Journal of Advanced Studies in Topology 6:3 (2015), 99–106

JOURNAL OF ADVANCED STUDIES IN TOPOLOGY
journal homepage: http://www.m-sciences.com/index.php?journal=jast

On slightly b-$\mathcal{I}$-continuous functions

A. Acikgoz[a,*], R. Balaji[b], N. Rajesh[c]

[a] Department of Mathematics, Faculty of Arts and Science, Balikesir University, Balikesir, Turkey.
[b] Department of Mathematics, Agni College of Technology, Chennai-600130,TamilNadu, India.
[c] Department of Mathematics, Rajah Serfoji Govt. College, Thanjavur-613005, Tamilnadu, India.

Abstract

In this paper a new class of functions called slightly b-$\mathcal{I}$-continuous functions has been defined and studied in ideal topological spaces.

Keywords: Ideal topological spaces, b-$\mathcal{I}$open sets, slightly b-$\mathcal{I}$-continuity.
2010 MSC: 54D10.

1. Introduction

The subject of ideals in topological spaces has been studied by Kuratowski [8] and Vaidyanathasamy [12]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies:

(i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and,
(ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$.

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [12] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for each neighbourhood U of $x\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $Cl^\star(.)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $\star$-topology, which is finer than τ, is defined by $Cl^\star(A) = A \cup A^\star(\tau, \mathcal{I})$. When there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X, Cl(A)$ and $Int(A)$ will denote the closure and interior of A in $(X, \mathcal{I})$, respectively. A subset A of a topological space (X, τ) is said to be b-open [1] or γ-open [5] if $A \subset Int(Cl(A)) \cup Cl(Int(A))$. The aim of this paper is to give a new class of functions called slightly b-$\mathcal{I}$-continuous functions in ideal topological space. Some characterizations and several basic properties of this class of functions are obtained.

*Corresponding author
Email addresses: ahuacikgoz@gmail.com (A. Acikgoz), balaji_2410@yahoo.co.in. (R. Balaji), nrajesh_topology@yahoo.co.in (N. