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Cluster analysis selecting tools using quadri partitioned Pythagorean neutrosophic normal interval-valued set with an aggregation operators

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Abstract

The goal of a quadri partitioned Pythagorean neutrosophic normal interval-valued fuzzy set (QPPNNIVFS) is to provide the neutrosophic sets a more comprehensive mathematical foundation. QPPNNIVFS divides the indeterminacy component into unknown and contradiction classes. The several aggregating operations that have been understood thus far are discussed here. The fuzzy weighted QPPNNIVFW averaging (QPPNNIVFWA), QPPNNIVFW geometric (QPPNNIVFWG), generalized QPPNNIVFW averaging (GQPPNNIVFWA) and generalized QPPNNIVFW geometric (GQPPNNIVFWG) are considered as a novel concept. We show that algebraic structures like associative, distributive, idempotent, bounded, commutative, and monotonic characteristics are satisfied by QPPNNIVFSs. We illustrate the practical applications of increased Euclidean distance, Hamming distance, score, and accuracy values. Unless there is a mathematical justification for selecting one cluster technique over another, the clustering strategy must be selected empirically. An algorithm that performs well on one set of data will not perform well on another. There are several approaches of conducting cluster analysis. These include social network analysis, distribution-based, density-based, centroid-based and hierarchical. Therefore, it is clear that the natural number θ has a big impact on the models. To illustrate the comparison analysis, sensitivity analysis and the validity of our suggested methodologies are also conducted. The outcomes will be very helpful to decision makers in handling uncertain and conflicting data effectively.

Keywords: Cluster analysis, decision making, weighted averaging, weighted geometric, generalized weighted averaging, generalized weighted geometric.

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Abbreviations

The following abbreviations are used throughout the paper for clarity and consistency.

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DM	Decision making
MADM	Multiple attribute decision-making
AO	Aggregation operator
ED	ED
HD	HD
AO	Aggregating operations
MD	Membership degree
NMD	Non membership degree
TD	Truth degree
ID	Indeterminacy degree
FD	False degree
IND	indeterminacy degree
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
NS	Neutrosophic set
PFS	Pythagorean fuzzy set
PIVFS	Pythagorean interval-valued fuzzy set
QPPNNIVFS	Quadri partitioned Pythagorean neutrosophic normal interval-valued fuzzy set
QPPNNIVFW	Quadri partitioned Pythagorean neutrosophic normal interval valued fuzzy weighted
QPPNNIVFWA	QPPNNIVFW averaging
QPPNNIVFWG	QPPNNIVFW geometric
GQPPNNIVFWA	generalized QPPNNIVFWA
GQPPNNIVFWG	generalized QPPNNIVFWG

1. Introduction

Intelligence gathering, machine learning, knowledge compilation, and other methods are used on a daily basis to settle disputes. The problem of proving that strategic planning works is enormous. The application of appropriate DM processes is made easier by mathematical. Since the DM concept analyzes and ranks many points of view based on their attributes, it could be helpful to enterprises. We can effectively choose, classify, develop and evaluate our options. To get the best answer, MADM examines all potential characteristics and variables. In the past, it was generally accepted that features and weights needed to be expressed quantitatively. Examining a range of criteria and indicators is necessary for several assessments and DM difficulties. Solving assessment and DM problems can be aided by evaluating and gathering data for evaluation indicators. Given that MADM problems are frequent in real-world systems because of their complexity, the assessment information is unclear. An unsupervised machine learning technique that has demonstrated effectiveness in a wide range of issues is clustering [1, 16]. Clustering techniques have been suggested by several scholars in the subject of load profiling in recent years. One technique for analyzing data and retrieving information is a clustering algorithm. Non-standard data detection and discarding may be referred to as data processing. Based on how similar they are, the load statistics are grouped together. The recorded load data is truly described by the load profiles of the load data clusters. A smaller collection of common load curves or load profiles can be used to illustrate the volume of data. As a result, the fundamental tool for analyzing data from smart meters might be load profiling based on clustering. Because the power systems community is aware of this, there is a lot of research being done to evaluate methods for clustering load data [36]. Although there are several clustering algorithm implementations in the load profiling literature, no study offers a generic framework that leads to safe algorithm selection findings. Clustering methods and load survey studies are used to generate load data profiles. In addition to data on consumption, load surveys aim to collect information on customer preferences, occupancy patterns, weather, and other factors [14, 44]. The qualities of the eligible customer sample determine how accurate load surveys are [12, 49]. The results of load surveys on the qualifying set are scaled up to include the other customers using a bottom-up methodology.

In Boolean logic, the numerical values one and zero stand for the truth and falsity values, respectively.

However, Zadeh introduced fuzzy logic, which built upon the idea of Boolean logic. Uncertainty exists everywhere in most everyday problems. The FS, IFS, PFS, interval valued FS [41], and interval valued IFS [3] and neutrosophic set (NSS) [39] are proposed to address the uncertainties. Wang et al. addressed the use of complex intuitionistic fuzzy DOMBI prioritized AOs [45]. FS logic, which addresses DM problems with solely MD presented by Zadeh. Atanassov introduced the idea of an IFS logic, which is defined by the requirement that the total of its MD and NMD values be less than or equal to one. When DM, we may run into troubles when the total of the MD and NMD values is more than or equal to one. The novel idea of PFS logic, which is an extension of IFS and is defined by the square sum of its MD and NMD for which the value is less than or equal to one by Yager. Indeterminacy is explicitly measured in NS, while MD, IND, and FD are independent. In many applications, such information fusion, where data is merged from many sensors, this assumption is crucial. The interval spherical fuzzy environment [5] and CRITIC COPRAS technique [4] are two examples of the DM problems. The IFS and paraconsistent set are generalized to the NS by Smarandache [40]. Chatterjee et al. [9] examined a real-world scenario and demonstrated how naturally such circumstances arise. Additionally, they demonstrated the applications capacity by solving a DM problems related to pattern recognition. In recent years, image processing has been the primary application of NSs [13, 22]. Recently, Chakraborty et al. [8] talked about the practical use of MCDM under NS. The set theoretic operators were suggested by Wang et al. [43] on an instance of NS known as IVNS. Deli et al. [19] combined bipolar FS and NS to create a more comprehensive framework known as bipolar NSs (BNS). DM problems were solved using the BNS Hamacher averaging operator by Jamil et al. [23]. Uncertain information may be handled reliably with the use of BNSs. It is an extension of the BFS and NS, which are capable of handling both positive and negative information in real-world situations. Chatterjee et al. discussed the extension of the NS via CRADIS method [10] and TOPSIS approach [11].

IVNS was generalized via soft set in DM problems by Broumi et al. [7]. Soft sets were used by Deli [18] to define the IVNS. The program developed by Veerappan et al. [15] uses IVNS similarity measurements to diagnose mental illnesses. The geometric operators linked to the linguistic IVNS number were proposed by Fahmi et al. [2]. The definition of IVNS-based similarity measurements may be found in [6]. Another similarity measure based on NS employing soft sets was proposed by Deli et al. [20]. The idea of a correlation measure for Pythagorean NSs were covered by Jansi et al. [24]. Peng et al. [32] deals that the AO based on IVPFS. The concept of PFS is interacted with by Yager [46]. We discover a preference reducing or rising order is rated under relative nearness in the TOPSIS approach, which uses two distances consists of positive ideal solution (PIS) and negative ideal solution (NIS). Peng et al. [31] covered the notion of neutrosophic MADM based on MABAC and TOPSIS. Hwang et al. communicated different real life applications [42]. Zulqarnain et al. [50] introduced the notion of TOPSIS extends to IVIFSS. He also discussed the correlation coefficient of IVIFSS. Linear Diophantine FS (LDFS) based on reference parameters pesented by Riaz et al. [35]. Kannan et al. [25] addressed the notion of the LDFS using CODAS. The Pentagonal fuzzy DEMATEL approach is used by Gazi et al. [21] to determine the most crucial components for women's empowerment in the sports industry. Zhang et al. [48] investigated geometric AOs in group DM using PIVFS. Liu et al. [26] presented AO based q-rung image FS. Seikh et al. [27, 37] presented the q-rung orthopair FS and its extension with AO. Seikh et al. [38] dealt with the Fermatean fuzzy CRADIS approach. Yang [47] used a normal set to characterize the PIVFS. Radha et al. [33] were the first to suggest quadri partitioned neutrosophic Pythagorean sets (QPSVNSs). The QPSVNS were introduced by Debnath [17] and also developed a variety of Dombi weighted operators. Additionally, scoring and accuracy elements were included to create a DM problem. However, the extraordinarily high processing cost of such sets is a major barrier to their acceptance by the broader academic community. Although prior research has utilized some of these dimensions, there are few works that investigate the distance measurements, similarity measures, and entropy metrics for QPSVNS.

Current research gaps and the reasons behind our investigation are as follows. The extension to quadri-partitioned Pythagorean neutrosophic set (QPPNST), which introduces a fourth possibility for uncertainty representation is still understudied. Pythagorean neutrosophic set theory (PNSST) and PFST extend spherical FS theory (SFST) by adding a fourth possibility. By improving the representation of uncertain information, FSTs and its expansions, such as IFSTs, PFSTs, and PIVNSSTs, have significantly advanced the field of DM. They do, however, have a unique set of restrictions. For example, when faced with a statement, a decision-maker still found it difficult to give a specific value of membership for his feelings of agreement, disagreement, both agreement and disagreement, and neither agreement nor disagreement. With the aid of its four separate components falsity, ignorance, contradiction, and truth QPPNNIVFNs are naturally able to convey the greatest amount of uncertain information. Four-valued logic (T, F, Both and F, Neither nor F) is the foundation of its idea. Normal numbers are used to convey the reliability (confidence) component of any uncertain information. We are the first to integrate these two prominent concepts QPPNNIVFNs and normal numbers into a single hybrid concept called Quadri partitioned Pythagorean neutrosophic normal interval valued fuzzy number (QPPNNIVFNs). This work is motivated by the subtle significance of these two concepts. A potential structure that may concurrently correlate a reliability index with a wide variety of information input from decision makers is QPPNNIVFN. As a result, QPPNNIVFNs not only more accurately convey the ambiguous information but also determine the degree of certainty in the collected data. Our main contributions and work objectives are following.

(i) With regard to truth, contradiction, ignorance, falsehood, and the associated reliability measures, our suggested concept of QPPNNIVFS may describe a large amount of uncertain information using four ordered pairs.
(ii) QPPNNIVFNs can also be used in very confusing and unclear situations. It is important to note that this work is the first to suggest QPPNNIVFNs.
(iii) Four extremely important factors in ranking QPPNNIVFNs are the new ED, HD, score, and accuracy values as well as the AOs suggested in this study. The four suggested operators are also shown to satisfy a few intriguing features.
(iv) Furthermore, the MADM approach based on our suggested AOs and ED, HD, score, and accuracy values is far more dependable, user friendly, and effective than the current MADM techniques because, in addition to collecting cognitive data from the real world, our method also assesses the accuracy of the information that is currently available.
(v) Building an algorithm based model with QPPNNIVFNs for the application of DM problems.

The MADM technique of QPPNNIVFNs uses weighted AOs, and score and accuracy function. This paper covers all of the real-world applications. The findings of our present study will be very helpful to decision makers in communicating the unclear information.

The structure of the paper is as follows. We review the ideas of PFNTS and NSSTs normal number and associated works in Section 2. Section 3 also suggests some appropriate arithmetic operations between QPPNNIVFNs and new ED, HD, score and accuracy values. Four AOs including QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG are defined in Section 4. Additionally, several features that these operators satisfy are established in this part and the connection between them is deduced. We present the method for the recently suggested MADM approach based on the AOs and the ED, HD, score and accuracy function in Section 5. This section appropriately outlines the validity requirements that each recently created MADM approach must meet. We illustrate the suitability of our approach in relation to case studies in Section 6. Every case study includes sufficient validation testing and comparative analysis. Additionally, Section 7 presents a thorough description of the final results, sensitivity analysis with our suggested AO, comparisons of our suggested technique. Section 8 provides concluding observations and opportunities for further study on our contributions to this research.

2. Basic preliminaries and initial results

In this section, we recall some basic definitions that are essential for the fulfillment of the proposed study.

In this case, the universal set is Ψ. We provide the concepts of IVPFS, PFS, and normal fuzzy number in this section.

Definition 2.1. Let Ψ be a non-empty set a FS J in Ψ is characterized by a MD $\mu_J : \Psi \to [0,1]$ such that

$$\mu_J(\varkappa) = \begin{cases} 1, & \text{if } \varkappa \in \Psi, \\ 0, & \text{if } \varkappa \notin \Psi, \\ (0,1), & \text{if } \varkappa \text{ is partly in } \Psi. \end{cases}$$

We write $J = \{\langle \varkappa, \mu_J(\varkappa)\rangle | \varkappa \in \Psi\}$.

Definition 2.2. An IFS $J = \{\langle x, \mu_J(\varkappa), \varphi_J(\varkappa)\rangle | \varkappa \in \Psi\}$ characterized by a $\mu_J(\varkappa), \varphi_J(\varkappa) : \Psi \to [0,1]$ and $0 \leqslant \mu_J(\varkappa) + \varphi_J(\varkappa) \leqslant 1$ and for each J in Ψ, $\pi_J(\varkappa) = 1 - \mu_J(\varkappa) - \varphi_J(\varkappa)$. Here, $\pi_J(\varkappa)$ denotes the degree of indeterminacy of $\varkappa \in \Psi$ and $\mu_J(\varkappa) + \varphi_J(\varkappa) + \pi_J(\varkappa) = 1$.

Definition 2.3 ([46])**.** The PFST $J = \left\{\varkappa, \left\langle \mu_J\langle\varkappa\rangle, \varphi_J\langle\varkappa\rangle\right\rangle \middle| \varkappa \in \Psi\right\}$, $\mu_J\langle\varkappa\rangle$ and $\varphi_J\langle\varkappa\rangle$ are called the MD and NMD of J, respectively, μ_J and φ_J belong to the Ψ into $[0,1]$ and $0 \preceq \langle\mu_J\langle\varkappa\rangle\rangle^2 + \langle\varphi_J\langle\varkappa\rangle\rangle^2 \preceq 1$. Here, $\left\langle \mu_J, \varphi_J \right\rangle$ is said to be Pythagorean fuzzy number (PFN).

Definition 2.4 ([32])**.** For any PIVFNs, $J = \left\langle \lfloor\mu^l, \mu^u\rfloor, \lfloor\xi^l, \xi^u\rfloor, \lfloor\varphi^l, \varphi^u\rfloor \right\rangle$, $J_1 = \left\langle \lfloor\mu_1^l, \mu_1^u\rfloor, \lfloor\xi_1^l, \xi_1^u\rfloor, \lfloor\varphi_1^l, \varphi_1^u\rfloor \right\rangle$, and $J_2 = \left\langle \lfloor\mu_2^l, \mu_2^u\rfloor, \lfloor\xi_2^l, \xi_2^u\rfloor, \lfloor\varphi_2^l, \varphi_2^u\rfloor \right\rangle$. Then

1. $J_1 \bigvee J_2 = \left(\left\lfloor \sqrt{\langle\mu_1^l\rangle^2 + \langle\mu_2^l\rangle^2 - \langle\mu_1^l\rangle^2 \cdot \langle\mu_2^l\rangle^2}, \sqrt{\langle\mu_1^u\rangle^2 + \langle\mu_2^u\rangle^2 - \langle\mu_1^u\rangle^2 \cdot \langle\mu_2^u\rangle^2} \right\rfloor, \left\lfloor \varphi_1^l \cdot \varphi_2^l, \varphi_1^u \cdot \varphi_2^u \right\rfloor \right)$;
2. $J_1 \bigwedge J_2 = \left(\left\lfloor \mu_1^l \cdot \mu_2^l, \mu_1^u \cdot \mu_2^u \right\rfloor, \left\lfloor \sqrt{\langle\varphi_1^l\rangle^2 + \langle\varphi_2^l\rangle^2 - \langle\varphi_1^l\rangle^2 \cdot \langle\varphi_2^l\rangle^2}, \sqrt{\langle\varphi_1^u\rangle^2 + \langle\varphi_2^u\rangle^2 - \langle\varphi_1^u\rangle^2 \cdot \langle\varphi_2^u\rangle^2} \right\rfloor \right)$;
3. $\delta \cdot J = \left(\left\lfloor \sqrt{1 - \left\langle 1 - \langle\mu^l\rangle^2 \right\rangle^\delta}, \sqrt{1 - \left\langle 1 - \langle\mu^u\rangle^2 \right\rangle^\delta} \right\rfloor, \left\lfloor \langle\varphi^l\rangle^\delta, \langle\varphi^u\rangle^\delta \right\rfloor \right)$;
4. $J^\delta = \left(\left\lfloor \langle\mu^l\rangle^\delta, \langle\mu^u\rangle^\delta \right\rfloor, \left\lfloor \sqrt{1 - \left\langle 1 - \langle\varphi^l\rangle^2 \right\rangle^\delta}, \sqrt{1 - \left\langle 1 - \langle\varphi^u\rangle^2 \right\rangle^\delta} \right\rfloor \right)$,

where μ and φ are called the MD and NMD of J, respectively.

Definition 2.5 ([32])**.** For any PIVFN $J = \left\langle \lfloor\mu^l, \mu^u\rfloor, \lfloor\varphi^l, \varphi^u\rfloor \right\rangle$ the score and accuracy values are defined as follows:

$$S\langle J\rangle = \frac{\left\langle \langle\mu^l\rangle^2 + \langle\mu^u\rangle^2 - \langle\varphi^l\rangle^2 - \langle\varphi^u\rangle^2 \right\rangle}{2}, \ S\langle J\rangle \in \lfloor -1, 1\rfloor,$$
$$H\langle J\rangle = \frac{\left\langle \langle\mu^l\rangle^2 + \langle\mu^u\rangle^2 + \langle\varphi^l\rangle^2 + \langle\varphi^u\rangle^2 \right\rangle}{2}, \ H\langle J\rangle \in [0, 1].$$

Definition 2.6 ([28])**.** Let J be a PNIVFS in Ψ then $\widetilde{J} = \left\{\varkappa, \left\langle \widetilde{\theta_J}\langle\varkappa\rangle, \widetilde{\mu_J}\langle\varkappa\rangle, \widetilde{\xi_J}\langle\varkappa\rangle, \widetilde{\varphi_J}\langle\varkappa\rangle \right\rangle \middle| \varkappa \in \Psi\right\}$, $\widetilde{\mu_J}\langle\varkappa\rangle = \left\lfloor \mu_J^l\langle\varkappa\rangle, \mu_J^u\langle\varkappa\rangle \right\rfloor$, $\widetilde{\xi_J}\langle\varkappa\rangle = \left\lfloor \xi_J^l\langle\varkappa\rangle, \xi_J^u\langle\varkappa\rangle \right\rfloor$, and $\widetilde{J_\varphi}\langle\varkappa\rangle = \left\lfloor \varphi_J^l\langle\varkappa\rangle, \varphi_J^u\langle\varkappa\rangle \right\rfloor$ denote TD, ID, and FD of J, respectively. Here $\widetilde{T_J}, \widetilde{I_J}$, and $\widetilde{F_J}$ are Ψ into $\mathbb{D}[0,1]$ and $0 \preceq \langle\mu_J^u\langle\varkappa\rangle\rangle^2 + \langle\xi_J^u\langle\varkappa\rangle\rangle^2 + \langle\varphi_J^u\langle\varkappa\rangle\rangle^2 \preceq 2$. Since, $\left\langle \left\lfloor \mu_J^l, \mu_J^u \right\rfloor, \left\lfloor \xi_J^l, \xi_J^u \right\rfloor, \left\lfloor \varphi_J^l, \varphi_J^u \right\rfloor \right\rangle$ is called the PNIVFN.

Definition 2.7. Let J be a QPNFS in Ψ then $J = \left\{\varkappa, \left\langle \theta_J\langle\varkappa\rangle, \mu_J\langle\varkappa\rangle, \xi_J\langle\varkappa\rangle, \varphi_J\langle\varkappa\rangle \right\rangle \middle| \varkappa \in \Psi\right\}$, where $\theta_J\langle\varkappa\rangle, \mu_J\langle\varkappa\rangle$, $\xi_J\langle\varkappa\rangle$, and $\varphi_J\langle\varkappa\rangle$ denote truth, contradiction, ignorance, and falsity fuzzy values of J, respectively. Here Q_J, T_J, I_J, and F_J are functions from Ψ into $\mathbb{D}[0,1]$ and $0 \preceq \langle\theta_J\langle\varkappa\rangle\rangle + \langle\mu_J\langle\varkappa\rangle\rangle + \langle\xi_J\langle\varkappa\rangle\rangle + \langle\varphi_J\langle\varkappa\rangle\rangle \preceq 4$.

Definition 2.8.

1. A real number R and fuzzy number $Q(\varkappa) = e^{-\left(\frac{\varkappa-\sigma}{\alpha}\right)^2}, (\alpha \succ 0)$, is called a normal fuzzy number (NFN) and denoted as $Q = (\sigma, \alpha)$.
2. If $P = (\sigma_1, \alpha_1) \in \mathbb{N}$ and $Q = (\sigma_2, \alpha_2) \in \mathbb{N}$, $(\alpha_1, \alpha_2 \succ 0)$, then the distance between P and Q is founded as $\mathbb{D}(P, Q) = \sqrt{(\sigma_1 - \sigma_2)^2 + \frac{1}{2}(\alpha_1 - \alpha_2)^2}$, where $\mathbb{N}$ is called the NFN.

3. Operations for QPPNNIVFN

In addition to the QPPNNIVFN score values, ED and HD measures, a few mathematical characteristics were incorporated. It is critical to understand the various distance metrics, since each has advantages and disadvantages while doing data analysis operations. The functions of the QPPNNIVFN have been specified.

Definition 3.1. Let J be a QPPNIVFS in Ψ and $\widetilde{J} = \left\{\varkappa, \left\langle \widetilde{\theta_J}\langle\varkappa\rangle, \widetilde{\mu_J}\langle\varkappa\rangle, \widetilde{\xi_J}\langle\varkappa\rangle, \widetilde{\varphi_J}\langle\varkappa\rangle \right\rangle \middle| \varkappa \in \Psi\right\}$, where $\widetilde{\theta_J}\langle\varkappa\rangle = \left\lfloor \theta_J^l\langle\varkappa\rangle, \theta_J^u\langle\varkappa\rangle \right\rfloor$, $\widetilde{\mu_J}\langle\varkappa\rangle = \left\lfloor \mu_J^l\langle\varkappa\rangle, \mu_J^u\langle\varkappa\rangle \right\rfloor$, $\widetilde{\xi_J}\langle\varkappa\rangle = \left\lfloor \xi_J^l\langle\varkappa\rangle, \xi_J^u\langle\varkappa\rangle \right\rfloor$, and $\widetilde{\varphi_J}\langle\varkappa\rangle = \left\lfloor \varphi_J^l\langle\varkappa\rangle, \varphi_J^u\langle\varkappa\rangle \right\rfloor$ denotes truth, contradiction, ignorance, and falsity fuzzy values of J, respectively. Here $\widetilde{Q_J}, \widetilde{T_J}, \widetilde{I_J}$, and $\widetilde{F_J}$ are function from Ψ into $\mathbb{D}[0,1]$ and $0 \preceq \langle\theta_J^u\langle\varkappa\rangle\rangle^2 + \langle\mu_J^u\langle\varkappa\rangle\rangle^2 + \langle\xi_J^u\langle\varkappa\rangle\rangle^2 + \langle\varphi_J^u\langle\varkappa\rangle\rangle^2 \preceq 2$.

Definition 3.2. For any QPPNIVFN $\widetilde{J}$, the score and accuracy values are

$$S\langle J\rangle = \frac{\sigma}{2}\left\langle \frac{\langle\theta^l\rangle^2 + \langle\theta^u\rangle^2}{2} + \frac{\langle\mu^l\rangle^2 + \langle\mu^u\rangle^2}{2} - \frac{\langle\xi^l\rangle^2 + \langle\xi^u\rangle^2}{2} - \frac{\langle\varphi^l\rangle^2 + \langle\varphi^u\rangle^2}{2} \right\rangle, \ S\langle J\rangle \in \lfloor -1, 1\rfloor,$$
$$H\langle J\rangle = \frac{\alpha}{2}\left\langle \frac{\langle\theta^l\rangle^2 + \langle\theta^u\rangle^2}{2} + \frac{\langle\mu^l\rangle^2 + \langle\mu^u\rangle^2}{2} + \frac{\langle\xi^l\rangle^2 + \langle\xi^u\rangle^2}{2} + \frac{\langle\varphi^l\rangle^2 + \langle\varphi^u\rangle^2}{2} \right\rangle, \ H\langle J\rangle \in [0, 1].$$

Definition 3.3. Let $\langle\sigma,\alpha\rangle \in \widetilde{\mathbb{N}}$, $\widetilde{J}$ is a QPPNNIVFN when its degree of truth, contradiction, ignorance, and falsity fuzzy values are defined as

$$\lfloor\theta_J^l, \theta_J^u\rfloor = \left\lfloor \theta_J^l e^{-\left\langle \frac{\varkappa-\sigma}{\alpha}\right\rangle^2}, \theta_J^u e^{-\left\langle \frac{\varkappa-\sigma}{\alpha}\right\rangle^2} \right\rfloor, \qquad \lfloor\mu_J^l, \mu_J^u\rfloor = \left\lfloor \mu_J^l e^{-\left\langle \frac{\varkappa-\sigma}{\alpha}\right\rangle^2}, \mu_J^u e^{-\left\langle \frac{\varkappa-\sigma}{\alpha}\right\rangle^2} \right\rfloor,$$
$$\lfloor\xi_J^l, \xi_J^u\rfloor = \left\lfloor \xi_J^l e^{-\left\langle \frac{\varkappa-\sigma}{\alpha}\right\rangle^2}, \xi_J^u e^{-\left\langle \frac{\varkappa-\sigma}{\alpha}\right\rangle^2} \right\rfloor, \text{ and } \lfloor\varphi_J^l, \varphi_J^u\rfloor = \left\lfloor 1 - \langle 1 - \varphi_J^l\rangle e^{-\left\langle \frac{\varkappa-\sigma}{\alpha}\right\rangle^2}, 1 - \langle 1 - \varphi_J^u\rangle e^{-\left\langle \frac{\varkappa-\sigma}{\alpha}\right\rangle^2} \right\rfloor,$$

$x \in X$, respectively, where X is a non-empty set and $\lfloor\theta_J^l, \theta_J^u\rfloor, \lfloor\mu_J^l, \mu_J^u\rfloor, \lfloor\xi_J^l, \xi_J^u\rfloor, \lfloor\varphi_J^l, \varphi_J^u\rfloor \in \mathbb{D}[0,1]$, and $0 \preceq \left\langle\theta_J^u\langle\varkappa\rangle\right\rangle^2 + \left\langle\mu_J^u\langle\varkappa\rangle\right\rangle^2 + \left\langle\xi_J^u\langle\varkappa\rangle\right\rangle^2 + \left\langle\varphi_J^u\langle\varkappa\rangle\right\rangle^2 \preceq 2$.

Definition 3.4. For any three QPPNNIVFNs $\widetilde{J}$, $\widetilde{J}_1$, $\widetilde{J}_2$, and δ being a positive integers, then

$$\widetilde{J}_1 \bigvee \widetilde{J}_2 = \begin{pmatrix} \langle\sigma_1+\sigma_2, \alpha_1+\alpha_2\rangle, \begin{pmatrix} \sqrt[2\delta]{\langle\theta_1^l\rangle^{2\delta} + \langle\theta_2^l\rangle^{2\delta} - \langle\theta_1^l\rangle^{2\delta}\cdot\langle\theta_2^l\rangle^{2\delta}}, \\ \sqrt[2\delta]{\langle\theta_1^u\rangle^{2\delta} + \langle\theta_2^u\rangle^{2\delta} - \langle\theta_1^u\rangle^{2\delta}\cdot\langle\theta_2^u\rangle^{2\delta}} \end{pmatrix}, \\ \begin{pmatrix} \sqrt[2\delta]{\langle\mu_1^l\rangle^{2\delta} + \langle\mu_2^l\rangle^{2\delta} - \langle\mu_1^l\rangle^{2\delta}\cdot\langle\mu_2^l\rangle^{2\delta}}, \\ \sqrt[2\delta]{\langle\mu_1^u\rangle^{2\delta} + \langle\mu_2^u\rangle^{2\delta} - \langle\mu_1^u\rangle^{2\delta}\cdot\langle\mu_2^u\rangle^{2\delta}} \end{pmatrix}, \begin{pmatrix} \sqrt[\delta]{\langle\xi_1^l\rangle^{\delta} + \langle\xi_2^l\rangle^{\delta} - \langle\xi_1^l\rangle^{\delta}\cdot\langle\xi_2^l\rangle^{\delta}}, \\ \sqrt[\delta]{\langle\xi_1^u\rangle^{\delta} + \langle\xi_2^u\rangle^{\delta} - \langle\xi_1^u\rangle^{\delta}\cdot\langle\xi_2^u\rangle^{\delta}} \end{pmatrix}, \\ \left\lfloor \varphi_1^l\cdot\varphi_2^l, \varphi_1^u\cdot\varphi_2^u \right\rfloor \end{pmatrix},$$

$$\widetilde{J}_1 \bigwedge \widetilde{J}_2 = \begin{pmatrix} \langle\sigma_1\cdot\sigma_2, \alpha_1\cdot\alpha_2\rangle, \left\lfloor \theta_1^l\cdot\theta_2^l, \theta_1^u\cdot\theta_2^u \right\rfloor, \begin{pmatrix} \sqrt[\delta]{\langle\mu_1^l\rangle^{\delta} + \langle\mu_2^l\rangle^{\delta} - \langle\mu_1^l\rangle^{\delta}\cdot\langle\mu_2^l\rangle^{\delta}}, \\ \sqrt[\delta]{\langle\mu_1^u\rangle^{\delta} + \langle\mu_2^u\rangle^{\delta} - \langle\mu_1^u\rangle^{\delta}\cdot\langle\mu_2^u\rangle^{\delta}} \end{pmatrix}, \\ \begin{pmatrix} \sqrt[2\delta]{\langle\xi_1^l\rangle^{2\delta} + \langle\xi_2^l\rangle^{2\delta} - \langle\xi_1^l\rangle^{2\delta}\cdot\langle\xi_2^l\rangle^{2\delta}}, \\ \sqrt[2\delta]{\langle\xi_1^u\rangle^{2\delta} + \langle\xi_2^u\rangle^{2\delta} - \langle\xi_1^u\rangle^{2\delta}\cdot\langle\xi_2^u\rangle^{2\delta}} \end{pmatrix}, \begin{pmatrix} \sqrt[2\delta]{\langle\varphi_1^l\rangle^{2\delta} + \langle\varphi_2^l\rangle^{2\delta} - \langle\varphi_1^l\rangle^{2\delta}\cdot\langle\varphi_2^l\rangle^{2\delta}}, \\ \sqrt[2\delta]{\langle\varphi_1^u\rangle^{2\delta} + \langle\varphi_2^u\rangle^{2\delta} - \langle\varphi_1^u\rangle^{2\delta}\cdot\langle\varphi_2^u\rangle^{2\delta}} \end{pmatrix} \end{pmatrix},$$

$$\delta\cdot\widetilde{J} = \begin{pmatrix} \langle\delta\cdot\sigma, \delta\cdot\alpha\rangle, \left\lfloor \sqrt[2\delta]{1 - \left\langle 1 - \langle\theta^l\rangle^{2\delta}\right\rangle^{\delta}}, \sqrt[2\delta]{1 - \left\langle 1 - \langle\theta^u\rangle^{2\delta}\right\rangle^{\delta}} \right\rfloor, \left\lfloor \sqrt[2\delta]{1 - \left\langle 1 - \langle\mu^l\rangle^{2\delta}\right\rangle^{\delta}}, \right. \\ \left. \sqrt[2\delta]{1 - \left\langle 1 - \langle\mu^u\rangle^{2\delta}\right\rangle^{\delta}} \right\rfloor, \left\lfloor \sqrt[\delta]{1 - \left\langle 1 - \langle\xi^l\rangle^{\delta}\right\rangle^{\delta}}, \sqrt[\delta]{1 - \left\langle 1 - \langle\xi^u\rangle^{\delta}\right\rangle^{\delta}} \right\rfloor, \left\lfloor \langle\varphi^l\rangle^{\delta}, \langle\varphi^u\rangle^{\delta} \right\rfloor \end{pmatrix},$$

$$\widetilde{J}^{\delta} = \begin{pmatrix} \langle\sigma^{\delta}, \alpha^{\delta}\rangle, \left\lfloor \langle\theta^l\rangle^{\delta}, \langle\theta^u\rangle^{\delta} \right\rfloor, \left\lfloor \sqrt[\delta]{1 - \left\langle 1 - \langle\mu^l\rangle^{\delta}\right\rangle^{\delta}}, \sqrt[\delta]{1 - \left\langle 1 - \langle\mu^u\rangle^{\delta}\right\rangle^{\delta}} \right\rfloor, \\ \left\lfloor \sqrt[2\delta]{1 - \left\langle 1 - \langle\xi^l\rangle^{2\delta}\right\rangle^{\delta}}, \sqrt[2\delta]{1 - \left\langle 1 - \langle\xi^u\rangle^{2\delta}\right\rangle^{\delta}} \right\rfloor, \left\lfloor \sqrt[2\delta]{1 - \left\langle 1 - \langle\varphi^l\rangle^{2\delta}\right\rangle^{\delta}}, \sqrt[2\delta]{1 - \left\langle 1 - \langle\varphi^u\rangle^{2\delta}\right\rangle^{\delta}} \right\rfloor \end{pmatrix}.$$

4. Distance measure for QPPNNIVFNs

Definition 4.1. For any two QPPNNIVFNs $\widetilde{J}_1$ and $\widetilde{J}_2$, then

$$\aleph_E\left\langle\widetilde{J}_1,\widetilde{J}_2\right\rangle=\frac{1}{2}\sqrt{\begin{array}{l}\left(\begin{array}{l}\frac{\shortmid+\langle\theta_1^l\rangle^2+\shortmid+\langle\mu_1^l\rangle^2-\langle\xi_1^l\rangle^2-\langle\varphi_1^l\rangle^2+\shortmid+\langle\theta_1^u\rangle^2+\shortmid+\langle\mu_1^u\rangle^2-\langle\xi_1^u\rangle^2-\langle\varphi_1^u\rangle^2}{2}\sigma_1\\-\frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2}\sigma_2\end{array}\right)^2\\+\frac{1}{2}\left(\begin{array}{l}\frac{\shortmid+\langle\theta_1^l\rangle^2+\shortmid+\langle\mu_1^l\rangle^2-\langle\xi_1^l\rangle^2-\langle\varphi_1^l\rangle^2+\shortmid+\langle\theta_1^u\rangle^2+\shortmid+\langle\mu_1^u\rangle^2-\langle\xi_1^u\rangle^2-\langle\varphi_1^u\rangle^2}{2}\alpha_1\\-\frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2}\alpha_2\end{array}\right)^2\end{array}},$$

where $\aleph_E\left\langle\widetilde{J}_1,\widetilde{J}_2\right\rangle$ denotes the ED between $\widetilde{J}_1$ and $\widetilde{J}_2$. Also

$$\aleph_H\left\langle\widetilde{J}_1,\widetilde{J}_2\right\rangle=\frac{1}{2}\left(\begin{array}{l}\left|\begin{array}{l}\frac{\shortmid+\langle\theta_1^l\rangle^2+\shortmid+\langle\mu_1^l\rangle^2-\langle\xi_1^l\rangle^2-\langle\varphi_1^l\rangle^2+\shortmid+\langle\theta_1^u\rangle^2+\shortmid+\langle\mu_1^u\rangle^2-\langle\xi_1^u\rangle^2-\langle\varphi_1^u\rangle^2}{2}\sigma_1\\-\frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2}\sigma_2\end{array}\right|\\+\frac{1}{2}\left|\begin{array}{l}\frac{\shortmid+\langle\theta_1^l\rangle^2+\shortmid+\langle\mu_1^l\rangle^2-\langle\xi_1^l\rangle^2-\langle\varphi_1^l\rangle^2+\shortmid+\langle\theta_1^u\rangle^2+\shortmid+\langle\mu_1^u\rangle^2-\langle\xi_1^u\rangle^2-\langle\varphi_1^u\rangle^2}{2}\alpha_1\\-\frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2}\alpha_2\end{array}\right|\end{array}\right),$$

where $\aleph_H\left\langle\widetilde{J}_1,\widetilde{J}_2\right\rangle$ denotes the HD between $\widetilde{J}_1$ and $\widetilde{J}_2$.

Theorem 4.2. *For any three QPPNNIVFNs* $\widetilde{J}_1$, $\widetilde{J}_2$, $\widetilde{J}_3$, *then* $\aleph_E\langle\widetilde{J}_1,\widetilde{J}_3\rangle\preceq\aleph_E\langle\widetilde{J}_1,\widetilde{J}_2\rangle+\aleph_E\langle\widetilde{J}_2,\widetilde{J}_3\rangle$.

Proof.

$$\left\langle\aleph_E\langle J_1,J_2\rangle+\aleph_E\langle J_2,J_3\rangle\right\rangle^2=\left(\begin{array}{l}\frac{1}{2}\sqrt{\begin{array}{l}\left(\begin{array}{l}\frac{\shortmid+\langle\theta_1^l\rangle^2+\shortmid+\langle\mu_1^l\rangle^2-\langle\xi_1^l\rangle^2-\langle\varphi_1^l\rangle^2+\shortmid+\langle\theta_1^u\rangle^2+\shortmid+\langle\mu_1^u\rangle^2-\langle\xi_1^u\rangle^2-\langle\varphi_1^u\rangle^2}{2}\sigma_1\\-\frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2}\sigma_2\end{array}\right)^2\\+\frac{1}{2}\left(\begin{array}{l}\frac{\shortmid+\langle\theta_1^l\rangle^2+\shortmid+\langle\mu_1^l\rangle^2-\langle\xi_1^l\rangle^2-\langle\varphi_1^l\rangle^2+\shortmid+\langle\theta_1^u\rangle^2+\shortmid+\langle\mu_1^u\rangle^2-\langle\xi_1^u\rangle^2-\langle\varphi_1^u\rangle^2}{2}\alpha_1\\-\frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2}\alpha_2\end{array}\right)^2\end{array}}\\+\frac{1}{2}\sqrt{\begin{array}{l}\left(\begin{array}{l}\frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2}\sigma_2\\-\frac{\shortmid+\langle\theta_3^l\rangle^2+\shortmid+\langle\mu_3^l\rangle^2-\langle\xi_3^l\rangle^2-\langle\varphi_3^l\rangle^2+\shortmid+\langle\theta_3^u\rangle^2+\shortmid+\langle\mu_3^u\rangle^2-\langle\xi_3^u\rangle^2-\langle\varphi_3^u\rangle^2}{2}\sigma_3\end{array}\right)^2\\+\frac{1}{2}\left(\begin{array}{l}\frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2}\alpha_2\\-\frac{\shortmid+\langle\theta_3^l\rangle^2+\shortmid+\langle\mu_3^l\rangle^2-\langle\xi_3^l\rangle^2-\langle\varphi_3^l\rangle^2+\shortmid+\langle\mu_3^u\rangle^2+\shortmid+\langle\theta_3^u\rangle^2-\langle\xi_3^u\rangle^2-\langle\varphi_3^u\rangle^2}{2}\alpha_3\end{array}\right)^2\end{array}}\end{array}\right)^2$$

implies

$$\frac{1}{4}\left\langle\langle\Phi_1\sigma_1-\Phi_2\sigma_2\rangle^2+\frac{1}{2}\langle\Phi_1\alpha_1-\Phi_2\alpha_2\rangle^2\right\rangle+\frac{1}{4}\left\langle\langle\Phi_2\sigma_2-\Phi_3\sigma_3\rangle^2+\frac{1}{2}\langle\Phi_2\alpha_2-\Phi_3\alpha_3\rangle^2\right\rangle$$
$$+\frac{1}{2}\left\langle\sqrt{\langle\Phi_1\sigma_1-\Phi_2\sigma_2\rangle^2+\frac{1}{2}\langle\Phi_1\alpha_1-\Phi_2\alpha_2\rangle^2}\times\sqrt{\langle\Phi_2\sigma_2-\Phi_3\sigma_3\rangle^2+\frac{1}{2}\langle\Phi_2\alpha_2-\Phi_3\alpha_3\rangle^2}\right\rangle,$$

where

$$\Phi_1=\frac{\shortmid+\langle\theta_1^l\rangle^2+\shortmid+\langle\mu_1^l\rangle^2-\langle\xi_1^l\rangle^2-\langle\varphi_1^l\rangle^2+\shortmid+\langle\theta_1^u\rangle^2+\shortmid+\langle\mu_1^u\rangle^2-\langle\xi_1^u\rangle^2-\langle\varphi_1^u\rangle^2}{2},$$

$$\Phi_2 = \frac{\shortmid+\langle\theta_2^l\rangle^2+\shortmid+\langle\mu_2^l\rangle^2-\langle\xi_2^l\rangle^2-\langle\varphi_2^l\rangle^2+\shortmid+\langle\theta_2^u\rangle^2+\shortmid+\langle\mu_2^u\rangle^2-\langle\xi_2^u\rangle^2-\langle\varphi_2^u\rangle^2}{2},$$
$$\Phi_3 = \frac{\shortmid+\langle\theta_3^l\rangle^2+\shortmid+\langle\mu_3^l\rangle^2-\langle\xi_3^l\rangle^2-\langle\varphi_3^l\rangle^2+\shortmid+\langle\theta_3^u\rangle^2+\shortmid+\langle\mu_3^u\rangle^2-\langle\xi_3^u\rangle^2-\langle\varphi_3^u\rangle^2}{2}.$$

Now,

$$\begin{aligned}
&\Big\langle \aleph_E\langle J_1,J_2\rangle+\aleph_E\langle J_2,J_3\rangle\Big\rangle^2\\
&\geqslant \frac{1}{4}\Big\langle\langle\Phi_1\sigma_1-\Phi_2\sigma_2\rangle^2+\frac{1}{2}\langle\Phi_1\alpha_1-\Phi_2\alpha_2\rangle^2\Big\rangle+\frac{1}{4}\Big\langle\langle\Phi_2\sigma_2-\Phi_3\sigma_3\rangle^2+\frac{1}{2}\langle\Phi_2\alpha_2-\Phi_3\alpha_3\rangle^2\Big\rangle\\
&\quad+\frac{1}{2}\Big\langle\langle\Phi_1\sigma_1-\Phi_2\sigma_2\rangle\times\langle\Phi_2\sigma_2-\Phi_3\sigma_3\rangle+\frac{1}{2}\langle\Phi_1\alpha_1-\Phi_2\alpha_2\rangle\times\langle\Phi_2\alpha_2-\Phi_3\alpha_3\rangle\Big\rangle\\
&=\frac{1}{4}\Big\langle\langle\Phi_1\sigma_1-\Phi_2\sigma_2\rangle^2+\langle\Phi_2\sigma_2-\Phi_3\sigma_3\rangle^2+2\langle\Phi_1\sigma_1-\Phi_2\sigma_2\rangle\times\langle\Phi_2\sigma_2-\Phi_3\sigma_3\rangle\Big\rangle\\
&\quad+\frac{1}{4}\Big\langle\frac{1}{2}\langle\Phi_1\alpha_1-\Phi_2\alpha_2\rangle^2+\frac{1}{2}\langle\Phi_2\alpha_2-\Phi_3\alpha_3\rangle^2+\langle\Phi_1\alpha_1-\Phi_2\alpha_2\rangle\times\langle\Phi_2\alpha_2-\Phi_3\alpha_3\rangle\Big\rangle\\
&=\frac{1}{4}\Big\lfloor\langle\Phi_1\sigma_1-\Phi_3\sigma_3\rangle^2+\frac{1}{2}\langle\Phi_1\alpha_1-\Phi_3\alpha_3\rangle^2\Big\rfloor=\aleph_E\langle J_1,J_3\rangle^2.
\end{aligned}$$

□

Remark 4.3. Since $\widetilde{J}_1\Big\langle\lfloor 1,1\rfloor,\lfloor 1,1\rfloor,\lfloor 1,1\rfloor,\lfloor 0,0\rfloor\Big\rangle$, $\widetilde{J}_2=\Big\langle\lfloor 1,1\rfloor,\lfloor 1,1\rfloor,\lfloor 1,1\rfloor,\lfloor 0,0\rfloor\Big\rangle$, the distance between QPPNNIVFNs is therefore transformed into the corresponding distance between NFNs.

5. AOs for QPPNNIVFNs

A variety of data types may be processed and integrated with AOs. AOs represent a set of modern techniques in engineering, economics, information analysis, and other fields. The lowest, maximum, average, or total value of a collection can be found by adding the number of objects or their numerical characteristics. Four weighted AOs for QPPNNIVFN information were proposed in this section consisting of QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG operator.

5.1. QPPNNIVF weighted averaging (QPPNNIVFWA) operator

Definition 5.1. Let $\widetilde{J}_a$ be the collection of QPPNNIVFNs, $\mathfrak{R}=\langle\mathfrak{R}_1,\mathfrak{R}_2,\dots,\mathfrak{R}_\eta\rangle$ be a weight of $\widetilde{J}_a$ and $\mathfrak{R}_a\geqslant 0$, $\Cup_{a\twoheadrightarrow 1}^{\eta}\mathfrak{R}_a=1$. Then QPPNNIVFWA $\langle\widetilde{J}_1,\widetilde{J}_2,\dots,\widetilde{J}_n\rangle=\Cup_{a\twoheadrightarrow 1}^{\eta}\mathfrak{R}_a\widetilde{J}_a$.

Theorem 5.2. *Let $\widetilde{J}_a$ be the collection of QPPNNIVFNs. Then* QPPNNIVFWA$\langle\widetilde{J}_1,\widetilde{J}_2,\dots,\widetilde{J}_n\rangle$ *is structured as*

$$\left(\begin{array}{c}\Big\langle\Cup_{a\twoheadrightarrow 1}^{\eta}\mathfrak{R}_a\sigma_a,\Cup_{a\twoheadrightarrow 1}^{\eta}\mathfrak{R}_a\alpha_a\Big\rangle,\ \left\lfloor\sqrt[2\delta]{\shortmid-\Cap_{a\twoheadrightarrow 1}^{\eta}\Big\langle\shortmid-\langle\theta_a^l\rangle^{2\delta}\Big\rangle^{\mathfrak{R}_a}},\sqrt[2\delta]{\shortmid-\Cap_{a\twoheadrightarrow 1}^{\eta}\Big\langle\shortmid-\langle\theta_a^u\rangle^{2\delta}\Big\rangle^{\mathfrak{R}_a}}\right\rfloor,\\ \left\lfloor\sqrt[2\delta]{\shortmid-\Cap_{a\twoheadrightarrow 1}^{\eta}\Big\langle\shortmid-\langle\mu_a^l\rangle^{2\delta}\Big\rangle^{\mathfrak{R}_a}},\sqrt[2\delta]{\shortmid-\Cap_{a\twoheadrightarrow 1}^{\eta}\Big\langle\shortmid-\langle\mu_a^u\rangle^{2\delta}\Big\rangle^{\mathfrak{R}_a}}\right\rfloor,\\ \left\lfloor\sqrt[\delta]{\shortmid-\Cap_{a\twoheadrightarrow 1}^{\eta}\Big\langle\shortmid-\langle\xi_a^l\rangle^{\delta}\Big\rangle^{\mathfrak{R}_a}},\sqrt[\delta]{\shortmid-\Cap_{a\twoheadrightarrow 1}^{\eta}\Big\langle\shortmid-\langle\xi_a^u\rangle^{\delta}\Big\rangle^{\mathfrak{R}_a}}\right\rfloor,\ \Big\lfloor\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\varphi_a^l\rangle^{\mathfrak{R}_a},\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\varphi_a^u\rangle^{\mathfrak{R}_a}\Big\rfloor\end{array}\right). \tag{5.1}$$

Proof. If $n=2$, then QPPNNIVFWA$\langle\widetilde{J}_1,\widetilde{J}_2\rangle=\mathfrak{R}_1\widetilde{J}_1\bigvee\mathfrak{R}_2\widetilde{J}_2$, where

$$\mathfrak{R}_1\widetilde{J}_1=\left(\begin{array}{c}\Big\langle\mathfrak{R}_1\sigma_1,\mathfrak{R}_1\alpha_1\Big\rangle,\ \left\lfloor\sqrt[2\delta]{\shortmid-\Big\langle\shortmid-\langle\theta_1^l\rangle^{2\delta}\Big\rangle^{\mathfrak{R}_1}},\sqrt[2\delta]{\shortmid-\Big\langle\shortmid-\langle\theta_1^u\rangle^{2\delta}\Big\rangle^{\mathfrak{R}_1}}\right\rfloor,\left\lfloor\sqrt[2\delta]{\shortmid-\Big\langle\shortmid-\langle\mu_1^l\rangle^{2\delta}\Big\rangle^{\mathfrak{R}_1}},\right.\\ \left.\sqrt[2\delta]{\shortmid-\Big\langle\shortmid-\langle\mu_1^u\rangle^{2\delta}\Big\rangle^{\mathfrak{R}_1}}\right\rfloor,\left\lfloor\sqrt[\delta]{\shortmid-\Big\langle\shortmid-\langle\xi_1^l\rangle^{\delta}\Big\rangle^{\mathfrak{R}_1}},\sqrt[\delta]{\shortmid-\Big\langle\shortmid-\langle\xi_1^u\rangle^{\delta}\Big\rangle^{\mathfrak{R}_1}}\right\rfloor,\Big\lfloor\langle\varphi_1^l\rangle^{\mathfrak{R}_1},\langle\varphi_1^u\rangle^{\mathfrak{R}_1}\Big\rfloor\end{array}\right),$$

$$\Re_2\widetilde{J}_2 = \left(\begin{array}{c} \left\langle \Re_2\sigma_2, \Re_2\alpha_2 \right\rangle, \left\lfloor \sqrt[2\delta]{1-\left\langle 1-\langle \theta_2^{l}\rangle^{2\delta}\right\rangle^{\Re_2}}, \sqrt[2\delta]{1-\left\langle 1-\langle \theta_2^{u}\rangle^{2\delta}\right\rangle^{\Re_2}} \right\rfloor, \left\lfloor \sqrt[2\delta]{1-\left\langle 1-\langle \mu_2^{l}\rangle^{2\delta}\right\rangle^{\Re_2}}, \right. \\ \left. \sqrt[2\delta]{1-\left\langle 1-\langle \mu_2^{u}\rangle^{2\delta}\right\rangle^{\Re_2}} \right\rfloor, \left\lfloor \sqrt[\delta]{1-\left\langle 1-\langle \xi_2^{l}\rangle^{\delta}\right\rangle^{\Re_2}}, \sqrt[\delta]{1-\left\langle 1-\langle \xi_2^{u}\rangle^{\delta}\right\rangle^{\Re_2}} \right\rfloor, \left\lfloor \langle \varphi_2^{l}\rangle^{\Re_2}, \langle \varphi_2^{u}\rangle^{\Re_2} \right\rfloor \end{array}\right).$$

Now,

$$\Re_1\widetilde{J}_1 \bigvee \Re_2\widetilde{J}_2 = \left(\begin{array}{c} \left\langle \Re_1\sigma_1+\Re_2\sigma_2, \Re_1\alpha_1+\Re_2\alpha_2 \right\rangle, \left(\begin{array}{c} \sqrt[2\delta]{\begin{array}{c}\left\langle 1-\left\langle 1-\langle \theta_1^{l}\rangle^{2\delta}\right\rangle^{\Re_1}\right\rangle + \left\langle 1-\left\langle 1-\langle \theta_2^{l}\rangle^{2\delta}\right\rangle^{\Re_2}\right\rangle \\ -\left\langle 1-\left\langle 1-\langle \theta_1^{l}\rangle^{2\delta}\right\rangle^{\Re_1}\right\rangle \cdot \left\langle 1-\left\langle 1-\langle \theta_2^{l}\rangle^{2\delta}\right\rangle^{\Re_2}\right\rangle\end{array}}, \\ \sqrt[2\delta]{\begin{array}{c}\left\langle 1-\left\langle 1-\langle \theta_1^{u}\rangle^{2\delta}\right\rangle^{\Re_1}\right\rangle + \left\langle 1-\left\langle 1-\langle \theta_2^{u}\rangle^{2\delta}\right\rangle^{\Re_2}\right\rangle \\ -\left\langle 1-\left\langle 1-\langle \theta_1^{u}\rangle^{2\delta}\right\rangle^{\Re_1}\right\rangle \cdot \left\langle 1-\left\langle 1-\langle \theta_2^{u}\rangle^{2\delta}\right\rangle^{\Re_2}\right\rangle\end{array}} \end{array}\right), \\ \left(\begin{array}{c} \sqrt[2\delta]{\begin{array}{c}\left\langle 1-\left\langle 1-\langle \mu_1^{l}\rangle^{2\delta}\right\rangle^{\Re_1}\right\rangle + \left\langle 1-\left\langle 1-\langle \mu_2^{l}\rangle^{2\delta}\right\rangle^{\Re_2}\right\rangle \\ -\left\langle 1-\left\langle 1-\langle \mu_1^{l}\rangle^{2\delta}\right\rangle^{\Re_1}\right\rangle \cdot \left\langle 1-\left\langle 1-\langle \mu_2^{l}\rangle^{2\delta}\right\rangle^{\Re_2}\right\rangle\end{array}}, \\ \sqrt[2\delta]{\begin{array}{c}\left\langle 1-\left\langle 1-\langle \mu_1^{u}\rangle^{2\delta}\right\rangle^{\Re_1}\right\rangle + \left\langle 1-\left\langle 1-\langle \mu_2^{u}\rangle^{2\delta}\right\rangle^{\Re_2}\right\rangle \\ -\left\langle 1-\left\langle 1-\langle \mu_1^{u}\rangle^{2\delta}\right\rangle^{\Re_1}\right\rangle \cdot \left\langle 1-\left\langle 1-\langle \mu_2^{u}\rangle^{2\delta}\right\rangle^{\Re_2}\right\rangle\end{array}} \end{array}\right), \\ \left(\begin{array}{c} \sqrt[\delta]{\begin{array}{c}\left\langle 1-\left\langle 1-\langle \xi_1^{l}\rangle^{\delta}\right\rangle^{\Re_1}\right\rangle + \left\langle 1-\left\langle 1-\langle \xi_2^{l}\rangle^{\delta}\right\rangle^{\Re_2}\right\rangle \\ -\left\langle 1-\left\langle 1-\langle \xi_1^{l}\rangle^{\delta}\right\rangle^{\Re_1}\right\rangle \cdot \left\langle 1-\left\langle 1-\langle \xi_2^{l}\rangle^{\delta}\right\rangle^{\Re_2}\right\rangle\end{array}}, \\ \sqrt[\delta]{\begin{array}{c}\left\langle 1-\left\langle 1-\langle \xi_1^{u}\rangle^{\delta}\right\rangle^{\Re_1}\right\rangle + \left\langle 1-\left\langle 1-\langle \xi_2^{u}\rangle^{\delta}\right\rangle^{\Re_2}\right\rangle \\ -\left\langle 1-\left\langle 1-\langle \xi_1^{u}\rangle^{\delta}\right\rangle^{\Re_1}\right\rangle \cdot \left\langle 1-\left\langle 1-\langle \xi_2^{u}\rangle^{\delta}\right\rangle^{\Re_2}\right\rangle\end{array}} \end{array}\right), \left\lfloor \langle \varphi_1^{l}\rangle^{\Re_1}\langle \varphi_2^{l}\rangle^{\Re_2}, \langle \varphi_1^{u}\rangle^{\Re_1}\langle \varphi_2^{u}\rangle^{\Re_2} \right\rfloor \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \Re_1\sigma_1+\Re_2\sigma_2, \Re_1\alpha_1+\Re_2\alpha_2 \right\rangle, \left(\begin{array}{c} \sqrt[2\delta]{1-\left\langle 1-\langle \theta_1^{l}\rangle^{2\delta}\right\rangle^{\Re_1}\left\langle 1-\langle \theta_2^{l}\rangle^{2\delta}\right\rangle^{\Re_2}}, \\ \sqrt[2\delta]{1-\left\langle 1-\langle \theta_1^{u}\rangle^{2\delta}\right\rangle^{\Re_1}\left\langle 1-\langle \theta_2^{u}\rangle^{2\delta}\right\rangle^{\Re_2}} \end{array}\right), \\ \left(\begin{array}{c} \sqrt[2\delta]{1-\left\langle 1-\langle \mu_1^{l}\rangle^{2\delta}\right\rangle^{\Re_1}\left\langle 1-\langle \mu_2^{l}\rangle^{2\delta}\right\rangle^{\Re_2}}, \\ \sqrt[2\delta]{1-\left\langle 1-\langle \mu_1^{u}\rangle^{2\delta}\right\rangle^{\Re_1}\left\langle 1-\langle \mu_2^{u}\rangle^{2\delta}\right\rangle^{\Re_2}} \end{array}\right), \left(\begin{array}{c} \sqrt[\delta]{1-\left\langle 1-\langle \xi_1^{l}\rangle^{\delta}\right\rangle^{\Re_1}\left\langle 1-\langle \xi_2^{l}\rangle^{\delta}\right\rangle^{\Re_2}}, \\ \sqrt[\delta]{1-\left\langle 1-\langle \xi_1^{u}\rangle^{\delta}\right\rangle^{\Re_1}\left\langle 1-\langle \xi_2^{u}\rangle^{\delta}\right\rangle^{\Re_2}} \end{array}\right), \\ \left\lfloor \langle \varphi_1^{l}\rangle^{\Re_1}\cdot\langle \varphi_2^{l}\rangle^{\Re_2}, \langle \varphi_1^{u}\rangle^{\Re_1}\cdot\langle \varphi_2^{u}\rangle^{\Re_2} \right\rfloor \end{array}\right),$$

$$\mathrm{QPPNNIVFWA}\langle\widetilde{J}_1, \widetilde{J}_2\rangle = \left(\begin{array}{c} \left\langle \Cup_{a\twoheadrightarrow 1}^{2} \Re_a\sigma_a, \Cup_{a\twoheadrightarrow 1}^{2} \Re_a\alpha_a \right\rangle, \\ \left\lfloor \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\langle \theta_a^{l}\rangle^{2\delta}\right\rangle^{\Re_a}}, \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\langle \theta_a^{u}\rangle^{2\delta}\right\rangle^{\Re_a}} \right\rfloor, \\ \left\lfloor \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\langle \mu_a^{l}\rangle^{2\delta}\right\rangle^{\Re_a}}, \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\langle \mu_a^{u}\rangle^{2\delta}\right\rangle^{\Re_a}} \right\rfloor, \\ \left\lfloor \sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\langle \xi_a^{l}\rangle^{\delta}\right\rangle^{\Re_a}}, \sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\langle \xi_a^{u}\rangle^{\delta}\right\rangle^{\Re_a}} \right\rfloor, \\ \left\lfloor \Cap_{a\twoheadrightarrow 1}^{2}\langle \varphi_a^{l}\rangle^{\Re_a}, \Cap_{a\twoheadrightarrow 1}^{2}\langle \varphi_a^{u}\rangle^{\Re_a} \right\rfloor. \end{array}\right).$$

In general

$$\mathrm{QPPNNIVFWA}\langle \widetilde{\mathrm{J}}_1, \widetilde{\mathrm{J}}_2, \ldots, \widetilde{\mathrm{J}}_l \rangle = \begin{pmatrix} \left\langle \Cup_{a \twoheadrightarrow 1}^{\ell} \mathfrak{R}_a \sigma_a, \Cup_{a \twoheadrightarrow 1}^{\ell} \mathfrak{R}_a \alpha_a \right\rangle, \\ \left\lfloor \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \langle \theta_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \langle \theta_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} \right\rfloor, \\ \left\lfloor \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \langle \mu_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \langle \mu_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} \right\rfloor, \\ \left\lfloor \sqrt[\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \langle \xi_a^l \rangle^{\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \langle \xi_a^u \rangle^{\delta} \right\rangle^{\mathfrak{R}_a}} \right\rfloor, \\ \left\lfloor \Cap_{a \twoheadrightarrow 1}^{\ell} \langle \varphi_a^l \rangle^{\mathfrak{R}_a}, \Cap_{a \twoheadrightarrow 1}^{\ell} \langle \varphi_a^u \rangle^{\mathfrak{R}_a} \right\rfloor. \end{pmatrix}.$$

If $n = l + 1$, then

$$\begin{aligned} &\mathrm{QPPNNIVFWA}\langle \widetilde{\mathrm{J}}_1, \widetilde{\mathrm{J}}_2, \ldots, \widetilde{\mathrm{J}}_l, \widetilde{\mathrm{J}}_{l+1} \rangle \\ &= \begin{pmatrix} \left\langle \Cup_{a \twoheadrightarrow 1}^{\ell} \mathfrak{R}_a \sigma_a + \mathfrak{R}_{l+1}\sigma_{l+1}, \Cup_{a \twoheadrightarrow 1}^{\ell} \mathfrak{R}_a \alpha_a + \mathfrak{R}_{l+1}\alpha_{l+1} \right\rangle, \\ \begin{pmatrix} \sqrt[2\delta]{\begin{matrix} \Cup_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \theta_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \theta_{l+1}^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \\ - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \theta_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \theta_{l+1}^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \end{matrix}}, \\ \sqrt[2\delta]{\begin{matrix} \Cup_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \theta_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \theta_{l+1}^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \\ - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \theta_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \theta_{l+1}^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \end{matrix}} \end{pmatrix}, \\ \begin{pmatrix} \sqrt[2\delta]{\begin{matrix} \Cup_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \mu_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \mu_{l+1}^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \\ - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \mu_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \mu_{l+1}^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \end{matrix}}, \\ \sqrt[2\delta]{\begin{matrix} \Cup_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \mu_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \mu_{l+1}^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \\ - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \mu_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \mu_{l+1}^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \end{matrix}} \end{pmatrix}, \\ \begin{pmatrix} \sqrt[\delta]{\begin{matrix} \Cup_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \xi_a^l \rangle^{\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \xi_{l+1}^l \rangle^{\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \\ - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \xi_a^l \rangle^{\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \xi_{l+1}^l \rangle^{\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \end{matrix}}, \\ \sqrt[\delta]{\begin{matrix} \Cup_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \xi_a^u \rangle^{\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \xi_{l+1}^u \rangle^{\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \\ - \Cap_{a \twoheadrightarrow 1}^{\ell} \left\langle 1 - \left\langle 1 - \langle \xi_a^u \rangle^{\delta} \right\rangle^{\mathfrak{R}_a} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \xi_{l+1}^u \rangle^{\delta} \right\rangle^{\mathfrak{R}_{l+1}} \right\rangle \end{matrix}} \end{pmatrix}, \\ \left\lfloor \Cap_{a \twoheadrightarrow 1}^{\ell} \langle \varphi_a^l \rangle^{\mathfrak{R}_a} \cdot \langle \varphi_{l+1}^l \rangle^{\mathfrak{R}_{l+1}}, \Cap_{a \twoheadrightarrow 1}^{\ell} \langle \varphi_a^u \rangle^{\mathfrak{R}_a} \cdot \langle \varphi_{l+1}^u \rangle^{\mathfrak{R}_{l+1}} \right\rfloor \end{pmatrix} \\ &= \begin{pmatrix} \left\langle \Cup_{a \twoheadrightarrow 1}^{\ell+1} \mathfrak{R}_a \sigma_a, \Cup_{a \twoheadrightarrow 1}^{\ell+1} \mathfrak{R}_a \alpha_a \right\rangle, \left(\sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell+1} \left\langle 1 - \langle \theta_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell+1} \left\langle 1 - \langle \theta_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} \right), \\ \left(\sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell+1} \left\langle 1 - \langle \mu_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell+1} \left\langle 1 - \langle \mu_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} \right), \\ \left(\sqrt[\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell+1} \left\langle 1 - \langle \xi_a^l \rangle^{\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\ell+1} \left\langle 1 - \langle \xi_a^u \rangle^{\delta} \right\rangle^{\mathfrak{R}_a}} \right), \left\lfloor \Cap_{a \twoheadrightarrow 1}^{\ell+1} \langle \varphi_a^l \rangle^{\mathfrak{R}_a}, \Cap_{a \twoheadrightarrow 1}^{\ell+1} \langle \varphi_a^u \rangle^{\mathfrak{R}_a} \right\rfloor \end{pmatrix}. \end{aligned}$$

□

Theorem 5.3. *If all* $\widetilde{\mathrm{J}}_a$ *are equal, then* $\mathrm{QPPNNIVFWA}\langle \widetilde{\mathrm{J}}_1, \widetilde{\mathrm{J}}_2, \ldots, \widetilde{\mathrm{J}}_n \rangle = \widetilde{\mathrm{J}}$ *(idempotency property).*

Proof. Given that $\langle \sigma_a, \alpha_a \rangle = \langle \sigma, \alpha \rangle$, $\lfloor \theta_a^l, \theta_a^u \rfloor = \lfloor \theta^l, \theta^u \rfloor$, $\lfloor \mu_a^l, \mu_a^u \rfloor = \lfloor \mu^l, \mu^u \rfloor$, $\lfloor \xi_a^l, \xi_a^u \rfloor = \lfloor \xi^l, \xi^u \rfloor$, and $\lfloor \varphi_a^l, \varphi_a^u \rfloor = \lfloor \varphi^l, \varphi^u \rfloor$ and $\Cup_{a \twoheadrightarrow 1}^{n} \mathfrak{R}_a = 1$, now,

$\mathrm{QPPNNIVFWA}\langle \widetilde{\mathrm{J}}_1, \widetilde{\mathrm{J}}_2, \ldots, \widetilde{\mathrm{J}}_n \rangle$

$$= \begin{pmatrix} \left\langle \Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \sigma_a, \Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \alpha_a \right\rangle, \left(\sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \theta_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \theta_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} \right), \\ \left(\sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \mu_a^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \mu_a^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} \right), \\ \left(\sqrt[\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \xi_a^l \rangle^{\delta} \right\rangle^{\mathfrak{R}_a}}, \sqrt[\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \xi_a^u \rangle^{\delta} \right\rangle^{\mathfrak{R}_a}} \right), \left\lfloor \Cap_{a\twoheadrightarrow 1}^{\eta} \langle \varphi_a^l \rangle^{\mathfrak{R}_a}, \Cap_{a\twoheadrightarrow 1}^{\eta} \langle \varphi_a^u \rangle^{\mathfrak{R}_a} \right\rfloor \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \sigma \Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a, \alpha \Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \right\rangle, \left(\sqrt[2\delta]{1 - \left\langle 1 - \langle \theta^l \rangle^{2\delta} \right\rangle^{\Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a}}, \sqrt[2\delta]{1 \left\langle 1 - \langle \theta^u \rangle^{2\delta} \right\rangle^{\Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a}} \right), \\ \left(\sqrt[2\delta]{1 - \left\langle 1 - \langle \mu^l \rangle^{2\delta} \right\rangle^{\Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a}}, \sqrt[2\delta]{1 \left\langle 1 - \langle \mu^u \rangle^{2\delta} \right\rangle^{\Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a}} \right), \\ \left(\sqrt[\delta]{1 - \left\langle 1 - \langle \xi^l \rangle^{\delta} \right\rangle^{\Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a}}, \sqrt[\delta]{1 \left\langle 1 - \langle \xi^u \rangle^{\delta} \right\rangle^{\Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a}} \right), \left(\langle \varphi^l \rangle^{\Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a}, \langle \varphi^u \rangle^{\Cup_{a\twoheadrightarrow 1}^{\eta} \mathfrak{R}_a} \right) \end{pmatrix}$$

$$= \begin{pmatrix} \langle \sigma, \alpha \rangle, \left(\sqrt[2\delta]{1 - \left\langle 1 - \langle \theta^l \rangle^{2\delta} \right\rangle}, \sqrt[2\delta]{1 - \left\langle 1 - \langle \theta^u \rangle^{2\delta} \right\rangle} \right), \left(\sqrt[2\delta]{1 - \left\langle 1 - \langle \mu^l \rangle^{2\delta} \right\rangle}, \sqrt[2\delta]{1 - \left\langle 1 - \langle \mu^u \rangle^{2\delta} \right\rangle} \right), \\ \left(\sqrt[\delta]{1 - \left\langle 1 - \langle \xi^l \rangle^{\delta} \right\rangle}, \sqrt[\delta]{1 - \left\langle 1 - \langle \xi^u \rangle^{\delta} \right\rangle} \right), \left\lfloor \langle \varphi^l \rangle, \langle \varphi^u \rangle \right\rfloor \end{pmatrix} = \widetilde{\mathrm{J}}.$$

□

Theorem 5.4. *Let* $\widetilde{\mathrm{J}}_a$ *be a collection QPPNNIVFWA, where* $\underline{\sigma} = \min \sigma_{ij}$, $\overline{\sigma} = \max \sigma_{ij}$, $\underline{\alpha} = \max \alpha_{ij}$, $\overline{\alpha} = \min \alpha_{ij}$, *and*

$$\begin{array}{llll} \underline{\theta^l} = \min \theta_{ij}^l, & \overline{\theta^l} = \max \theta_{ij}^l, & \underline{\theta^u} = \min \theta_{ij}^u, & \overline{\theta^u} = \max \theta_{ij}^u, \\ \underline{\mu}^l = \min \mu_{ij}^l, & \overline{\mu}^l = \max \mu_{ij}^l, & \underline{\mu}^u = \min \mu_{ij}^u, & \overline{\mu}^u = \max \mu_{ij}^u, \\ \underline{\xi}^l = \min \xi_{ij}^l, & \overline{\xi^l} = \max \xi_{ij}^l, & \underline{\xi}^u = \min \xi_{ij}^u, & \overline{\xi^u} = \max \xi_{ij}^u, \\ \underline{\varphi}^l = \min \varphi_{ij}^l, & \overline{\varphi^l} = \max \varphi_{ij}^l, & \underline{\varphi}^u = \min \varphi_{ij}^u, & \overline{\varphi^u} = \max \varphi_{ij}^u, \end{array}$$

$1 \preceq i \preceq n, j{=}1,2,\ldots,i_j$, *then,* $\left\langle \langle \underline{\sigma}, \underline{\alpha} \rangle, \lfloor \underline{\theta^l}, \underline{\theta^u} \rfloor, \lfloor \underline{\mu}^l, \underline{\mu}^u \rfloor, \lfloor \underline{\xi}^l, \underline{\xi}^u \rfloor, \lfloor \overline{\varphi^l}, \overline{\varphi^u} \rfloor \right\rangle \preceq \mathrm{QPPNNIVFWA}\langle \widetilde{\mathrm{J}}_1, \widetilde{\mathrm{J}}_2, \ldots, \widetilde{\mathrm{J}}_n \rangle$ $\preceq \left\langle \langle \overline{\sigma}, \overline{\alpha} \rangle, \lfloor \overline{\theta^l}, \overline{\theta^u} \rfloor, \lfloor \overline{\mu}^l, \overline{\mu}^u \rfloor, \lfloor \overline{\xi^l}, \overline{\xi^u} \rfloor, \lfloor \underline{\varphi}^l, \underline{\varphi}^u \rfloor \right\rangle$ *(boundedness property).*

Proof. Since, $\underline{\theta^l} = \min \theta_{ij}^l$, $\overline{\theta^l} = \max \theta_{ij}^l$, $\underline{\theta^u} = \min \theta_{ij}^u$, $\overline{\theta^u} = \max \theta_{ij}^u$, and $\underline{\theta^l} \preceq \theta_{ij}^l \preceq \overline{\theta^l}$ and $\underline{\theta^u} \preceq \theta_{ij}^u \preceq \overline{\theta^u}$, we have

$$\begin{aligned} \underline{\theta^l} + \underline{\theta^u} &= \sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \underline{\theta^l} \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} + \sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \underline{\theta^u} \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} \\ &\preceq \sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \theta_{ij}^l \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} + \sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \theta_{ij}^u \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} \\ &\preceq \sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \overline{\theta^l} \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} + \sqrt[2\delta]{1 - \Cap_{a\twoheadrightarrow 1}^{\eta} \left\langle 1 - \langle \overline{\theta^u} \rangle^{2\delta} \right\rangle^{\mathfrak{R}_a}} = \overline{\theta^l} + \overline{\theta^u}. \end{aligned}$$

Since, $\underline{\varphi}^l = \min \varphi_{ij}^l$, $\overline{\varphi^l} = \max \varphi_{ij}^l$, $\underline{\varphi}^u = \min \varphi_{ij}^u$, $\overline{\varphi^u} = \max \varphi_{ij}^u$, and $\underline{\varphi}^l \preceq \varphi_{ij}^l \preceq \overline{\varphi^l}$ and $\underline{\varphi}^u \preceq \varphi_{ij}^u \preceq \overline{\varphi^u}$, we have

$$\begin{aligned} \underline{\varphi}^l + \underline{\varphi}^u &= \Cap_{a\twoheadrightarrow 1}^{\eta} \langle \underline{\varphi}^l \rangle^{\mathfrak{R}_a} + \Cap_{a\twoheadrightarrow 1}^{\eta} \langle \underline{\varphi}^u \rangle^{\mathfrak{R}_a} \\ &\preceq \Cap_{a\twoheadrightarrow 1}^{\eta} \langle \varphi_{ij}^l \rangle^{\mathfrak{R}_a} + \Cap_{a\twoheadrightarrow 1}^{\eta} \langle \varphi_{ij}^u \rangle^{\mathfrak{R}_a} \preceq \Cap_{a\twoheadrightarrow 1}^{\eta} \langle \overline{\varphi^l} \rangle^{\mathfrak{R}_a} + \Cap_{a\twoheadrightarrow 1}^{\eta} \langle \overline{\varphi^u} \rangle^{\mathfrak{R}_a} = \overline{\varphi^l} + \overline{\varphi^u}. \end{aligned}$$

Since, $\underline{\sigma} = \min \sigma_{ij}$, $\overline{\sigma} = \max \sigma_{ij}$, $\underline{\alpha} = \max \alpha_{ij}$, $\overline{\alpha} = \min \alpha_{ij}$, and $\underline{\sigma} \preceq \sigma_{ij} \preceq \overline{\sigma}$ and $\overline{\alpha} \preceq \alpha_{ij} \preceq \underline{\alpha}$. Hence,

$\mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\underline{\sigma} \preceq \mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\sigma_{ij} \preceq \mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\overline{\sigma}$ and $\mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\overline{\alpha} \preceq \mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\alpha_{ij} \preceq \mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\underline{\alpha}$. Therefore,

$$
\begin{aligned}
&\frac{\mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\underline{\sigma}}{2}\times\left(\cfrac{\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\underline{\theta}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\underline{\theta}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2+\cfrac{\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\underline{\mu}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\underline{\mu}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2-\cfrac{\left\langle\sqrt[\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\underline{\xi}^{l}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\underline{\xi}^{u}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2-\cfrac{\left\langle\Cap^{\eta}_{a\twoheadrightarrow 1}\langle\overline{\varphi}^{l}\rangle^{\mathfrak{R}_a}\right\rangle^{2}+\left\langle\Cap^{\eta}_{a\twoheadrightarrow 1}\langle\overline{\varphi}^{u}\rangle^{\mathfrak{R}_a}\right\rangle^{2}}{2}}}}\right)\\
&\preceq\frac{\mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\sigma_{ij}}{2}\times\left(\cfrac{\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\theta^{l}_{ij}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\theta^{u}_{ij}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2+\cfrac{\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\mu^{l}_{ij}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\mu^{u}_{ij}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2-\cfrac{\left\langle\sqrt[\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\xi^{l}_{ij}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\xi^{u}_{ij}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2-\cfrac{\left\langle\Cap^{\eta}_{a\twoheadrightarrow 1}\langle\varphi^{l}_{ij}\rangle^{\mathfrak{R}_a}\right\rangle^{2}+\left\langle\Cap^{\eta}_{a\twoheadrightarrow 1}\langle\varphi^{u}_{ij}\rangle^{\mathfrak{R}_a}\right\rangle^{2}}{2}}}}\right)\\
&\preceq\frac{\mathbb{W}^{\eta}_{a\twoheadrightarrow 1}\mathfrak{R}_a\overline{\sigma}}{2}\times\left(\cfrac{\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\overline{\theta}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\overline{\theta}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2+\cfrac{\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\overline{\mu}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[2\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\overline{\mu}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2-\cfrac{\left\langle\sqrt[\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\overline{\xi}^{l}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}+\left\langle\sqrt[\delta]{1-\Cap^{\eta}_{a\twoheadrightarrow 1}\left\langle 1-\langle\overline{\xi}^{u}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2}}{2-\cfrac{\left\langle\Cap^{\eta}_{a\twoheadrightarrow 1}\langle\underline{\varphi}^{l}\rangle^{\mathfrak{R}_a}\right\rangle^{2}+\left\langle\Cap^{\eta}_{a\twoheadrightarrow 1}\langle\underline{\varphi}^{u}\rangle^{\mathfrak{R}_a}\right\rangle^{2}}{2}}}}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left\langle\langle\underline{\sigma},\underline{\alpha}\rangle,\lfloor\underline{\theta^{l}},\underline{\theta^{u}}\rfloor,\lfloor\underline{\mu}^{l},\underline{\mu}^{u}\rfloor,\lfloor\underline{\xi^{l}},\underline{\xi^{u}}\rfloor,\lfloor\overline{\varphi^{l}},\overline{\varphi^{u}}\rfloor\right\rangle &\preceq \mathrm{QPPNNIVFWA}\langle\widetilde{J}_1,\widetilde{J}_2,\dots,\widetilde{J}_n\rangle\\
&\preceq\left\langle\langle\overline{\sigma},\overline{\alpha}\rangle,\lfloor\overline{\theta^{l}},\overline{\theta^{u}}\rfloor,\lfloor\overline{\mu}^{l},\overline{\mu}^{u}\rfloor,\lfloor\overline{\xi^{l}},\overline{\xi^{u}}\rfloor,\lfloor\underline{\varphi^{l}},\underline{\varphi^{u}}\rfloor\right\rangle.
\end{aligned}
$$

