such as engineering, business, and public policy to address complex decision problems. By allowing decision-makers to assess and compare diverse options based on specific criteria, TOPSIS facilitates the selection of the most suitable choice.

4.1 Case study

We considered seven (7) smartphone brands for selection; these smartphones are exemplified by a set given as: $P = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$. The smartphones are kept anonymous due to conflict of interest. Nine (9) criteria are considered for the selection process as presented in the set: $F = \{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$, where F_1 represents price, F_2 represents brand reputation, F_3 represents camera quality, F_4 represents battery life, F_5 represents storage capacity, F_6 represents display size/quality, F_7 represents processing speed, F_8 represents software features, and F_9 represents durability. The criteria are described as follows:

Price (F_1): This refers to the cost of the smartphone, which often influences purchasing decisions based on budget and perceived value for money. Budget phones are affordable with basic features, while premium phones offer advanced features at higher prices.

Brand reputation (F_2): This is the trustworthiness and popularity of the smartphone manufacturer, which can affect consumer confidence and post-purchase satisfaction. Brands with good reputations often have reliable customer support and consistent product quality.

Camera quality (F_3): This represents the capability of the phone's camera system, including resolution (measured in megapixels), sensor quality, and features like optical zoom, night mode, and AI enhancements, which are vital for users who prioritize photography.

Battery life (F_4): This refers to the phone's ability to last through a day or more of use, determined by battery capacity (measured in mAh) and software optimizations. Phones with larger batteries or fast-charging capabilities are highly preferred.

Storage capacity (F_5): This is the phone's internal storage space for apps, media, and files, usually ranging from 32 gigabyte to 1 terabyte. Expandable storage via Secure Digital (SD) cards is also a key consideration for users who need more space.

Display size/quality (F_6): This refers to the dimensions of the screen and its visual clarity, measured by resolution such as high definition (HD), quad HD (QHD), full HD (FHD), pixel density, and technologies like AMOLED or liquid crystal display (LCD). Larger, high-quality displays are ideal for gaming, streaming, and multitasking.

Processing speed (F_7): This determines the phone's performance, influenced by the random-access memory (RAM), central processing unit (CPU), and graphics processing unit (GPU). Faster processors enable smooth multitasking, gaming, and app usage, with popular chipsets including Qualcomm Snapdragon and MediaTek Dimensity.

The smartphone brands and their selection criteria are represented in Figure 3.

A knowledge-based system of data collection is employed where data is randomly collected from five smartphone maintenance experts via linguistic variables. The smartphone maintenance experts were selected because of their broad knowledge regarding different phone brands. The linguistic terms (LTs) and the PF numbers (PFNs) to be used by the smartphones' experts are presented in Table 3.

4.2 Algorithm for the TOPSIS technique

Step I: Obtain the opinions of the five smartphone maintenance experts based on the LVs in Table 4.

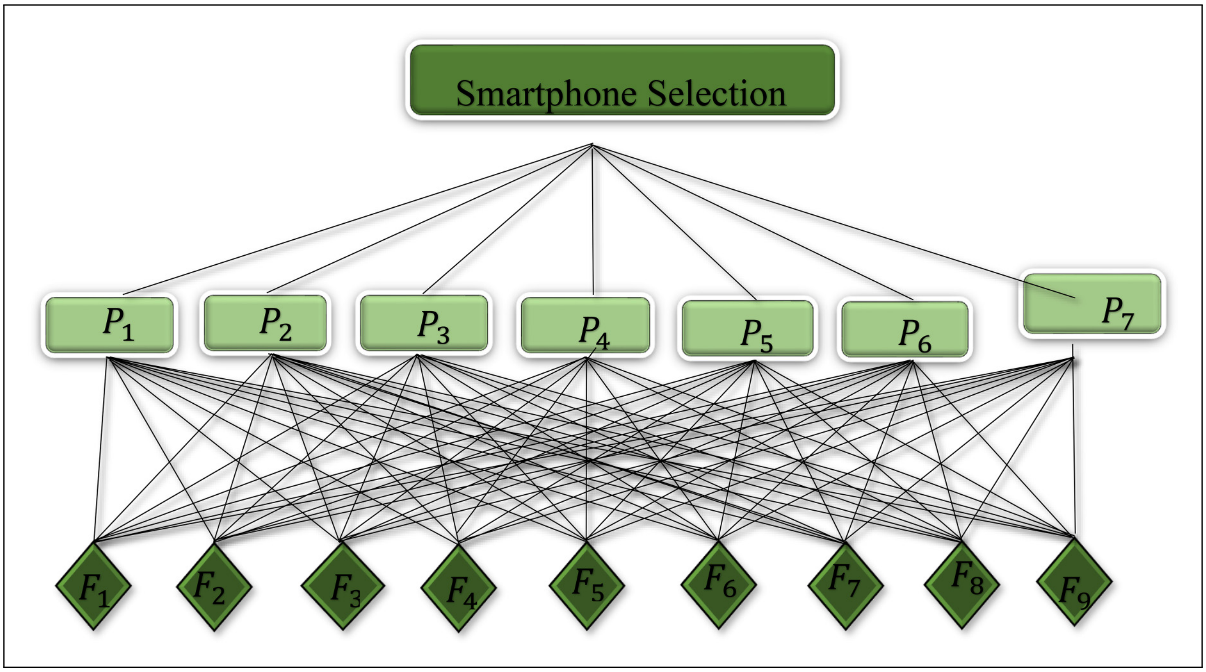

Figure 3. Selection involving the alternative smartphones and selection criteria.

Table 3. Smartphone selection linguistic variables.