□

Theorem 5.5. *Suppose $\widetilde{J}_1,\widetilde{J}_2,\dots,\widetilde{J}_n$ and $\widetilde{W}_1,\widetilde{W}_2,\dots,\widetilde{W}_n$ are two sets of QPPNNIVFWA and*

$$\widetilde{J}_a=\left\langle\langle\sigma_{t_{ij}},\alpha_{t_{ij}}\rangle,\lfloor\theta^{l}_{t_{ij}},\theta^{u}_{t_{ij}}\rfloor,\lfloor\mu^{l}_{t_{ij}},\mu^{u}_{t_{ij}}\rfloor,\lfloor\xi^{l}_{t_{ij}},\xi^{u}_{t_{ij}}\rfloor\lfloor\varphi^{l}_{t_{ij}},\varphi^{u}_{t_{ij}}\rfloor\right\rangle$$

and

$$\widetilde{W}_a=\left\langle\langle\sigma_{h_{ij}},\alpha_{h_{ij}}\rangle,\lfloor\theta^{l}_{h_{ij}},\theta^{u}_{h_{ij}}\rfloor,,\lfloor\mu^{l}_{h_{ij}},\mu^{u}_{h_{ij}}\rfloor,\lfloor\xi^{l}_{h_{ij}},\xi^{u}_{h_{ij}}\rfloor\lfloor\varphi^{l}_{h_{ij}},\varphi^{u}_{h_{ij}}\rfloor\right\rangle.$$

For any i, if there is $\sigma_{t_{ij}}\preceq\alpha_{h_{ij}}$, $\left\langle\theta^{l}_{t_{ij}}\right\rangle^{2}+\left\langle\theta^{u}_{t_{ij}}\right\rangle^{2}\preceq\left\langle\theta^{l}_{h_{ij}}\right\rangle^{2}+\left\langle\theta^{u}_{h_{ij}}\right\rangle^{2}$, $\left\langle\mu^{l}_{t_{ij}}\right\rangle^{2}+\left\langle\mu^{u}_{t_{ij}}\right\rangle^{2}\preceq\left\langle\mu^{l}_{h_{ij}}\right\rangle^{2}+\left\langle\mu^{u}_{h_{ij}}\right\rangle^{2}$ and $\left\langle\xi^{l}_{t_{ij}}\right\rangle^{2}+\left\langle\xi^{u}_{t_{ij}}\right\rangle^{2}\preceq\left\langle\xi^{l}_{h_{ij}}\right\rangle^{2}+\left\langle\xi^{u}_{h_{ij}}\right\rangle^{2}$ and $\left\langle\varphi^{l}_{t_{ij}}\right\rangle^{2}+\left\langle\varphi^{u}_{t_{ij}}\right\rangle^{2}\geqslant\left\langle\varphi^{l}_{h_{ij}}\right\rangle^{2}+\left\langle\varphi^{u}_{h_{ij}}\right\rangle^{2}$ or $\widetilde{J}_a\preceq\widetilde{W}_a$, then $\mathrm{QPPNNIVFWA}\left\langle\widetilde{J}_1,\widetilde{J}_2,\dots,\widetilde{J}_n\right\rangle\preceq\mathrm{QPPNNIVFWA}\left\langle\widetilde{W}_1,\widetilde{W}_2,\dots,\widetilde{W}_n\right\rangle$ (monotonicity property).

Proof. For any i, $\sigma_{t_{ij}} \preceq \alpha_{h_{ij}}$. Therefore, $\Cup_{a\twoheadrightarrow 1}^{\eta}\sigma_{t_{ij}} \preceq \Cup_{a\twoheadrightarrow 1}^{\eta}\alpha_{h_{ij}}$. For any i, $\left\langle\theta_{t_{ij}}^{l}\right\rangle^2+\left\langle\theta_{t_{ij}}^{u}\right\rangle^2 \preceq \left\langle\theta_{h_{ij}}^{l}\right\rangle^2+\left\langle\theta_{h_{ij}}^{u}\right\rangle^2$. Therefore, $1-\langle\theta_{t_a}^{l}\rangle^2+1-\langle\theta_{t_a}^{u}\rangle^2 \geqslant 1-\left\langle\theta_{h_a}^{l}\right\rangle^2+1-\left\langle\theta_{h_a}^{u}\right\rangle^2$. Hence, $\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\theta_{t_a}^{l}\rangle^2\right\rangle^{\mathfrak{R}_a}+\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\theta_{t_a}^{u}\rangle^2\right\rangle^{\mathfrak{R}_a} \geqslant \Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\left\langle\theta_{h_a}^{l}\right\rangle^2\right\rangle^{\mathfrak{R}_a}+\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\left\langle\theta_{h_a}^{u}\right\rangle^2\right\rangle^{\mathfrak{R}_a}$ and

$$\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\theta_{t_a}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}+\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\theta_{t_a}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}$$
$$\preceq \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\left\langle\theta_{h_a}^{l}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}+\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\left\langle\theta_{h_a}^{u}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}.$$

For any i, $\left\langle\varphi_{t_{ij}}^{l}\right\rangle^2+\left\langle\varphi_{t_{ij}}^{u}\right\rangle^2 \geqslant \left\langle\varphi_{h_{ij}}^{l}\right\rangle^2+\left\langle\varphi_{h_{ij}}^{u}\right\rangle^2$. Therefore,

$$-\frac{\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\varphi_{t_{ij}}^{l}\right\rangle^2+\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\varphi_{t_{ij}}^{u}\right\rangle^2}{2} \preceq -\frac{\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\varphi_{h_{ij}}^{l}\right\rangle^2+\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\varphi_{h_{ij}}^{u}\right\rangle^2}{2},$$

$$\frac{\Cup_{a\twoheadrightarrow 1}^{\eta}\sigma_{tij}}{2}\times\left(\frac{\frac{\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\theta_{ti}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2+\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\theta_{ti}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2}{2}}{\frac{\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\mu_{ti}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2+\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\mu_{ti}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2}{2}-\frac{\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\xi_{ti}^{l}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2+\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\xi_{ti}^{u}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2}{2}-\frac{\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\varphi_{tij}^{l}\rangle\right\rangle^2+\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\varphi_{tij}^{u}\rangle\right\rangle^2}{2}}\right)$$

$$\preceq \frac{\Cup_{a\twoheadrightarrow 1}^{\eta}\sigma_{hij}}{2}\times\left(\frac{\frac{\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\mu_{hi}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2+\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\mu_{hi}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2}{2}}{\frac{\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\theta_{hi}^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2+\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\theta_{hi}^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2}{2}-\frac{\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\xi_{hi}^{l}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2+\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\xi_{hi}^{u}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^2}{2}-\frac{\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\varphi_{hij}^{l}\rangle\right\rangle^2+\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\varphi_{hij}^{u}\rangle\right\rangle^2}{2}}\right).$$

Hence, $\text{QPPNNIVFWA}\left\langle\widetilde{J}_1,\widetilde{J}_2,\ldots,\widetilde{J}_n\right\rangle \preceq \text{QPPNNIVFWA}\left\langle\widetilde{W}_1,\widetilde{W}_2,\ldots,\widetilde{W}_n\right\rangle$. □

5.2. QPPNNIVF weighted geometric (QPPNNIVFWG) operator

Definition 5.6. Let $\widetilde{J}_a$ be the collection of QPPNNIVFNs. Then $\text{QPPNNIVFWG}\langle\widetilde{J}_1,\widetilde{J}_2,\ldots,\widetilde{J}_n\rangle = \Cap_{a\twoheadrightarrow 1}^{\eta}\widetilde{J}_a^{\mathfrak{R}_a}$.

Theorem 5.7. *Let $\widetilde{J}_a$ be the collection of QPPNNIVFNs. Then*

$$\text{QPPNNIVFWG}\langle\widetilde{J}_1,\widetilde{J}_2,\ldots,\widetilde{J}_n\rangle = \left(\begin{array}{c}\left\langle\Cap_{a\twoheadrightarrow 1}^{\eta}\sigma_a^{\mathfrak{R}_a},\Cap_{a\twoheadrightarrow 1}^{\eta}\alpha_a^{\mathfrak{R}_a}\right\rangle,\left(\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\theta_a^{l}\rangle^{\mathfrak{R}_a},\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\theta_a^{u}\rangle^{\mathfrak{R}_a}\right),\\ \left(\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\mu_a^{l}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}},\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\mu_a^{u}\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right),\\ \left(\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\xi_a^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}},\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\xi_a^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right),\\ \left(\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\varphi_a^{l}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}},\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\langle\varphi_a^{u}\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right)\end{array}\right) \quad (5.2)$$

Proof. Proof follows from Theorem 5.2. □

Theorem 5.8. *If all* $\widetilde{J}_a$ *are equal, then* $\text{QPPNNIVFWG}\langle \widetilde{J}_1, \widetilde{J}_2, \dots, \widetilde{J}_n \rangle = \widetilde{J}$.

The boundedness and monotonicity conditions are satisfied in the case of the QPPNNIVFWG operator.

5.3. Generalized QPPNNIVFWA (GQPPNNIVFWA) operator

Definition 5.9. Let $\widetilde{J}_a$ be the collection of QPPNNIVFN. Then GQPPNNIVFWA $\langle \widetilde{J}_1, \widetilde{J}_2, \dots, \widetilde{J}_n \rangle = \Big\langle \Cup_{a \twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \widetilde{J}_a^{\delta} \Big\rangle^{1/\delta}$ is called a GQPPNNIVFWA operator.

Theorem 5.10. *Let* $\widetilde{J}_a$ *be the collection of QPPNNIVFNs. Then*

$$
\begin{aligned}
&\text{GQPPNNIVFWA}\langle \widetilde{J}_1, \widetilde{J}_2, \dots, \widetilde{J}_n \rangle \\
&= \left(\begin{array}{c}
\Big\langle \Big\langle \Cup_{a \twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \sigma_a^{\delta} \Big\rangle^{1/\delta}, \Big\langle \Cup_{a \twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \alpha_a^{\delta} \Big\rangle^{1/\delta} \Big\rangle, \\
\left(\Big\langle \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \theta_a^{l} \rangle^{\delta} \Big\rangle^{2\delta} \Big\rangle^{\mathfrak{R}_a}} \Big\rangle^{1/\delta}, \Big\langle \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \theta_a^{u} \rangle^{\delta} \Big\rangle^{2\delta} \Big\rangle^{\mathfrak{R}_a}} \Big\rangle^{1/\delta} \right), \\
\left(\Big\langle \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \mu_a^{l} \rangle^{\delta} \Big\rangle^{2\delta} \Big\rangle^{\mathfrak{R}_a}} \Big\rangle^{1/\delta}, \Big\langle \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \mu_a^{u} \rangle^{\delta} \Big\rangle^{2\delta} \Big\rangle^{\mathfrak{R}_a}} \Big\rangle^{1/\delta} \right), \\
\left(\Big\langle \sqrt[\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \xi_a^{l} \rangle^{\delta} \Big\rangle^{\delta} \Big\rangle^{\mathfrak{R}_a}} \Big\rangle^{1/\delta}, \Big\langle \sqrt[\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \xi_a^{u} \rangle^{\delta} \Big\rangle^{\delta} \Big\rangle^{\mathfrak{R}_a}} \Big\rangle^{1/\delta} \right), \\
\left(\begin{array}{c}
\sqrt[2\delta]{1 - \Big\langle 1 - \Big\langle \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle \sqrt[2\delta]{1 - \Big\langle 1 - \langle \varphi_a^{l} \rangle^{2\delta} \Big\rangle^{\delta}} \Big\rangle^{\mathfrak{R}_a} \Big\rangle^{2\delta} \Big\rangle^{1/\delta}}, \\
\sqrt[2\delta]{1 - \Big\langle 1 - \Big\langle \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle \sqrt[2\delta]{1 - \Big\langle 1 - \langle \varphi_a^{u} \rangle^{2\delta} \Big\rangle^{\delta}} \Big\rangle^{\mathfrak{R}_a} \Big\rangle^{2\delta} \Big\rangle^{1/\delta}}
\end{array} \right)
\end{array} \right)
\end{aligned}
$$

Proof. It is necessary to prove that,

$$
\Cup_{a \twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \widetilde{J}_a^{\delta} = \left(\begin{array}{c}
\Big\langle \Big\langle \Cup_{a \twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \sigma_a^{\delta} \Big\rangle, \Big\langle \Cup_{a \twoheadrightarrow 1}^{\eta} \mathfrak{R}_a \alpha_a^{\delta} \Big\rangle \Big\rangle, \\
\left(\sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \theta_a^{l} \rangle^{\delta} \Big\rangle^{2\delta} \Big\rangle^{\mathfrak{R}_a}}, \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \theta_a^{u} \rangle^{\delta} \Big\rangle^{2\delta} \Big\rangle^{\mathfrak{R}_a}} \right), \\
\left(\sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \mu_a^{l} \rangle^{\delta} \Big\rangle^{2\delta} \Big\rangle^{\mathfrak{R}_a}}, \sqrt[2\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \mu_a^{u} \rangle^{\delta} \Big\rangle^{2\delta} \Big\rangle^{\mathfrak{R}_a}} \right), \\
\left(\sqrt[\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \xi_a^{l} \rangle^{\delta} \Big\rangle^{\delta} \Big\rangle^{\mathfrak{R}_a}}, \sqrt[\delta]{1 - \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle 1 - \Big\langle \langle \xi_a^{u} \rangle^{\delta} \Big\rangle^{\delta} \Big\rangle^{\mathfrak{R}_a}} \right), \\
\left(\Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle \sqrt[2\delta]{1 - \Big\langle 1 - \langle \varphi_a^{l} \rangle^{2\delta} \Big\rangle^{\delta}} \Big\rangle^{\mathfrak{R}_a}, \Cap_{a \twoheadrightarrow 1}^{\eta} \Big\langle \sqrt[2\delta]{1 - \Big\langle 1 - \langle \varphi_a^{u} \rangle^{2\delta} \Big\rangle^{\delta}} \Big\rangle^{\mathfrak{R}_a} \right)
\end{array} \right).
$$

Putting $n = 2$,

$$
\left(\begin{array}{c}
\left\langle \Re_1\sigma_1^{\delta}+\Re_2\sigma_2^{\delta},\Re_1\alpha_1^{\delta}+\Re_2\alpha_2^{\delta}\right\rangle,
\left(\begin{array}{c}
\sqrt[2\delta]{\begin{array}{c}\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\theta_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}+\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\theta_2^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\\ -\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\theta_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\cdot\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\theta_2^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\end{array}},\\
\sqrt[2\delta]{\begin{array}{c}\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\theta_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}+\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\theta_2^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\\ -\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\theta_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\cdot\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\theta_2^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\end{array}}
\end{array}\right),\\
\left(\begin{array}{c}
\sqrt[2\delta]{\begin{array}{c}\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\mu_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}+\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\mu_2^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\\ -\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\mu_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\cdot\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\mu_2^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\end{array}},\\
\sqrt[2\delta]{\begin{array}{c}\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\mu_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}+\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\mu_2^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\\ -\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\mu_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\cdot\left\langle \sqrt[2\delta]{1-\left\langle 1-\left\langle \langle\mu_2^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_1}}\right\rangle^{2\delta}\end{array}}
\end{array}\right),\\
\left(\begin{array}{c}
\sqrt[\delta]{\begin{array}{c}\left\langle \sqrt[\delta]{1-\left\langle 1-\left\langle \langle\xi_1^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_1}}\right\rangle^{\delta}+\left\langle \sqrt[\delta]{1-\left\langle 1-\left\langle \langle\xi_2^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_1}}\right\rangle^{\delta}\\ -\left\langle \sqrt[\delta]{1-\left\langle 1-\left\langle \langle\xi_1^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_1}}\right\rangle^{\delta}\cdot\left\langle \sqrt[\delta]{1-\left\langle 1-\left\langle \langle\xi_2^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_1}}\right\rangle^{\delta}\end{array}},\\
\sqrt[\delta]{\begin{array}{c}\left\langle \sqrt[\delta]{1-\left\langle 1-\left\langle \langle\xi_1^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_1}}\right\rangle^{\delta}+\left\langle \sqrt[\delta]{1-\left\langle 1-\left\langle \langle\xi_2^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_1}}\right\rangle^{\delta}\\ -\left\langle \sqrt[\delta]{1-\left\langle 1-\left\langle \langle\xi_1^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_1}}\right\rangle^{\delta}\cdot\left\langle \sqrt[\delta]{1-\left\langle 1-\left\langle \langle\xi_2^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_1}}\right\rangle^{\delta}\end{array}}
\end{array}\right),\\
\left(\left\langle \sqrt[2\delta]{1-\left\langle 1-\langle\varphi_1^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_1}\cdot\left\langle \sqrt[2\delta]{1-\left\langle 1-\langle\varphi_2^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_1},\left\langle \sqrt[2\delta]{1-\left\langle 1-\langle\varphi_1^{u}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_1}\cdot\left\langle \sqrt[2\delta]{1-\left\langle 1-\langle\varphi_2^{u}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_1}\right)
\end{array}\right)
$$

$$
=\left(\begin{array}{c}
\left\langle \uplus_{a\twoheadrightarrow 1}^{2}\Re_a\sigma_a^{\delta},\uplus_{a\twoheadrightarrow 1}^{2}\Re_a\alpha_a^{\delta}\right\rangle,\\
\left(\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\left\langle \langle\theta_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}},\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\left\langle \langle\theta_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right),\\
\left(\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\left\langle \langle\mu_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}},\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\left\langle \langle\mu_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right),\\
\left(\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\left\langle \langle\xi_1^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_a}},\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{2}\left\langle 1-\left\langle \langle\xi_1^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_a}}\right),\\
\left(\Cap_{a\twoheadrightarrow 1}^{2}\left\langle \sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_a},\Cap_{a\twoheadrightarrow 1}^{2}\left\langle \sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{u}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_a}\right)
\end{array}\right).
$$

In general,

$$\Cup_{a\twoheadrightarrow 1}^{\ell}\mathfrak{R}_a\widetilde{J}_a^{\delta}=\begin{pmatrix}\left\langle \Cup_{a\twoheadrightarrow 1}^{\ell}\mathfrak{R}_a\sigma_a^{\delta},\Cup_{a\twoheadrightarrow 1}^{\ell}\mathfrak{R}_a\alpha_a^{\delta}\right\rangle,\\ \left(\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\theta_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}},\ \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\theta_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right),\\ \left(\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\mu_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}},\ \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\mu_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right),\\ \left(\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\xi_1^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}},\ \sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\xi_1^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right),\\ \left(\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\mathfrak{R}_a},\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\mathfrak{R}_a}\right)\end{pmatrix}.$$