Linguistic Terms	PFNs
Extremely Good (EG)	(1.00, 0.00)
Very Good (VG)	(0.91, 0.05)
Good (G)	(0.75, 0.20)
Average (A)	(0.55, 0.35)
Poor (P)	(0.35, 0.40)
Very Poor (VP)	(0.15, 0.60)
Extremely Poor (EP)	(0.00, 1.00)

Step II: Convert the LVs in Step I, to PFNs to obtain the PF decision matrices represented by $\mathcal{M}_K$, which is defined in (14).

$$\mathcal{M}_K = (P_j(F_i)\ )_{7\times 9} = \begin{array}{c c} & \begin{array}{ccc} F_1 & \dots & F_9 \end{array} \\ \begin{array}{c} P_1 \\ \vdots \\ P_7 \end{array} & \begin{pmatrix} (m,n)_{1,1} & \cdots & (m,n)_{1,9} \\ \vdots & \ddots & \vdots \\ (m,n)_{7,1} & \cdots & (m,n)_{7,9} \end{pmatrix} \end{array} \tag{14}$$

where P_j indicates the available phone brands ($j = 1,\ 2, \dots, 7$), F_i denotes criteria ($i = 1,\ 2, \dots, 9$), and k denotes the number of experts ($k = 1,\ 2,\ \dots,\ 5$).

Step III: Obtain the average PF decision matrix from the PF decision matrices in Step II using (15).

$$\tilde{\mathcal{M}}_K = \frac{(P_j(F_i)\ )_{7\times 9}}{K}. \tag{15}$$

Step IV: Identify the least F_i called the cost criterion (CC) and the rest of F_i are called the benefit criteria (BC). The CC is obtained by (16).

$$\wedge_{1\le j\le 7}\left\{\sum_{i=1}^{9} P_{j_m}(F_i)\right\} \tag{16}$$

Table 4. Linguistic variable from smartphone maintenance experts.

P/F	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
Expert I									
P_1	EG	EG	VG	VG	VG	EG	G	G	A
P_2	G	VG	VG	A	G	G	G	A	A
P_3	VG	A	A	VG	A	A	A	G	A
P_4	VG	EG	VG	EG	VG	VG	EG	G	G
P_5	A	G	VG	G	G	VG	G	G	G
P_6	A	VG	VG	G	G	G	G	A	A
P_7	A	VG	VG	EG	G	G	G	G	A
Expert II									
P_1	VG	EG	G	G	G	VG	VG	P	G
P_2	G	EG	G	G	G	EG	VG	G	G
P_3	VG	VG	G	G	G	VG	VG	G	VG
P_4	VG	VG	A	G	G	VG	EG	P	A
P_5	A	EG	VG	VG	VG	VG	VG	VG	VG
P_6	A	VG	VG	EG	VG	EG	VG	VG	VG
P_7	VG	EG	VG	VG	VG	VG	VG	A	EG
Expert III									
P_1	G	G	A	A	VG	G	G	G	G
P_2	G	G	VG	A	G	G	G	G	G
P_3	VG	A	G	VG	G	G	G	G	G
P_4	VG	G	A	G	VG	G	G	VG	G
P_5	A	VG	G	EG	G	VG	VG	G	EG
P_6	A	EG	VG	G	G	G	VG	G	EG
P_7	G	VG	A	G	G	A	G	G	VG
Expert IV									
P_1	VG	VG	VG	EG	VG	VG	VG	VG	VG
P_2	G	VG	G	VG	VG	VG	G	G	VG
P_3	VG	A	G	VG	A	G	G	G	G
P_4	VG	VG	VG	EG	VG	VG	VG	VG	EG
P_5	VG	EG	VG	VG	VG	VG	EG	VG	VG
P_6	VG	G	G	VG	G	G	G	VG	G
P_7	VG	VG	VG	EG	VG	EG	VG	VG	EG
Expert V									
P_1	VG	VG	G	VG	G	EG	VG	VG	VG
P_2	VG	EG	G	EG	G	VG	VG	G	EG
P_3	EG	VG	G	VG	G	VG	EG	G	VG
P_4	VG	VG	G	G	G	EG	G	G	G
P_5	A	EG	VG	G	VG	VG	G	VG	VG
P_6	A	VG	G	G	G	VG	A	G	G
P_7	A	VG	VG	G	VG	EG	VG	VG	EG

where is $\wedge$ a minimum operator.

Step V: Use the average PF decision matrix to obtain the normalized PF decision matrix $\check{\mathcal{M}}_K$ defined by (17).

$$\check{\mathcal{M}}_K = \begin{cases} (P_{j_m}(F_i), P_{j_n}(F_i)), & \textit{for BC of } P_j \\ (P_{j_n}(F_i), P_{j_m}(F_i)), & \textit{for CC of } P_j \end{cases}. \tag{17}$$

Step VI: Formulate the positive ideal solution (PIS) represented by $\check{\mathcal{M}}^+ = \{\check{\mathcal{M}}_1^+ \ldots, \check{\mathcal{M}}_k^+\}$ and the negative ideal solution (NIS) represented by $\check{\mathcal{M}}^- = \{\check{\mathcal{M}}_1^- \ldots, \check{\mathcal{M}}_k^-\}$ defined by (18).

$$\left.\begin{aligned} \check{\mathcal{M}}^+ &= \begin{cases} (\vee_{1\le j\le 7}\{P_{j_m}(F_i)\}, \wedge_{1\le j\le 7}\{P_{j_n}(F_i)\}), & \textit{if } F_i \textit{ is the BC} \\ (\wedge_{1\le j\le 7}\{P_{j_m}(F_i)\}, \vee_{1\le j\le 7}\{P_{j_n}(F_i)\}), & \textit{if } F_i \textit{ is the CC} \end{cases} \\ \check{\mathcal{M}}^- &= \begin{cases} (\wedge_{1\le j\le 7}\ \{P_{j_m}(F_i)\}, \vee_{1\le j\le 7}\ \{P_{j_n}(F_i)\}), & \textit{if } F_i \textit{ is the BC} \\ (\vee_{1\le j\le 7}\ \{P_{j_m}(F_i)\}, \vee_{1\le j\le 7}\ \{P_{j_n}(F_i)\}), & \textit{if } F_i \textit{ is the CC} \end{cases} \end{aligned}\right\} \tag{18}$$

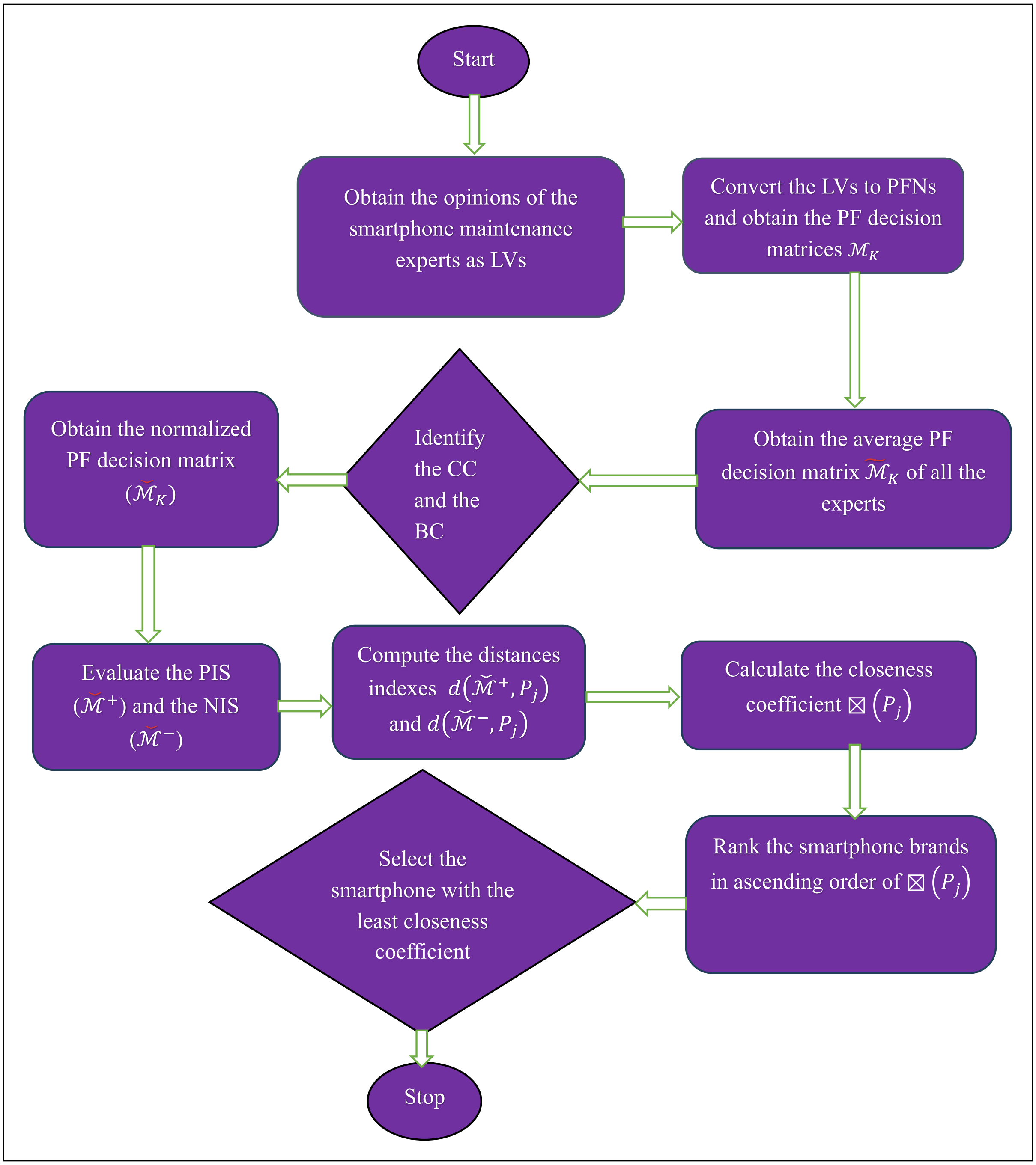

Figure 4. Flowchart for the TOPSIS technique.

where $\vee$ is a maximum operator.

Step VII: Compute the distances, $d_*\left(\check{\mathcal{M}}^+, P_j\right)$ and $d_*\left(\check{\mathcal{M}}^-, P_j\right)$ using the PF distance metrics.

Step VIII: Calculate the closeness coefficient $\boxtimes(P_j)$ for each alternative P_j using (19).

$$\boxtimes(P_j) = \frac{d_*\left(\check{\mathcal{M}}^+, P_j\right)}{d_*\left(\check{\mathcal{M}}^+, P_j\right) + d_*\left(\check{\mathcal{M}}^-, P_j\right)}. \tag{19}$$

Step IX: Rank the smartphone brands in ascending order of $\boxtimes(P_j)$ to ascertain the most suitable smartphone to buy.

Step X: Select the smartphone with the least closeness coefficient.

This algorithm is captured in Figure 4.