If $n=l+1$, then $\Cup_{a\twoheadrightarrow 1}^{\ell}\mathfrak{R}_a\widetilde{J}_a^{\delta}+\mathfrak{R}_{l+1}\widetilde{J}_{l+1}^{\delta}=\Cup_{a\twoheadrightarrow 1}^{\ell+1}\mathfrak{R}_a\widetilde{J}_a^{\delta}$. Now, $\Cup_{a\twoheadrightarrow 1}^{\ell}\mathfrak{R}_a\widetilde{J}_a^{\delta}+\mathfrak{R}_{l+1}\widetilde{J}_{l+1}^{\delta}=\mathfrak{R}_1\widetilde{J}_1^{\delta}\bigvee\mathfrak{R}_2\widetilde{J}_2^{\delta}\bigvee\cdots\bigvee\mathfrak{R}_l\widetilde{J}_l^{\delta}\bigvee\mathfrak{R}_{l+1}\widetilde{J}_{l+1}^{\delta}$,

$$=\begin{pmatrix}\left\langle \Cup_{a\twoheadrightarrow 1}^{\ell}\mathfrak{R}_a\sigma_a^{\delta}+\mathfrak{R}_{l+1}\sigma_{l+1}^{\delta},\Cup_{a\twoheadrightarrow 1}^{\ell}\mathfrak{R}_a\alpha_a^{\delta}+\mathfrak{R}_{l+1}\alpha_{l+1}^{\delta}\right\rangle,\\
\left(\sqrt[2\delta]{\begin{array}{c}\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\theta_a^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2\delta}+\left\langle\sqrt[2\delta]{1-\left\langle 1-\left\langle\langle\theta_{l+1}^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{2\delta},\\ -\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\theta_a^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2\delta}\cdot\left\langle\sqrt[2\delta]{1-\left\langle 1-\left\langle\langle\theta_{l+1}^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{2\delta}\end{array}},\ \sqrt[2\delta]{\begin{array}{c}\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\theta_a^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2\delta}+\left\langle\sqrt[2\delta]{1-\left\langle 1-\left\langle\langle\theta_{l+1}^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{2\delta}\\ -\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\theta_a^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2\delta}\cdot\left\langle\sqrt[2\delta]{1-\left\langle 1-\left\langle\langle\theta_{l+1}^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{2\delta}\end{array}}\right),\\
\left(\sqrt[2\delta]{\begin{array}{c}\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\mu_a^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2\delta}+\left\langle\sqrt[2\delta]{1-\left\langle 1-\left\langle\langle\mu_{l+1}^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{2\delta},\\ -\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\mu_a^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2\delta}\cdot\left\langle\sqrt[2\delta]{1-\left\langle 1-\left\langle\langle\mu_{l+1}^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{2\delta}\end{array}},\ \sqrt[2\delta]{\begin{array}{c}\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\mu_a^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2\delta}+\left\langle\sqrt[2\delta]{1-\left\langle 1-\left\langle\langle\mu_{l+1}^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{2\delta}\\ -\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\mu_a^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{2\delta}\cdot\left\langle\sqrt[2\delta]{1-\left\langle 1-\left\langle\langle\mu_{l+1}^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{2\delta}\end{array}}\right),\\
\left(\sqrt[\delta]{\begin{array}{c}\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\xi_a^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{\delta}+\left\langle\sqrt[\delta]{1-\left\langle 1-\left\langle\langle\xi_{l+1}^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{\delta},\\ -\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\xi_a^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{\delta}\cdot\left\langle\sqrt[\delta]{1-\left\langle 1-\left\langle\langle\xi_{l+1}^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{\delta}\end{array}},\ \sqrt[\delta]{\begin{array}{c}\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\xi_a^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{\delta}+\left\langle\sqrt[\delta]{1-\left\langle 1-\left\langle\langle\xi_{l+1}^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{\delta}\\ -\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle 1-\left\langle\langle\xi_a^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\mathfrak{R}_a}}\right\rangle^{\delta}\cdot\left\langle\sqrt[\delta]{1-\left\langle 1-\left\langle\langle\xi_{l+1}^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\mathfrak{R}_1}}\right\rangle^{\delta}\end{array}}\right),\\
\begin{pmatrix}\Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\mathfrak{R}_a}\cdot\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\varphi_{l+1}^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\mathfrak{R}_1},\\ \Cap_{a\twoheadrightarrow 1}^{\ell}\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{u}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\mathfrak{R}_a}\cdot\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\varphi_{l+1}^{u}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\mathfrak{R}_1}\end{pmatrix}\end{pmatrix}$$

$$\Cup_{a\twoheadrightarrow 1}^{\ell+1}\Re_a\widetilde{J}_a^{\delta} = \begin{pmatrix} \left\langle \Cup_{a\twoheadrightarrow 1}^{\ell+1}\Re_a\sigma_a^{\delta}, \Cup_{a\twoheadrightarrow 1}^{\ell+1}\Re_a\alpha_a^{\delta}\right\rangle, \\ \left(\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\theta_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}, \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\theta_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right), \\ \left(\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\mu_1^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}, \sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\mu_1^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right), \\ \left(\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\xi_1^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_a}}, \sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\xi_1^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_a}}\right), \\ \left(\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle \sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_a}, \Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle \sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_a}\right) \end{pmatrix}$$

and

$$\left\langle\Cup_{a\twoheadrightarrow 1}^{\ell+1}\Re_a\widetilde{J}_a^{\delta}\right\rangle^{1/\delta} = \begin{pmatrix} \left\langle \left\langle\Cup_{a\twoheadrightarrow 1}^{\ell+1}\Re_a\sigma_a^{\delta}\right\rangle^{1/\delta}, \left\langle\Cup_{a\twoheadrightarrow 1}^{\ell+1}\Re_a\alpha_a^{\delta}\right\rangle^{1/\delta}\right\rangle, \\ \left(\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\theta_a^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}, \left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\theta_a^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}\right), \\ \left(\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\mu_a^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}, \left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\mu_a^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}\right), \\ \left(\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\xi_a^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}, \left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle 1-\left\langle\langle\xi_a^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}\right), \\ \left(\begin{array}{c} \sqrt[2\delta]{1-\left\langle 1-\left\langle \Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_a}\right\rangle^{2}\right\rangle^{1/\delta}}, \\ \sqrt[2\delta]{1-\left\langle 1-\left\langle \Cap_{a\twoheadrightarrow 1}^{\ell+1}\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\varphi_a^{u}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_a}\right\rangle^{2}\right\rangle^{1/\delta}} \end{array}\right) \end{pmatrix}.$$

□

Theorem 5.11. *If all* $\widetilde{J}_a$ *are equal, then* GQPPNNIVFWA$\langle\widetilde{J}_1, \widetilde{J}_2, \dots, \widetilde{J}_n\rangle = \widetilde{J}$.

The monotonicity and boundedness characteristics are satisfied for the GQPPNNIVFWA operator.

5.4. Generalized QPPNNIVFWG (GQPPNNIVFWG) operator

Definition 5.12. Let $\widetilde{J}_a$ be a collection of QPPNNIVFNs. Then GQPPNNIVFWG $\langle\widetilde{J}_1, \widetilde{J}_2, \dots, \widetilde{J}_n\rangle = \frac{1}{\delta}\left\langle \Cap_{a\twoheadrightarrow 1}^{\eta} \langle\delta\widetilde{J}_a\rangle^{\Re_a}\right\rangle$ is called a GQPPNNIVFWG operator.

Theorem 5.13. *Let* $\widetilde{J}_a$ *be the collection of QPPNNIVFNs. Then*

$$\text{GQPPNNIVFWG}\langle\widetilde{J}_1, \widetilde{J}_2, \dots, \widetilde{J}_n\rangle = \begin{pmatrix} \left\langle \frac{1}{\delta}\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\delta\sigma_a\rangle^{\Re_a}, \frac{1}{\delta}\Cap_{a\twoheadrightarrow 1}^{\eta}\langle\delta\alpha_a\rangle^{\Re_a}\right\rangle, \\ \left(\begin{array}{c} \sqrt[2\delta]{1-\left\langle 1-\left\langle \Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\theta_a^{l}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_a}\right\rangle^{2\delta}\right\rangle^{1/\delta}}, \\ \sqrt[2\delta]{1-\left\langle 1-\left\langle \Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle\sqrt[2\delta]{1-\left\langle 1-\langle\theta_a^{u}\rangle^{2\delta}\right\rangle^{\delta}}\right\rangle^{\Re_a}\right\rangle^{2\delta}\right\rangle^{1/\delta}} \end{array}\right), \\ \left(\left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\left\langle\langle\mu_a^{l}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}, \left\langle\sqrt[\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\left\langle\langle\mu_a^{u}\rangle^{\delta}\right\rangle^{\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}\right), \\ \left(\left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\left\langle\langle\xi_a^{l}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}, \left\langle\sqrt[2\delta]{1-\Cap_{a\twoheadrightarrow 1}^{\eta}\left\langle 1-\left\langle\langle\xi_a^{u}\rangle^{\delta}\right\rangle^{2\delta}\right\rangle^{\Re_a}}\right\rangle^{1/\delta}\right) \end{pmatrix}.$$

i. The relationship between QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG operators is straightforward. If $\delta = 1$, then GQPPNNIVFWA (GQPPNNIVFWG) operator is converted to the QPPNNIVFWA (QPPNNIVFWG) operator.
ii. The monotonicity and boundedness conditions are satisfied in the case of the GQPPNNIVFWG operator.

Theorem 5.14. *If all* $\widetilde{J}_a$ *are equal, then* $\text{GQPPNNIVFWG}(\widetilde{J}_1, \widetilde{J}_2, \ldots, \widetilde{J}_n) = \widetilde{J}$.

6. MADM approach for QPPNNIVFS

In this part, we apply the similarity measures to MADM and use the similarity measures between QPPNNIVFSs to provide a handling approach for the MADM problem in the IVNS setting. The ideal point notion has been applied in MADM contexts to help in determining which option in the decision set is the best. The perfect alternative provides a helpful theoretical framework for comparing alternatives, even if it does not exist in reality. Using our suggested QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG operators, as well as the ED, HD, score, and accuracy function, which takes advantage of the preference evaluation data in the QPPNNIVFS setting. Let $J = \{J_1, J_2, \ldots, J_n\}$ be the set of n-alternatives, $C = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m\}$ be the set of m-attributes, $w = \{\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_m\}$ be the weights of attributes, for $a \twoheadrightarrow 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ and $\widetilde{J}_{ij}$ denotes QPPNNIVFN of alternative J_a in attribute $\mathscr{C}_j$ and $0 \preceq (\theta^u_{ij}(\varkappa))^2 + (\mu^u_{ij}(\varkappa))^2 + (\xi^u_{ij}(\varkappa))^2 + (\varphi^u_{ij}(\varkappa))^2 \preceq 2$.

6.1. *Algorithm for QPPNNIVF*

The algorithm for DM is as follows.

1. Enter the decision values for QPPNNIVFS.
2. Find the choice values for normalization. The decision matrix $\aleph = (\widetilde{J}_{ij})_{n\times m}$ is normalized into $\widehat{\aleph} = (\widehat{J_{ij}})_{n\times m}$ and $\widehat{J_{ij}} = \left\langle (\widehat{\sigma_{ij}}, \widehat{\alpha_{ij}}), \lfloor \widehat{\theta^l_{ij}}, \widehat{\theta^u_{ij}} \rfloor, \lfloor \widehat{\mu^l_{ij}}, \widehat{\mu^u_{ij}} \rfloor, \lfloor \widehat{\xi^l_{ij}}, \widehat{\xi^u_{ij}} \rfloor, \lfloor \widehat{\varphi^l_{ij}}\ \widehat{\varphi^u_{ij}} \rfloor \right\rangle$, $\widehat{\sigma_{ij}} = \frac{\sigma_{ij}}{\max_a(\sigma_{ij})}$, $\widehat{\alpha_{ij}} = \frac{\alpha_{ij}}{\max_a(\alpha_{ij})} \cdot \frac{\alpha_{ij}}{\sigma_{ij}}$, and
$$\widehat{\theta^l_{ij}} = \theta^l_{ij}, \widehat{\theta^u_{ij}} = \theta^u_{ij}, \widehat{\mu^l_{ij}} = \mu^l_{ij}, \widehat{\mu^u_{ij}} = \mu^u_{ij}, \widehat{\xi^l_{ij}} = \xi^l_{ij}, \widehat{\xi^u_{ij}} = \xi^u_{ij}, \widehat{\varphi^l_{ij}} = \varphi^l_{ij}, \widehat{\varphi^u_{ij}} = \varphi^u_{ij}. \tag{6.1}$$
3. Determine aggregate values for each alternative. Using QPPNNIVFS information, find attribute $\mathscr{C}_j$ in $\widetilde{J}_a$ and $\widehat{J_{ij}}$ is aggregated into $\widehat{J_a}$.
4. Determine each alternative's ideal values, both positive and negative, including
$$\begin{aligned} \widehat{J}^+ &= \left\langle \left(\max_{1\preceq i\preceq n} (\widehat{\sigma_{ij}}), \min_{1\preceq i\preceq n} (\widehat{\alpha_{ij}}) \right), \lfloor 1,1 \rfloor, \lfloor 1,1 \rfloor, \lfloor 1,1 \rfloor, \lfloor 0,0 \rfloor \right\rangle, \\ \widehat{J}^- &= \left\langle \left(\min_{1\preceq i\preceq n} (\widehat{\sigma_{ij}}), \max_{1\preceq i\preceq n} (\widehat{\alpha_{ij}}) \right), \lfloor 0,0 \rfloor, \lfloor 0,0 \rfloor, \lfloor 0,0 \rfloor, \lfloor 1,1 \rfloor \right\rangle. \end{aligned} \tag{6.2}$$
5. Determine the EDs (HDs) between each option using two ideal values, for example
$$\aleph^+_a = \aleph_E\left(\widehat{J}_i, \widehat{J}^+\right), \quad \aleph^-_a = \aleph_E\left(\widehat{J}_i, \widehat{J}^-\right), \quad \aleph^+_a = \aleph_H\left(\widehat{J}_i, \widehat{J}^+\right), \quad \aleph^-_a = \aleph_H\left(\widehat{J}_i, \widehat{J}^-\right). \tag{6.3}$$
The accuracy and score values of each choice are then calculated using our recommended methodology in the manner described below:
$$S(J) = \frac{\sigma}{2}\left(\frac{(\theta^l)^2 + (\theta^u)^2}{2} + \frac{(\mu^l)^2 + (\mu^u)^2}{2} - \frac{(\xi^l)^2 + (\xi^u)^2}{2} - \frac{(\varphi^l)^2 + (\varphi^u)^2}{2} \right), \ S(J) \in \lfloor -1, 1 \rfloor,$$
and
$$H(J) = \frac{\alpha}{2}\left(\frac{(\theta^l)^2 + (\theta^u)^2}{2} + \frac{(\mu^l)^2 + (\mu^u)^2}{2} + \frac{(\xi^l)^2 + (\xi^u)^2}{2} + \frac{(\varphi^l)^2 + (\varphi^u)^2}{2} \right), \ H(J) \in [0, 1].$$

6. Determine the relative proximity values as

$$\aleph_{a}^{*} = \frac{\aleph_{a}^{-}}{\aleph_{a}^{+} + \aleph_{a}^{-}}. \tag{6.4}$$

7. The option with the highest score value is the best or ideal choice. The other selections are similarly sorted in decreasing order based on their score values. This is the best solution to the problem at hand since the result indicates that max $\aleph_{a}^{*}$ is the ideal value.

Flowchart of the algorithm for the MADM utilizing QPPNNIVFS is displayed in Figure 1.

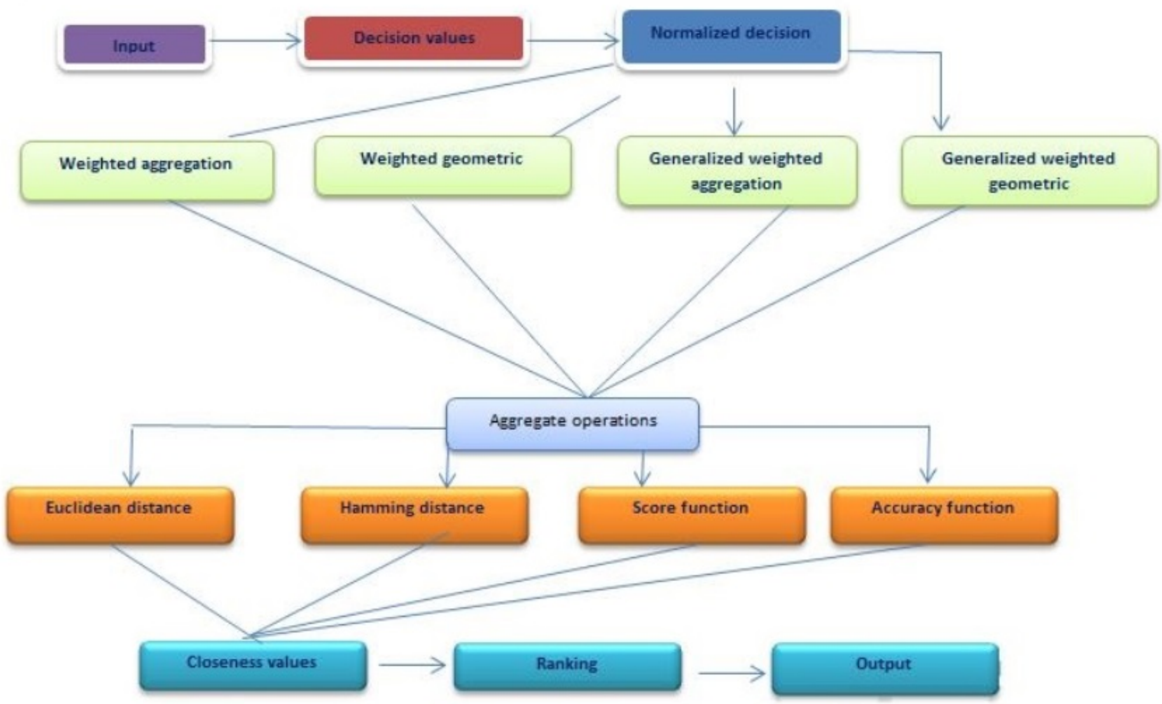

Figure 1: Flowchart of the algorithm.

The following example stated algorithm's real-world application. We examine the following problem in order to properly apply the suggested method in a practical setting:

7. Applications in cluster analysis related scenarios

The use of the proposed DM approach and its effectiveness are illustrated in this section using an example for the MADM problem of choices. Cluster analysis (CA) can be used as a data reduction tool or as a technique for exploratory data analysis (EDA). It can be used to group cases based on their profile on a set of variables so that clusters of similar cases can be examined separately or as the cases for additional analysis. The clusters are seen as indicating significant subgroups of the modeled cases, and CA is occasionally employed for creation or confirmation. These kinds of applications are widespread in psychology and education. Numerous heuristic optimization techniques and criteria for determining optimal solutions have been developed within each of these broad categories of clustering models. Alternative approaches to fitting partitions include, for instance, multivariate mixture models and latent class or latent profile cluster algorithms. A quick overview of these methodological definitions and differences is given in the section that follows. Unless there is a mathematical justification for selecting one cluster technique over another, the clustering strategy must be selected empirically. Note that a method that performs well on one set of data will not perform well on another. A cluster analysis can be performed using a variety of techniques, which fall into the following categories.

1. In hierarchical cluster analysis, a cluster is first formed, then it is then joined with the closest and most similar cluster to form a single cluster. This method is repeated until all of the subjects are in a single cluster. This particular method is called an agglomerative technique. Creating clusters out of individual items is the first step in agglomerative clustering. The divisive strategy, which starts by clustering the full data set before starting to divide into divisions, is another kind of hierarchical approach. Hierarchical cluster analysis (HCA), sometimes referred to as hierarchical clustering, groups objects into a cluster hierarchy without forcing them into a linear order. Numerous disciplines, such as the social sciences, biology, and image analysis, use hierarchical clustering techniques to analyze and spot patterns in data. Examples of use include classifying populations in clinical research, segmenting customers, and finding communities of nodes in network models. Hierarchical clustering technique has significant computational costs. Although a heap might expedite computation, it also raises memory requirements. Both agglomerative and divisive clustering are "greedy," meaning that the algorithm selects the clusters that should split or combine based on the locally optimal choice at each stage. Using a stop condition is an additional choice, whereby the algorithm stops cluster agglomeration or splitting when it reaches a predetermined number of clusters.

2. Centroid-based Clustering: This kind of clustering uses a central entity, which may or may not be present in the data collection, to represent the clusters. This method employs the K-Means clustering technique, in which items are assigned to the closest cluster centers, with k serving as the cluster centers. The goal of centroid-based clustering, a crucial technique in unsupervised machine learning, is to divide a collection of data points into discrete clusters, each of which is identified by its centroid, such as the cluster mean. This method is noteworthy because it is straightforward and effective, which makes it appropriate for big datasets in a variety of fields, including anomaly identification, market segmentation, and picture compression. The most well-known technique in this category is k-means clustering, which minimizes the variance within each cluster by repeatedly improving cluster centroids. Further study on more robust variants and starting techniques is necessary to improve centroid-based clustering's effectiveness and usability in complex datasets, since it can be vulnerable to outliers and initial centroid selection despite its computing efficiency. The idea of clustering has been around for a few decades. Its first formal applications in computer science appeared in the 1960s, and it gained significant popularity in the 1980s with the release of the k-means algorithm, especially because of its application in a variety of real-world and academic contexts.

3. Distribution-based Clustering: It is a kind of clustering model that has a tight relationship to statistics based on distribution modalities. A single cluster is created from items that are part of the same distribution.Certain complicated object qualities, such as correlation and attribute dependency, can be captured by this kind of grouping. One kind of clustering method is distribution-based clustering, which divides data points into clusters according to the data's probability distribution. This method makes the assumption that a variety of underlying probability distributions, each of which corresponds to a cluster, are used to create the data. Finding the underlying distributions and allocating each data point to the cluster most likely to have produced it are the objectives of distribution-based clustering.

4. Density-based Clustering: Areas of density that are bigger than the rest of the data set generate clusters in this kind of clustering. In sparse regions, clusters are frequently separated by items. The noise and the graph's border points are typically found in these sparse points.DBSCAN is the most widely used technique for this kind of clustering. Cluster analysis is one of the primary problems with data analysis. Clustering is a technique used by data scientists to find broken servers, group genes with related expression patterns, spot irregularities in biomedical imaging, and more. Of the various families of data clustering algorithms, DBSCAN and K-means are the most well-known, and you may be familiar with them. K-means clusters the data and finds k centroids, or the center of a data cluster, by assigning points to the nearest centroid.

5. Social network analysis: is a method for studying social structures by examining relationships (ties, edges, or links) between actors (individuals, groups, or organizations) represented as nodes in a network. It leverages graph and mathematical principles to analyze the patterns and dynamics of these relationships, revealing insights into how information, resources, and influence flow within a social system.

The clustering analysis uses five sorts of criteria:

1. Minimum number of parameters that need to be specified: The complexity of the system and the kind of model being employed determine how many parameters are required to represent it. Slope and intercept are the only two parameters required for basic linear models, such as linear regression. On the other hand, more intricate models or systems, such as neural networks, may need dozens or even billions of parameters.
2. Superior performance as measured by the most validity indicators: Excellent performance, as measured by several validity indicators, indicates that the measurement instrument or technique is successfully capturing the desired construct or trait, reducing the impact of unrelated variables, and guaranteeing that the observed outcomes accurately represent the phenomenon of interest. It basically indicates that the evaluation is precise, trustworthy, and offers insightful information about the variable being assessed.
3. High execution speed/minimum time requirement: A key component of performance optimization in computer science and software engineering is high execution speed, or decreasing execution time. It describes a system's or algorithm's capacity to do tasks as fast as feasible, frequently within a certain time limit. This is especially crucial for high-frequency trading, real-time systems, and any other application where quick reaction times are necessary.
4. Software availability: A method for classifying related software modules or components according to their availability attributes is called software availability cluster analysis. This aids in comprehending a software system's dependencies, possible weak areas, and general resilience. Developers can enhance system resilience, prioritize optimization efforts, and perhaps anticipate future availability problems by locating groups of modules with comparable availability patterns.

The weights $w = \{0.4, 0.3, 0.2, 0.1\}$ represent their respective values. The five possible choices will be evaluated using the four criteria listed above in the form of QPPNNIVFNs. Our goal is to select the best option from a selected number of alternatives by comparing the criteria. Our objective is to choose the most effective cluster analysis approaches.

Tables 1 to 2 demonstrate the following choice matrix.

Table 1: QPPNNIVFN preference information of the alternatives.

Choice	$\mathscr{C}_1$	$\mathscr{C}_2$
$\mathscr{P}$	$\langle(0.9, 0.8), [0.6, 0.65], [0.55, 0.65], [0.65, 0.7], [0.45, 0.55]\rangle$	$\langle(0.55, 0.5), [0.45, 0.55] [0.25, 0.35], [0.55, 0.7], [0.65, 0.7]\rangle$
$\mathscr{Q}$	$\langle(0.75, 0.65), [0.55, 0.6], [0.6, 0.7], [0.55, 0.6], [0.4, 0.45]\rangle$	$\langle(0.6, 0.45), [0.55, 0.6] [0.45, 0.5], [0.45, 0.6], [0.55, 0.6]\rangle$
$\mathscr{R}$	$\langle(0.65, 0.6), [0.45, 0.5], [0.7, 0.75], [0.3, 0.35], [0.55, 0.6]\rangle$	$\langle(0.45, 0.35), [0.35, 0.45] [0.55, 0.6], [0.75, 0.8], [0.45, 0.5]\rangle$
$\mathscr{S}$	$\langle(0.8, 0.7), [0.65, 0.7], [0.45, 0.5], [0.45, 0.5], [0.35, 0.4]\rangle$	$\langle(0.35, 0.25), [0.25, 0.35] [0.65, 0.7], [0.55, 0.65], [0.65, 0.75]\rangle$
$\mathscr{T}$	$\langle(0.85, 0.75), [0.75, 0.8], [0.5, 0.55], [0.4, 0.45], [0.25, 0.35]\rangle$	$\langle(0.65, 0.5), [0.45, 0.5] [0.7, 0.75], [0.45, 0.55], [0.6, 0.65]\rangle$

Table 2: QPPNNIVFN preference information of the alternatives.

Choice	$\mathscr{C}_3$	$\mathscr{C}_4$
$\mathscr{P}$	$\langle(0.45, 0.4), [0.55, 0.6],$ $[0.25, 0.4], [0.5, 0.55], [0.5, 0.6]\rangle$	$\langle(0.25, 0.2), [0.35, 0.45]$ $[0.65, 0.75], [0.45, 0.55], [0.55, 0.6]\rangle$
$\mathscr{Q}$	$\langle(0.4, 0.35), [0.45, 0.5],$ $[0.3, 0.5], [0.55, 0.6], [0.55, 0.65]\rangle$	$\langle(0.3, 0.25), [0.55, 0.6]$ $[0.7, 0.75], [0.6, 0.65], [0.45, 0.5]\rangle$
$\mathscr{R}$	$\langle(0.35, 0.3), [0.7, 0.75],$ $[0.45, 0.6], [0.45, 0.55], [0.65, 0.7]\rangle$	$\langle(0.6, 0.55), [0.65, 0.7]$ $[0.55, 0.6], [0.65, 0.7], [0.35, 0.4]\rangle$
$\mathscr{S}$	$\langle(0.6, 0.55), [0.8, 0.85],$ $[0.6, 0.7], [0.6, 0.65], [0.7, 0.75]\rangle$	$\langle(0.35, 0.3), [0.6, 0.65]$ $[0.45, 0.55], [0.45, 0.55], [0.3, 0.45]\rangle$
$\mathscr{T}$	$\langle(0.35, 0.3), [0.65, 0.7],$ $[0.75, 0.8], [0.7, 0.75], [0.45, 0.55]\rangle$	$\langle(0.4, 0.35), [0.7, 0.75]$ $[0.5, 0.65], [0.55, 0.6], [0.4, 0.5]\rangle$

By equation (6.1), Tables 3 and 4 provide a normalized decision matrix.

Table 3: Normalized QPPNNIVFN preference information of the alternatives.

Choice	$\mathscr{C}_1$	$\mathscr{C}_2$
$\mathscr{P}$	$\langle(1, 0.8889), [0.6, 0.65],$ $[0.55, 0.65], [0.65, 0.7], [0.45, 0.55]\rangle$	$\langle(0.8462, 0.9091), [0.45, 0.55]$ $[0.25, 0.35], [0.55, 0.7], [0.65, 0.7]\rangle$
$\mathscr{Q}$	$\langle(0.8333, 0.7042), [0.55, 0.6],$ $[0.6, 0.7], [0.55, 0.6], [0.4, 0.45]\rangle$	$\langle(0.9231, 0.6750), [0.55, 0.6]$ $[0.45, 0.5], [0.45, 0.6], [0.55, 0.6]\rangle$
$\mathscr{R}$	$\langle(0.7222, 0.6923), [0.45, 0.5],$ $[0.7, 0.75], [0.3, 0.35], [0.55, 0.6]\rangle$	$\langle(0.6923, 0.5444), [0.35, 0.45]$ $[0.55, 0.6], [0.75, 0.8], [0.45, 0.5]\rangle$
$\mathscr{S}$	$\langle(0.8889, 0.7656), [0.65, 0.7],$ $[0.45, 0.5], [0.45, 0.5], [0.35, 0.4]\rangle$	$\langle(0.5385, 0.3570), [0.25, 0.35]$ $[0.65, 0.7], [0.55, 0.65], [0.65, 0.75]\rangle$
$\mathscr{T}$	$\langle(0.9444, 0.8272), [0.75, 0.8],$ $[0.5, 0.55], [0.4, 0.45], [0.25, 0.35]\rangle$	$\langle(1, 0.7692), [0.45, 0.5]$ $[0.7, 0.75], [0.45, 0.55], [0.6, 0.65]\rangle$

Table 4: Normalized QPPNNIVFN preference information of the alternatives.

Choice	$\mathscr{C}_3$	$\mathscr{C}_4$
$\mathscr{P}$	$\langle(0.75, 0.6465), [0.55, 0.6],$ $[0.25, 0.4], [0.5, 0.55], [0.5, 0.6]\rangle$	$\langle(0.4167, 0.2909), [0.35, 0.45]$ $[0.65, 0.75], [0.45, 0.55], [0.55, 0.6]\rangle$
$\mathscr{Q}$	$\langle(0.6667, 0.5568), [0.45, 0.5],$ $[0.3, 0.5], [0.55, 0.6], [0.55, 0.65]\rangle$	$\langle(0.5, 0.3788), [0.55, 0.6]$ $[0.7, 0.75], [0.6, 0.65], [0.45, 0.5]\rangle$
$\mathscr{R}$	$\langle(0.5833, 0.4675), [0.7, 0.75],$ $[0.45, 0.6], [0.45, 0.55], [0.65, 0.7]\rangle$	$\langle(1, 0.9167), [0.65, 0.7]$ $[0.55, 0.6], [0.65, 0.7], [0.35, 0.4]\rangle$
$\mathscr{S}$	$\langle(1, 0.9167), [0.8, 0.85],$ $[0.6, 0.7], [0.6, 0.65], [0.7, 0.75]\rangle$	$\langle(0.5833, 0.4675), [0.6, 0.65]$ $[0.45, 0.55], [0.45, 0.55], [0.3, 0.45]\rangle$
$\mathscr{T}$	$\langle(0.5833, 0.4675), [0.65, 0.7],$ $[0.75, 0.8], [0.7, 0.75], [0.45, 0.55]\rangle$	$\langle(0.6667, 0.5568), [0.7, 0.75]$ $[0.5, 0.65], [0.55, 0.6], [0.4, 0.5]\rangle$

By equation (5.1), and using the QPPNNIVFWA operation we aggregate the data for each possibility as indicated in Table 5.

Table 5: Weighted QPPNNIVFN information of the alternatives.

Choice	QPPNNIVFWA operator ($\delta = 1$)
$\widehat{\mathscr{P}}$	$\langle(0.8455, 0.7867), [0.5312, 0.5964], [0.4544, 0.5600], [0.5759, 0.6612], [0.5236, 0.6069]\rangle$
$\widehat{\mathscr{Q}}$	$\langle(0.7936, 0.6334), [0.5325, 0.5826], [0.5319, 0.6253], [0.5277, 0.6053], [0.4746, 0.5336]\rangle$
$\widehat{\mathscr{R}}$	$\langle(0.7132, 0.6254), [0.5228, 0.5832], [0.6065, 0.6711], [0.5430, 0.6075], [0.5118, 0.5626]\rangle$
$\widehat{\mathscr{S}}$	$\langle(0.7754, 0.6435), [0.6214, 0.6797], [0.5543, 0.6213], [0.5141, 0.5861], [0.4767, 0.5542]\rangle$
$\widehat{\mathscr{T}}$	$\langle(0.8611, 0.7108), [0.6616, 0.7138], [0.6326, 0.6918], [0.5056, 0.5715], [0.3832, 0.4780]\rangle$

Using equation (6.2), the ideal values for the following options are

$$\langle(0.8611, 0.6254), [1, 1], [1, 1], [1, 1], [0, 0]\rangle, \quad \langle(0.7132, 0.7867), [0, 0], [0, 0], [0, 0], [1, 1]\rangle.$$

With the help of the decision matrix $\aleph = (\widetilde{J}_{ij})_{n\times m}$, we can determine the optimal option. By using equation (6.3), the ED of each choice with ideal values are

$$\aleph_1^+ = 0.5196, \quad \aleph_2^+ = 0.5105, \quad \aleph_3^+ = 0.5803, \quad \aleph_4^+ = 0.4681, \quad \aleph_5^+ = 0.2490,$$

and

$$\aleph_1^- = 0.4985, \quad \aleph_2^- = 0.5008, \quad \aleph_3^- = 0.4320, \quad \aleph_4^- = 0.5437, \quad \aleph_5^- = 0.7677.$$

Equation (6.4) may then be used to determine the relative closeness values of $\aleph_i^*(i = 1, 2, 3, 4, 5)$ as follows:

$$\aleph_1^* = 0.4896, \quad \aleph_2^* = 0.4952, \quad \aleph_3^* = 0.4268, \quad \aleph_4^* = 0.5374, \quad \aleph_5^* = 0.7551.$$

Thus, the five options are ranked in the following order: $\mathscr{T}$, $\mathscr{S}$, $\mathscr{Q}$, $\mathscr{P}$, and $\mathscr{R}$. Clearly, $\mathscr{T}$ is the most successful clustering analysis technique.

7.1. Comparison of the proposed and existing models with discussion

The significance of score values in the MADM process cannot be overstated. However, there have been very few research on MADM. Score values have been utilized in these research as selection criteria for alternatives such vague sets, NSs, IVNSs. We plan to examine other methods of score value calculation in this study to see if the proposed DM approach can be used in real-world scenarios. In addition to MADM and clustering analysis, fuzzy information measures may be utilized for picture segmentation. However, occasionally it seems like the results of both activities are similar. In any case, the results will probably vary. One way to rank alternatives in MADM situations is to use fuzzy entropy or fuzzy knowledge measures. To illustrate the value and benefits of our models, we contrast them with those that already exist. The approaches that we employ are QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG with corresponding ED, HD, score, and accuracy values. Using ED, HD, score, and accuracy values as in the Table 11 devoted the various AOs for QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG. Yang et al. [47] dealt with the IVPFNN idea for MADM. The concept of PNNIVFS was put out by Palanikumar et al. [28]. Palanikumar et al. [29] wrote about the concept of MADM based on spherical vague NFNs and their AOs. Palanikumar et al. [30] dealt with the generalized Fermatean normals that are used by the different AOs. Razaka et al. [34] discussed the concept of IVPNSS and its characteristics.

We might try several strategies in the previously given scenario. The data comes from Tables 6-11.

Table 6: Proposed methods.

Proposed $\delta = 1$	WA	WG	GWA	GWG
TOPSIS based on Euclidean distance	$\Gamma \succ \Xi \succ \daleth \succ \beth \succ \gimel$	$\Gamma \succ \daleth \succ \beth \succ \Xi \succ \gimel$	$\Gamma \succ \Xi \succ \daleth \succ \beth \succ \gimel$	$\Gamma \succ \daleth \succ \beth \succ \Xi \succ \gimel$
TOPSIS based on Hamming distance	$\Gamma \succ \Xi \succ \beth \succ \daleth \succ \gimel$	$\Gamma \succ \daleth \succ \beth \succ \Xi \succ \gimel$	$\Gamma \succ \Xi \succ \beth \succ \daleth \succ \gimel$	$\Gamma \succ \daleth \succ \beth \succ \Xi \succ \gimel$
Score values	$\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$	$\Gamma \succ \daleth \succ \gimel \succ \Xi \succ \beth$	$\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$	$\Gamma \succ \daleth \succ \gimel \succ \Xi \succ \beth$
Accuracy values	$\Gamma \succ \beth \succ \gimel \succ \Xi \succ \daleth$	$\Gamma \succ \beth \succ \gimel \succ \Xi \succ \daleth$	$\Gamma \succ \beth \succ \gimel \succ \Xi \succ \daleth$	$\Gamma \succ \beth \succ \gimel \succ \Xi \succ \daleth$

Table 7: Existing methods.

$\delta = 1$	WA	WG	GWA	GWG
TOPSIS based on ED [28]	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$
TOPSIS based on HD	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$
Score values	$\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$	$\Gamma \succ \daleth \succ \beth \succ \gimel \succ \Xi$	$\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$	$\Gamma \succ \daleth \succ \beth \succ \gimel \succ \Xi$
Accuracy values	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$

Table 8: Existing methods.

$\delta = 1$	WA	WG	GWA	GWG
TOPSIS based on ED [47]	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$
TOPSIS based on HD	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$
Score values	$\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$	$\Gamma \succ \daleth \succ \gimel \succ \Xi \succ \beth$	$\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$	$\Gamma \succ \daleth \succ \gimel \succ \Xi \succ \beth$
Accuracy values	$\Gamma \succ \Xi \succ \beth \succ \gimel \succ \daleth$	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$	$\Gamma \succ \Xi \succ \beth \succ \gimel \succ \daleth$	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$

Table 9: Existing methods.

$\delta = 1$	WA	WG	GWA	GWG
TOPSIS based on ED [29]	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$
TOPSIS based on HD	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth \succ \gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi \succ \gimel \succ \beth$
Score values	$\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$	$\Gamma \succ \daleth \succ \beth \succ \gimel \succ \Xi$	$\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$	$\Gamma \succ \daleth \succ \beth \succ \gimel \succ \Xi$
Accuracy values	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$	$\Gamma \succ \beth \succ \Xi \succ \gimel \succ \daleth$

Table 10: Existing methods.

$\delta = 1$	WA	WG	GWA	GWG
TOPSIS based on ED [34]	$\Gamma \succ \Xi \succ \daleth$ $\gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi$ $\gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth$ $\gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi$ $\gimel \succ \beth$
TOPSIS based on Hamming distance	$\Gamma \succ \Xi \succ \daleth$ $\gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi$ $\gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth$ $\gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi$ $\gimel \succ \beth$
Score values	$\Gamma \succ \Xi \succ \gimel$ $\daleth \succ \beth$	$\Gamma \succ \daleth \succ \gimel$ $\Xi \succ \beth$	$\Gamma \succ \Xi \succ \gimel$ $\daleth \succ \beth$	$\Gamma \succ \daleth \succ \gimel$ $\Xi \succ \beth$
Accuracy values	$\Gamma \succ \Xi \succ \gimel$ $\beth \succ \daleth$	$\Gamma \succ \gimel \succ \Xi$ $\beth \succ \daleth$	$\Gamma \succ \Xi \succ \gimel$ $\beth \succ \daleth$	$\Gamma \succ \gimel \succ \Xi$ $\beth \succ \daleth$

Table 11: Existing methods.