4.3 Algorithm implementation

The data for the implementation is collected from five smartphone maintenance experts in terms of LTs as presented in Table 4. To determine the most suitable smartphone brand in P to buy considering the criteria F, we utilize the TOPSIS algorithm. With Step II, the LVs from the five smartphone maintenance experts are converted to PFNs in Table 5. From Step III, the average PF decision matrix is presented in Table 6.

Evaluating Step IV, the least criterion is software features (F_8) which is called the CC, and the rest of the criteria are the BC. Using Steps V, we obtain the normalized PF decision matrix, which is obtainable in Table 7. By Step VI, we obtain the PIS and NIS, which is displayed in Table 8.

By Step VII, the distance indexes based on the new distance scheme using the information and Tables 6 and 8 are given as follows:

$$\begin{aligned}
&d_*\left(\check{\mathcal{M}}^+, P_1\right) = 0.0448, d_*\left(\check{\mathcal{M}}^+,\ P_2\right) = 0.0489, d_*\left(\check{\mathcal{M}}^+,\ P_3\right) = 0.0624, d_*\left(\check{\mathcal{M}}^+,\ P_4\right) = 0.0463,\\
&d_*\left(\check{\mathcal{M}}^+,\ P_5\right) = 0.0629, d_*\left(\check{\mathcal{M}}^+,\ P_6\right) = 0.0610, \mathit{and}\ d_*\left(\check{\mathcal{M}}^+,\ P_7\right) = 0.0510.\\
&d_*\left(\check{\mathcal{M}}^-,\ P_1\right) = 0.0619, d_*\left(\check{\mathcal{M}}^-,\ P_2\right) = 0.0501, d_*\left(\check{\mathcal{M}}^-,\ P_3\right) = 0.0538, d_*\left(\check{\mathcal{M}}^-,\ P_4\right) = 0.0622,\\
&d_*\left(\check{\mathcal{M}}^-,\ P_5\right) = 0.0698, d_*\left(\check{\mathcal{M}}^-,\ P_6\right) = 0.0524, \mathit{and}\ d_*\left(\check{\mathcal{M}}^-,\ P_7\right) = 0.0611.
\end{aligned}$$

Adopting step VIII by using the given distance indexes, the closeness coefficients are:

$$\begin{aligned}
&\boxtimes(P_1) = 0.4199, \boxtimes(P_2) = 0.4939, \boxtimes(P_3) = 0.5370, \boxtimes(P_4) = 0.4267,\\
&\boxtimes(P_5) = 0.4740, \boxtimes(P_6) = 0.5379 \quad \mathit{and} \quad \boxtimes(P_7) = 0.4550.
\end{aligned}$$

Using Steps IX and X, the ranking of the given closeness coefficients as: $P_1 \prec P_4 \prec P_7 \prec P_5 \prec P_2 \prec P_3 \prec P_6$. Using the ranking of the smartphones, the best brand of smartphone to select is P_1, which is the most suitable smartphone to buy, considering the criteria in F, via the TOPSIS technique. P_4 is also a good alternative as it possesses an equally low closeness coefficient and is ranked next after P_1.

4.4 Sensitivity analysis

The sensitivity analysis approach is adopted to estimate the effect of the criteria-weight on the distances indexes and the selection process based on the new distance metric. Using the information in Table 6, based on (13), the criteria-weight are computed and presented as follows:

$$w_i = \begin{Bmatrix} (w_1, 0.1068), (w_2, 0.1223), (w_3, 0.1076),\\ (w_4, 0.1140), (w_5, 0.1084), (w_6, 0.1175),\\ (w_7, 0.1126), (w_8, 0.1019), (w_9, 0.1088) \end{Bmatrix}.$$

Next, we compute the distances concerning the smartphone alternatives and NIS/PIS with the criteria-weight and without the criteria-weight and the results are presented in Table 9 and Figure 5.

The information in Table 9, Figure 5 shows the impact of weights on the exactness of the distance functions. From the information, we see that the distance function constructed without the weight-element gives a better distance value in comparison with the distance function which considered the weight-element. Based on this distance scheme, it is advisable to compute the distance without the criteria-weight to achieve a precise output.

We now compute the closeness coefficients and rankings of the smartphone alternatives with criteria-weight and without criteria-weight and the outcomes are presented in Table 10.

Table 10, which gives the sensitivity analysis closeness coefficients of the smartphones for new PFDM with weights d_*^w and without weights d_* shows that, the weighted version ranks the smartphones in the following order: $P_1 \prec P_7 \prec P_4 \prec P_5 \prec P_2 \prec P_3 \prec P_6$ and non-weighted version ranks the phone in the following order: $P_1 \prec P_4 \prec P_7 \prec P_5 \prec P_2 \prec P_3 \prec P_6$ both versions ranking smartphone P_1 as the best smartphone to buy and P_6 as the least option to consider. From the information, we see that the better closeness coefficients are achieved with the criteria-weight inclusion (i.e., it yields the smallest closeness coefficients by comparison). Thus, to attain accurate closeness coefficients, it is better to consider the weight-criteria in the distance scheme.

Table 5. PFNs from smartphone maintenance experts.

P/F	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
Expert I									
P_1	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$
P_2	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$
P_3	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$
P_4	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_5	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_6	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$
P_7	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$
Expert II									
P_1	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.35\\0.40\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_2	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_3	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_4	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.35\\0.40\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$
P_5	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_6	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_7	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$
Expert III									
P_1	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_2	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_3	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_4	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_5	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$
P_6	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$
P_7	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
Expert IV									
P_1	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_2	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_3	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$

(contined)

Table 5. Continued.

P/F	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
Expert IV									
P_4	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$
P_5	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_6	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_7	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
Expert V									
P_1	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_2	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_3	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_4	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_5	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$
P_6	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$
P_7	$\begin{pmatrix}0.55\\0.35\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.75\\0.20\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}0.91\\0.05\end{pmatrix}$	$\begin{pmatrix}1.00\\0.00\end{pmatrix}$

Table 6. Average PF decision matrix.

P/F	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
P_1	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.9140\\0.0600\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.8240\\0.1300\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.9140\\0.0600\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.7340\\0.1800\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$
P_2	$\begin{pmatrix}0.7820\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.9140\\0.0600\end{pmatrix}$	$\begin{pmatrix}0.8140\\0.1400\end{pmatrix}$	$\begin{pmatrix}0.7520\\0.1900\end{pmatrix}$	$\begin{pmatrix}0.7820\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.8640\\0.1000\end{pmatrix}$	$\begin{pmatrix}0.8140\\0.1400\end{pmatrix}$	$\begin{pmatrix}0.7100\\0.2300\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$
P_3	$\begin{pmatrix}0.9280\\0.0400\end{pmatrix}$	$\begin{pmatrix}0.6940\\0.2300\end{pmatrix}$	$\begin{pmatrix}0.7100\\0.2300\end{pmatrix}$	$\begin{pmatrix}0.8780\\0.0800\end{pmatrix}$	$\begin{pmatrix}0.6700\\0.2600\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.7920\\0.1600\end{pmatrix}$	$\begin{pmatrix}0.7500\\0.2000\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$
P_4	$\begin{pmatrix}0.9100\\0.0500\end{pmatrix}$	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.7340\\0.2000\end{pmatrix}$	$\begin{pmatrix}0.8500\\0.1200\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.8820\\0.0900\end{pmatrix}$	$\begin{pmatrix}0.7340\\0.1800\end{pmatrix}$	$\begin{pmatrix}0.7600\\0.1900\end{pmatrix}$
P_5	$\begin{pmatrix}0.6220\\0.2900\end{pmatrix}$	$\begin{pmatrix}0.9320\\0.0500\end{pmatrix}$	$\begin{pmatrix}0.8780\\0.0800\end{pmatrix}$	$\begin{pmatrix}0.8640\\0.1000\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.9100\\0.0500\end{pmatrix}$	$\begin{pmatrix}0.8640\\0.1000\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$
P_6	$\begin{pmatrix}0.6220\\0.2900\end{pmatrix}$	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8320\\0.1300\end{pmatrix}$	$\begin{pmatrix}0.7820\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.8320\\0.1300\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.7920\\0.1600\end{pmatrix}$
P_7	$\begin{pmatrix}0.7340\\0.2000\end{pmatrix}$	$\begin{pmatrix}0.9280\\0.0400\end{pmatrix}$	$\begin{pmatrix}0.8380\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8820\\0.0900\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8420\\0.1200\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.8740\\0.0900\end{pmatrix}$

4.5 Comparative analysis

In this segment, a comparative analysis of the distance methods modelled in the TOPSIS algorithm is considered to show the effectiveness and efficiency of the newly developed scheme. The comparative distance indexes between P_j and PIS/NIS in Tables 8 and 9 are captured in Table 11.

In addition, the comparative $\boxtimes(P_j)$ and the ranking are presented in Tables 12 and 13, respectively.

The information in Table 11 shows the accuracy and precision of the new distance scheme by comparison to the existing ones. Although the method in[58] shows extreme accuracy, it method is deficient because it is based on 2D. Table 12, which gives the closeness coefficient, and Table 13, which gives the ranking of the smartphones indicate that both the novel distance method and existing ones select smartphone brand P_1 as the most suitable smartphone to buy, considering the criteria in F.

Table 7. Normalized pf decision matrix.

P/F	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
P_1	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.9140\\0.0600\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.8240\\0.1300\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.9140\\0.0600\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.1800\\0.7340\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$
P_2	$\begin{pmatrix}0.7820\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.9140\\0.0600\end{pmatrix}$	$\begin{pmatrix}0.8140\\0.1400\end{pmatrix}$	$\begin{pmatrix}0.7520\\0.1900\end{pmatrix}$	$\begin{pmatrix}0.7820\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.8640\\0.1000\end{pmatrix}$	$\begin{pmatrix}0.8140\\0.1400\end{pmatrix}$	$\begin{pmatrix}0.2300\\0.7100\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$
P_3	$\begin{pmatrix}0.9280\\0.0400\end{pmatrix}$	$\begin{pmatrix}0.6940\\0.2300\end{pmatrix}$	$\begin{pmatrix}0.7100\\0.2300\end{pmatrix}$	$\begin{pmatrix}0.8780\\0.0800\end{pmatrix}$	$\begin{pmatrix}0.6700\\0.2600\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.7920\\0.1600\end{pmatrix}$	$\begin{pmatrix}0.2000\\0.7500\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$
P_4	$\begin{pmatrix}0.9100\\0.0500\end{pmatrix}$	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.7340\\0.2000\end{pmatrix}$	$\begin{pmatrix}0.8500\\0.1200\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.8820\\0.0900\end{pmatrix}$	$\begin{pmatrix}0.1800\\0.7340\end{pmatrix}$	$\begin{pmatrix}0.7600\\0.1900\end{pmatrix}$
P_5	$\begin{pmatrix}0.6220\\0.2900\end{pmatrix}$	$\begin{pmatrix}0.9320\\0.0500\end{pmatrix}$	$\begin{pmatrix}0.8780\\0.0800\end{pmatrix}$	$\begin{pmatrix}0.8640\\0.1000\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.9100\\0.0500\end{pmatrix}$	$\begin{pmatrix}0.8640\\0.1000\end{pmatrix}$	$\begin{pmatrix}0.1100\\0.8460\end{pmatrix}$	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$
P_6	$\begin{pmatrix}0.6220\\0.2900\end{pmatrix}$	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8320\\0.1300\end{pmatrix}$	$\begin{pmatrix}0.7820\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.8320\\0.1300\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$	$\begin{pmatrix}0.1700\\0.7740\end{pmatrix}$	$\begin{pmatrix}0.7920\\0.1600\end{pmatrix}$
P_7	$\begin{pmatrix}0.7340\\0.2000\end{pmatrix}$	$\begin{pmatrix}0.9280\\0.0400\end{pmatrix}$	$\begin{pmatrix}0.8380\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8820\\0.0900\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.8420\\0.1200\end{pmatrix}$	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.1700\\0.7740\end{pmatrix}$	$\begin{pmatrix}0.8740\\0.0900\end{pmatrix}$