$\delta = 1$	WA	WG	GWA	GWG
TOPSIS based on ED [30]	$\Gamma \succ \Xi \succ \daleth$ $\gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi$ $\gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth$ $\gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi$ $\gimel \succ \beth$
TOPSIS based on HD	$\Gamma \succ \Xi \succ \daleth$ $\gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi$ $\gimel \succ \beth$	$\Gamma \succ \Xi \succ \daleth$ $\gimel \succ \beth$	$\Gamma \succ \daleth \succ \Xi$ $\gimel \succ \beth$
Score values	$\Gamma \succ \Xi \succ \gimel$ $\daleth \succ \beth$	$\Gamma \succ \daleth \succ \gimel$ $\Xi \succ \beth$	$\Gamma \succ \Xi \succ \gimel$ $\daleth \succ \beth$	$\Gamma \succ \daleth \succ \gimel$ $\Xi \succ \beth$
Accuracy values	$\Gamma \succ \beth \succ \Xi$ $\gimel \succ \daleth$	$\Gamma \succ \beth \succ \gimel$ $\Xi \succ \daleth$	$\Gamma \succ \beth \succ \Xi$ $\gimel \succ \daleth$	$\Gamma \succ \beth \succ \gimel$ $\Xi \succ \daleth$

Figure 2 shows the EDs of the proposed and existing approaches.

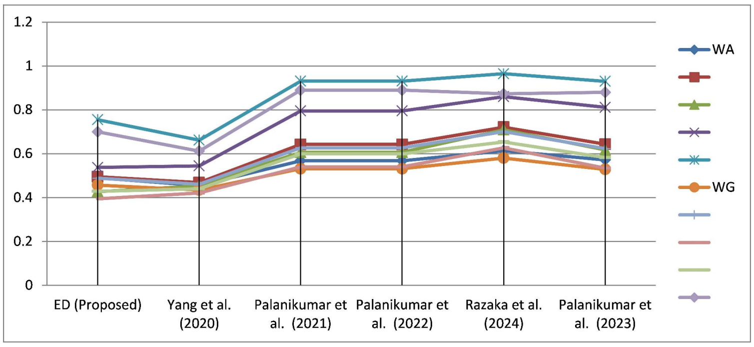

Figure 2: Comparison of the suggested and current ED-based models.

Figure 3 shows the HDs of the proposed and existing approaches.

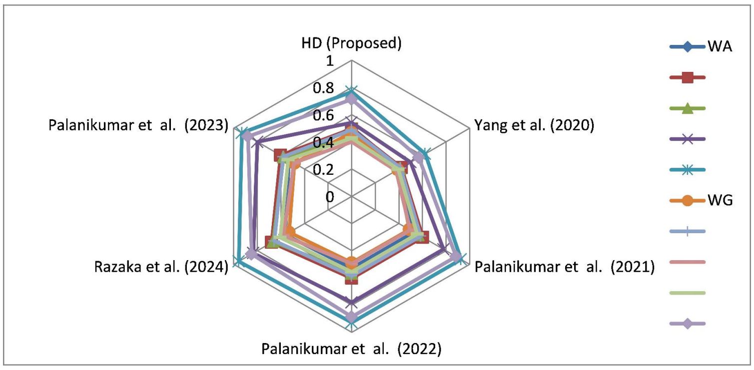

Figure 3: Comparison of the suggested and current HD-based models.

Figure 4 shows the score values of the proposed and existing approaches.

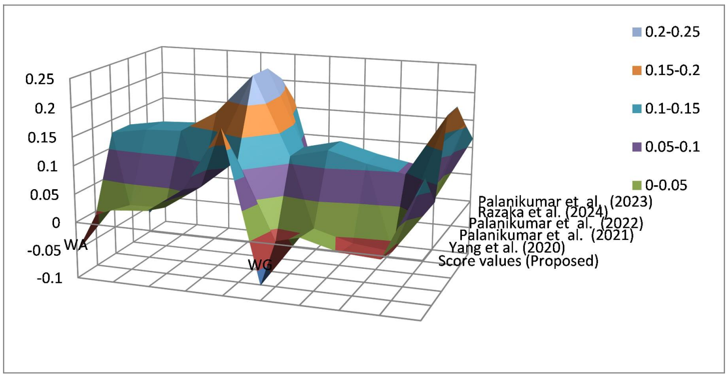

Figure 4: Comparison of the suggested and current score values

Figure 5 shows the accuracy values of the proposed and existing approaches.

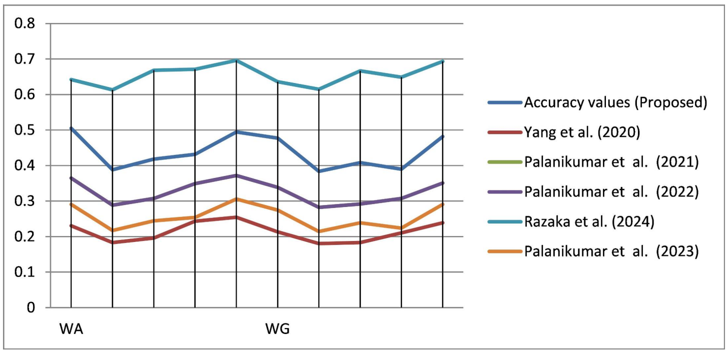

Figure 5: Comparison of the suggested and current accuracy values.

The advantages of the novel approach are demonstrated by comparing the proposed algorithm with a few existing methods. This shows whether or not the operators property is satisfied using the symbols ✓ and ×. Furthermore, it was claimed that the existing approaches excluded data on DM problems and only addressed MADM problems.

The validity of the proposed and existing models may be compared and evaluated using the ED, HD, score, and accuracy values in Table 12.

Table 12: Validity of the AO.

(ED, HD)	*WA*	*WG*	*GWA*	*GWG*
Palanikumar et al. [28]	✓	✓	✓	✓
Yang et al. [47]	✓	✓	✓	✓
Palanikumar et al. [29]	✓	✓	✓	✓
Razaka et al. [34]	✓	✓	✓	✓
Palanikumar et al. [30]	✓	✓	✓	✓

To ascertain the validity and applicability of proposed and current models, we may assess them using score and accuracy values from Table 13.

Table 13: Validity of the AO.

Score (accuracy) values	*WA*	*WG*	*GWA*	*GWG*
Palanikumar et al. [28]	✓	✓	✓	✓
Yang et al. [47]	✓	✓	✓	✓
Palanikumar et al. [29]	✓	✓	✓	✓
Razaka et al. [34]	✓	✓	✓	✓
Palanikumar et al. [30]	✓	✓	✓	✓

According to the results, the proposed method exceeds the existing ones in solving DM problems.

7.2. Effectiveness test

The dependability ratings of multiple options for the MADM approach. Several testing criteria must be met. The QPPNNIVFWA technique can be used to determine the following proximity values and rankings when different δ values are obtained. We must aggregate the data for each possibility using the different δ choices based on the QPPNNIVFWA operator. Adjust the δ values using the QPPNNIVFWA method. The δ = 2 to δ = 15 values of the QPPNNIVFWA methods are displayed in Figure 6.

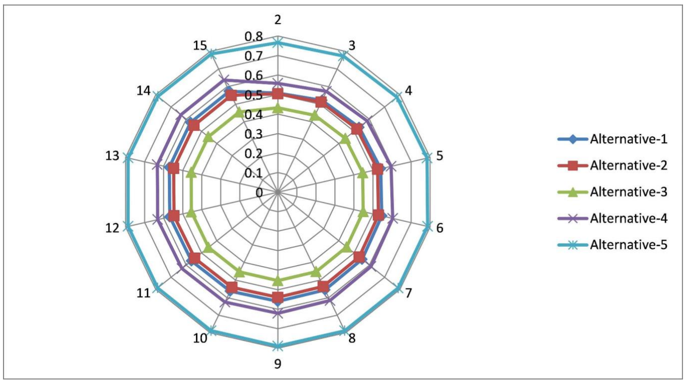

Figure 6: Various parameter values.

The δ = 16 to δ = 35 values of the QPPNNIVFWA methods are displayed in Figure 7.

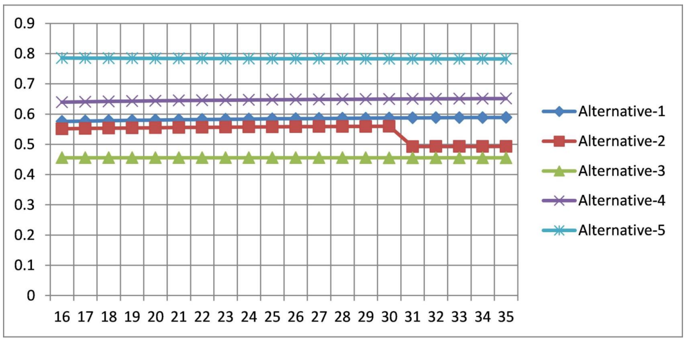

Figure 7: Various parameter values.

The δ = 36 to δ = 40 values of the QPPNNIVFWA methods are displayed in Figure 8.

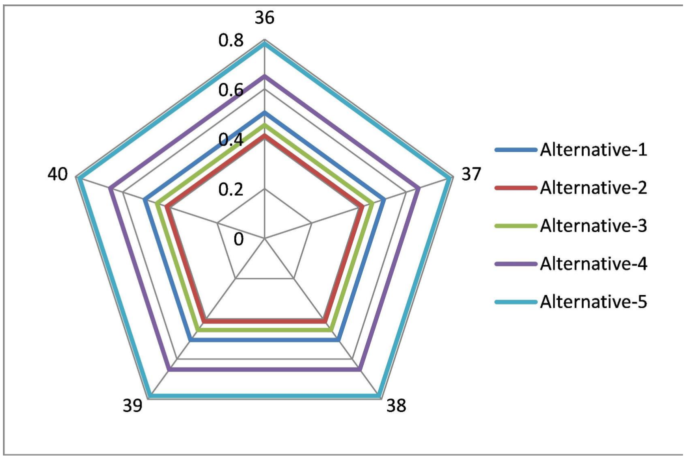

Figure 8: Various parameter values.

The $\delta = 41$ to $\delta = 50$ values of the QPPNNIVFWA methods are displayed in Figure 9.

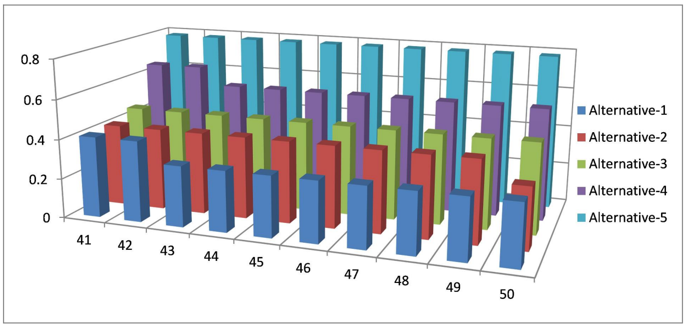

Figure 9: Various parameter values.

The $\delta = 51$ to $\delta = 60$ values of the QPPNNIVFWA methods are displayed in Figure 10.

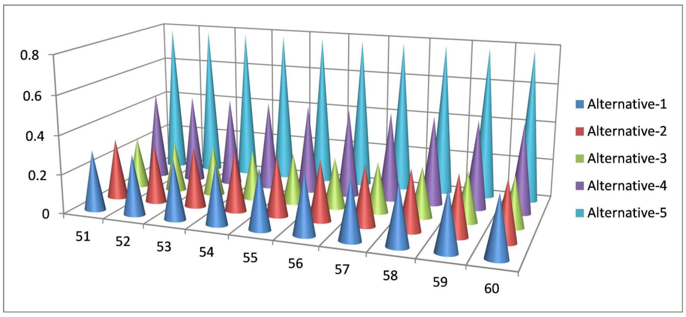

Figure 10: Various parameter values.

The $\delta = 61$ to $\delta = 69$ values of the QPPNNIVFWA methods are displayed in Figure 11.

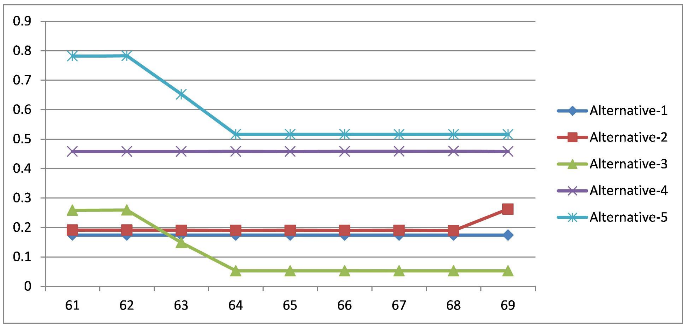

Figure 11: Various parameter values.

The $\delta = 70$ to $\delta = 71$ values of the QPPNNIVFWA methods are displayed in Figure 12.

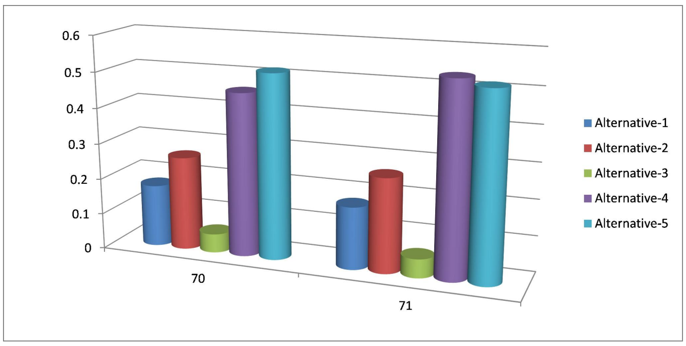

Figure 12: Various parameter values.

It is evident from the above research that Ξ is the best option for the decision maker. With QPPNNIVFN information, the suggested approach can solve the DM problem. Additionally, by highlighting the distinct similarities between the options, the greater similarity score yields a more accurate technique. When the previously described MADM problem is compared in a fuzzy system, it is shown that the same conclusion is achieved at $\delta = 71$ and that the outcome varies between Γ and Ξ. As a result, we believe that the suggested approach can be quite helpful in DM. The best outcomes with the biggest value were achieved for each procedure. To prove their superiority and validity, the suggested aggregation procedures were put to the test with the tools at hand. The precision and dependability of the suggested AO are superior to those of the current process. We proposed a new approach to solve the MADM problem by identifying the optimal solution.

7.3. *Sensitivity analysis*

The QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG operators satisfy the associativity, boundedness, and monotonicity conditions. When $\delta = 1$, this study converts the GQPPNNIVFWA operator to the QPPNNIVFWA operator. When $\delta = 1$, this study converts the GQPPNNIVFWG operator to the QPPNNIVFWG operator. The earlier information indicates that the alternate ranking is based on the QPPNNIVFWA operator. If $2 \preceq \delta \preceq 35$, then ranking of alternative is $\Gamma \succ \Xi \succ \beth \succ \daleth \succ \gimel$, if $36 \preceq \delta \preceq 40$, then the new alternative ranking is $\Gamma \succ \Xi \succ \beth \succ \gimel \succ \daleth$, if $41 \preceq \delta \preceq 50$, then ranking of alternative again changes into $\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$. If $51 \preceq \delta \preceq 60$, then ranking of alternative new changes is $\Gamma \succ \Xi \succ \daleth \succ \beth \succ \gimel$. If $61 \preceq \delta \preceq 62$, then the new alternative ranking is $\Gamma \succ \Xi \succ \gimel \succ \daleth \succ \beth$. If $63 \preceq \delta \preceq 70$, then ranking of alternative again changes into $\Gamma \succ \Xi \succ \daleth \succ \beth \succ \gimel$. If $\delta = 71$, then ranking of alternative in a new order is $\Xi \succ \Gamma \succ \daleth \succ \beth \succ \gimel$. Thus, the best option is to go from Γ to Ξ. Similarly, the alternate ranks are based on QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG operators with δ values. Using δ, alternative rankings are obtained from the QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG. The earlier approach is advantageous since it considers the connections among the various qualities. Thus, the recommended method produces superior results. Thus, as shown in [28–30, 34, 47], this method is more effective. This study established the ED, HD, score, and accuracy values for QPPNNIVFN. It was demonstrated through comparison that the accuracy, score, HD, and ED values were better. We developed an innovative concept for the ED, HD, score, and accuracy numbers for QPPNNIVFN. These numbers are provided in a straightforward mathematical manner that makes practical computations easier. As a result, a numerical illustration was provided showing the superiority of the ED, HD, score, and accuracy values. A real-world application demonstrates the importance of ED, HD, score and accuracy numbers.

7.4. *Limitations*

Distance is an important aspect while choosing the best options. In the end, the strategy with the highest value generated the best results. Using current methodologies, the proposed aggregating procedures validity and superiority were assessed. The proposed AOs exceed current approaches in terms of accuracy and reliability. This approach is effective because it evaluates the relationships between the various qualities. As a consequence, the advised approach yielded higher ranking results. Using the provided data, we can observe that our technique achieved exactly the same ranking outcomes as the existing aggregating method utilized by [28–30, 34, 47]. The ED, HD, score, and accuracy numbers for QPPNNIVFN are computed in this work. As a result, a numerical example demonstrated that including both of these variables improved the ED, HD, score, and accuracy values. A real-world example demonstrates the actual application of ED, HD, score, and accuracy measurements. To address the MADM problem optimally, we recommend using these AOs. This example helps comparison by identifying numerous options and alternatives based on a particular collection of characteristics.

7.5. *Advantages*

The benefits and advantages of the proposed proposal are discussed in this article. We provide the operators QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG. This method's ability to enable several specialists to assess DM operations both objectively and subjectively is one of its main advantages. Cluster analysis is a fascinating manner to look at decisions as a random collection of rules and processes. To make comparing the available alternatives simple, a variety of options and possibilities that depend on the decision-makers of an ideal set of attributes will be provided as an example. The main component of this approach is the generalized weighted operator, which is derived from the weighted aggregation model. By utilizing weighted operators that accept ED, HD, score values and accuracy, decision makers may compare their choices or alternatives to an ideal set of preferences. The main objective of this approach is to address a wider range of complex problems. In order to compare an ideal set of preferences with alternative and option choices, weighted operators introduce ED, HD, score values, and accuracy values. Along with the validity of the recommended technique and its general adaptability to handle a range of inputs and outputs, the influence of score functions, analysis, superiority, and a comparison of the proposed methodology with the present methodologies are all discussed.

8. Conclusion

In this study, we suggested AO rules for QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG. The fundamental properties and impacts of a MADM model called QPPNNIVFNs are examined in this paper. The utilization of QPPNNIVFS data can assist users in selecting the optimal option among the available options in situations with unclear and inconsistent data. The QPPNNIVFSs were given the concepts of ED, HD, score, and accuracy values. A distinct ranking of alternatives may be generated by the QPPNNIVFWA, QPPNNIVFWG, GQPPNNIVFWA, and GQPPNNIVFWG operators with δ. The scores of each alternative according to each criterion were expressed using QPPNNIVFNs. The similarity measures provide for the immediate selection of the best choice and for the development of the ranking order of each alternative. The example demonstrated the use of the developed approach. Finally, it shows that the generalized values of δ have an important impact on the ranking of alternatives. Calculating δ values according to the actual criteria for the greatest acceptable ranking allows decision makers to make the best decisions. Having δ may be necessary for the decision makers to evaluate the options in an accurate manner. Based on the values of δ, the decision maker can estimate the number of DM required to calculate the conclusion. Based on these score values, a technique for addressing DM problems in the QPPNNIVFS environment has been developed. Finally, a real-world example is conducted to evaluate the proposed method and show its usefulness. More information is provided on the following topics: IVPFSs and uncertain sets are connected by a Pythagorean hesitant form. The problem is solved by analyzing normal complex QPFSs in terms of vague sets, q-rung vague sets. We will discuss the following subjects in further detail. (1) QPPNNIVFS for interaction with AOs. (2) The complex hesitant PNNIVFS, complex QPPNNIVFS, sine trigonometric QPPNNIVFS, and Diophantine QPPNNIVFS are among them. Further research is also needed to maximize the benefits of these operators consists of FS, IVFS and other modeling tools. We will establish guidelines that will enable the most effective selection of these operators.

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