Table 8. PIS and NIS.

Criteria	PIS	NIS
F_1	$\begin{pmatrix}0.9280\\0.0400\end{pmatrix}$	$\begin{pmatrix}0.6220\\0.2900\end{pmatrix}$
F_2	$\begin{pmatrix}0.9280\\0.0400\end{pmatrix}$	$\begin{pmatrix}0.6940\\0.2300\end{pmatrix}$
F_3	$\begin{pmatrix}0.8780\\0.0800\end{pmatrix}$	$\begin{pmatrix}0.7100\\0.2300\end{pmatrix}$
F_4	$\begin{pmatrix}0.8820\\0.0900\end{pmatrix}$	$\begin{pmatrix}0.7520\\0.1900\end{pmatrix}$
F_5	$\begin{pmatrix}0.8460\\0.1100\end{pmatrix}$	$\begin{pmatrix}0.6700\\0.2600\end{pmatrix}$
F_6	$\begin{pmatrix}0.9140\\0.0500\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$
F_7	$\begin{pmatrix}0.8820\\0.0900\end{pmatrix}$	$\begin{pmatrix}0.7740\\0.1700\end{pmatrix}$
F_8	$\begin{pmatrix}0.1100\\0.8460\end{pmatrix}$	$\begin{pmatrix}0.2300\\0.7100\end{pmatrix}$
F_9	$\begin{pmatrix}0.8960\\0.0700\end{pmatrix}$	$\begin{pmatrix}0.7600\\0.1900\end{pmatrix}$

Table 9. Distance indexes with and without criteria-weight.

Schemes	$(\check{\mathcal{M}}^+, P_1)$	$(\check{\mathcal{M}}^+, P_2)$	$(\check{\mathcal{M}}^+, P_3)$	$(\check{\mathcal{M}}^+, P_4)$	$(\check{\mathcal{M}}^+, P_5)$	$(\check{\mathcal{M}}^+, P_6)$	$(\check{\mathcal{M}}^+, P_7)$
d_*	0.0448	0.0489	0.0624	0.0463	0.0629	0.0610	0.0510
d_*^w	0.1299	0.1429	0.1856	0.1343	0.1822	0.1780	0.1479
	$(\check{\mathcal{M}}^-, P_1)$	$(\check{\mathcal{M}}^-, P_2)$	$(\check{\mathcal{M}}^-, P_3)$	$(\check{\mathcal{M}}^-, P_4)$	$(\check{\mathcal{M}}^-, P_5)$	$(\check{\mathcal{M}}^-, P_6)$	$(\check{\mathcal{M}}^-, P_7)$
d_*	0.0619	0.0501	0.0538	0.0622	0.0698	0.0524	0.0611
d_*^w	0.1840	0.1492	0.1570	0.1845	0.2069	0.1547	0.1817

Table 10. Sensitivity analysis closeness coefficients with and without criteria-weights.

Closeness coefficients	P_1	P_2	P_3	P_4	P_5	P_6	P_7	Best Smartphone
$\boxtimes(P_j)$	0.4199	0.4939	0.5370	0.4267	0.4740	0.5379	0.4550	P_1
$\boxtimes^{w_i}(P_j)$	0.4138	0.4892	0.5282	0.4531	0.4683	0.5350	0.4487	P_1

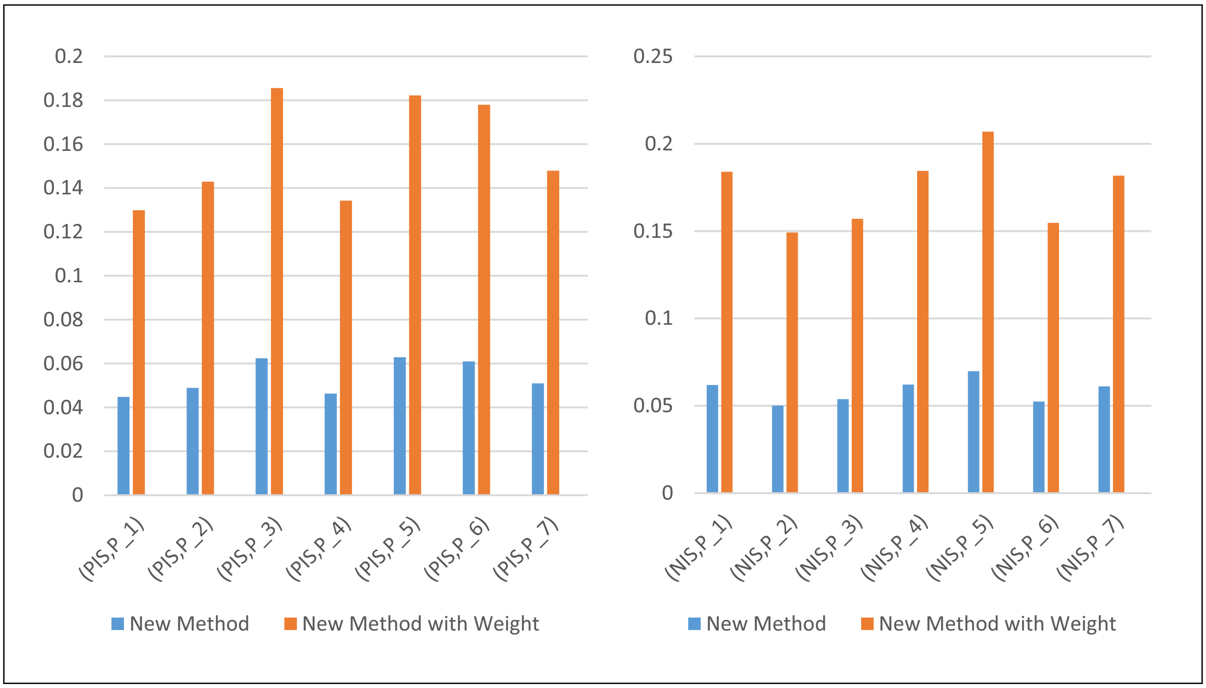

Figure 5. Distances of (PIS, P_j) and (NIS, P_j) with criteria-weight and without criteria-weight.

Table 11. Comparative distance indexes.

Methods	$(\check{\mathcal{M}}^+, P_1)$	$(\check{\mathcal{M}}^+, P_2)$	$(\check{\mathcal{M}}^+, P_3)$	$(\check{\mathcal{M}}^+, P_4)$	$(\check{\mathcal{M}}^+, P_5)$	$(\check{\mathcal{M}}^+, P_6)$	$(\check{\mathcal{M}}^+, P_7)$
d_{ZX}[37]	0.0187	0.0162	0.0843	0.0203	0.1073	0.1184	0.0717
d_{HY}[55]	0.1275	0.1778	0.2290	0.1293	0.1395	0.2069	0.1340
d_{XD}[56]	0.0207	0.0795	0.1189	0.0218	0.1463	0.1475	0.1017
d_E[70]	0.0094	0.0306	0.0422	0.0102	0.0536	0.0592	0.0358
d_{MP}[58]	0.0003	0.0012	0.0017	0.0004	0.0023	0.0025	0.0014
d_{W1}[71]	0.0967	0.1312	0.1621	0.0978	0.0931	0.1453	0.0967
d_{W2}[71]	0.0010	0.0034	0.0047	0.0011	0.0060	0.0066	0.0040
d_{EO}[57]	0.0173	0.0585	0.0835	0.0179	0.1025	0.1037	0.0722
d_{He}[1]	0.0704	0.0710	0.0813	0.0713	0.0851	0.0815	0.0750
d_{Te}[67]	0.1003	0.1266	0.1565	0.1018	0.1082	0.1439	0.1034
d_*	0.0448	0.0489	0.0624	0.0463	0.0629	0.0610	0.0510
	$(\check{\mathcal{M}}^-, P_1)$	$(\check{\mathcal{M}}^-, P_2)$	$(\check{\mathcal{M}}^-, P_3)$	$(\check{\mathcal{M}}^-, P_4)$	$(\check{\mathcal{M}}^-, P_5)$	$(\check{\mathcal{M}}^-, P_6)$	$(\check{\mathcal{M}}^-, P_7)$
d_{ZX}[37]	0.1710	0.1285	0.1054	0.1694	0.0860	0.0714	0.1181
d_{HY}[55]	0.2339	0.1759	0.1377	0.2321	0.2629	0.1679	0.2409
d_{XD}[56]	0.1688	0.1296	0.1463	0.1684	0.1215	0.1000	0.1272
d_E[70]	0.0855	0.0643	0.0527	0.0847	0.0430	0.0357	0.0590
d_{MP}[58]	0.0042	0.0034	0.0029	0.0042	0.0023	0.0020	0.0032
d_{W1}[71]	0.1559	0.1173	0.0918	0.1548	0.1753	0.1119	0.1606
d_{W2}[71]	0.0095	0.0071	0.0059	0.0094	0.0048	0.0040	0.0066
d_{EO}[57]	0.1160	0.0887	0.1025	0.1164	0.0847	0.0683	0.0892
d_{He}[1]	0.0714	0.0615	0.0652	0.0717	0.0795	0.0650	0.0715
d_{Te}[67]	0.1509	0.1188	0.0965	0.1494	0.1646	0.1140	0.1545
d_*	0.0619	0.0501	0.0538	0.0622	0.0698	0.0524	0.0611

Table 12. Comparative closeness coefficients.

Methods	⊠(P_1)	⊠(P_2)	⊠(P_3)	⊠(P_4)	⊠(P_5)	⊠(P_6)	⊠(P_7)
d_{ZX}[37]	0.0986	0.1120	0.4444	0.1070	0.5551	0.6238	0.3778
d_{HY}[55]	0.3528	0.5027	0.6245	0.3578	0.3467	0.5520	0.3574
d_{XD}[56]	0.1092	0.3802	0.4483	0.1146	0.5463	0.5960	0.4443
d_E[70]	0.0991	0.3224	0.4447	0.1075	0.5549	0.6238	0.3776
d_{MP}[58]	0.0667	0.2609	0.3696	0.0870	0.5000	0.5556	0.3043
d_{W1}[71]	0.3828	0.5280	0.6384	0.3871	0.3469	0.5649	0.3758
d_{W2}[71]	0.0952	0.3238	0.4434	0.1048	0.5556	0.6226	0.3774
d_{EO}[57]	0.1298	0.3974	0.4489	0.1333	0.5475	0.6029	0.4473
d_{He}[1]	0.4965	0.5358	0.5549	0.4986	0.5170	0.5563	0.5119
d_{Te}[67]	0.3999	0.5159	0.6185	0.4053	0.3966	0.5580	0.4009
d_*	0.4199	0.4939	0.5370	0.4267	0.4740	0.5379	0.4550

Table 13. Comparative ranking of smartphones.

Methods	Ordering	Decision
d_{ZX}[37]	$P_1 \prec P_4 \prec P_2 \prec P_7 \prec P_3 \prec P_5 \prec P_6$	P_1
d_{HY}[55]	$P_5 \prec P_1 \prec P_7 \prec P_4 \prec P_2 \prec P_3 \prec P_6$	P_5
d_{XD}[56]	$P_1 \prec P_4 \prec P_2 \prec P_7 \prec P_3 \prec P_5 \prec P_6$	P_1
d_E[70]	$P_1 \prec P_4 \prec P_2 \prec P_7 \prec P_3 \prec P_5 \prec P_6$	P_1
d_{MP}[58]	$P_1 \prec P_4 \prec P_2 \prec P_7 \prec P_3 \prec P_5 \prec P_6$	P_1
d_{W1}[71]	$P_5 \prec P_7 \prec P_1 \prec P_4 \prec P_2 \prec P_6 \prec P_3$	P_5
d_{W2}[71]	$P_1 \prec P_4 \prec P_2 \prec P_7 \prec P_3 \prec P_5 \prec P_6$	P_1
d_{EO}[57]	$P_1 \prec P_4 \prec P_2 \prec P_7 \prec P_3 \prec P_5 \prec P_6$	P_1
d_{He}[1]	$P_1 \prec P_4 \prec P_7 \prec P_5 \prec P_2 \prec P_3 \prec P_6$	P_1
d_{Te}[67]	$P_5 \prec P_1 \prec P_7 \prec P_4 \prec P_2 \prec P_5 \prec P_3$	P_5
d_*	$P_1 \prec P_4 \prec P_7 \prec P_5 \prec P_2 \prec P_3 \prec P_6$	P_1

Similarly, Tables 12 and 13 show that the new distance scheme along with the methods in[1,37,58–60,70,71] all select P_1 as the most suitable smartphone to buy, considering the criteria, F. We also observe that, the methods in[55,67] and d_{W1},[71] all select P_5 as the most suitable smartphone to buy, though it is worthy to note that, the methods in[55,67] did not consider the HD of PFSs.

4.6 Advantages of the new distance models

The limitations of the existing PFDMs are classified into four aspects:

(i) The distance method in[37] does not include the cardinality of the underlying set of the considered PFSs, which possibly will affect the result of the computation.
(ii) The distance methods in[1,55,58,67] disregards the effect of HD in its scheme, which may result to error of exclusion.
(iii) The methods in[56,70,71] discard the number of the complete parametric differences, which may perhaps result to error of omission.
(iv) The distance measuring approach in[57] might have no mathematical limitation, but lacks computational precision.

On the contrary, the new PF distance model surmounts the inadequacies stated above. To begin with, the new PF distance model includes the cardinality of the underlying set of the considered PFSs, which if omitted could possibly affect the result of the computation. In addition, the new PF distance model considers all the descriptive parameters of PFSs in its model unlike the distance models in.[1,55,58,67] This inclusion enhances reliability and precision. Furthermore, the new PF distance model takes into account the number of the complete parametric differences, whose exclusion may perhaps result to error. Finally, the new PF distance model possess high level of accuracy which promote the confidence of its interpretation.

5 Conclusion

This study introduced a novel 3D PFDM that incorporates MD, NMD and HD into a comprehensive mathematical framework to enhance the precision and reliability of the MCDM processes. The novel 3D PFDM has been comparatively demonstrated to be more reliable, yielding more precise distance outcomes. By embedding the proposed distance metric into the TOPSIS technique, the study effectively addressed a real-world decision-making problem of smartphone selection, demonstrating improved accuracy and robustness when compared to the existing PFDMs. The theoretical reliability of the new method, its satisfaction of the axioms of distance metric, and its practical application showcase its contribution to the domain of fuzzy decision-making. Beyond smartphone selection, the proposed model has strong potential for broader industrial applications, such as healthcare diagnostics, supply chain management, waste management, and smart manufacturing, where decision-making under uncertainty is crucial. By offering an interpretable and reliable solutions, the new PFDM will support organizations in achieving optimal choices when faced with conflicting criteria and vague information. Despite its strengths, the proposed method presents certain limitations. Currently, it is restricted to classical PFSs and does not accommodate more generalized or extended fuzzy models such as circular PFSs, spherical FSs and Fermatean fuzzy sets among others. To overcome these constraints, future research should focus on extending the 3D PFDM to accommodate other FS variants. Also, integrating the new method with other MCDM techniques such as VIKOR and ELECTRE with real-time data processing capabilities, will boost application potentials. Such enhancements aim to enhance the flexibility, adaptability, and computational effectiveness of the new distance scheme across complex and dynamic decision-making scenarios.

Ethical approval and informed consent statements

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This paper has no associated data. All the data used are contained in the paper.

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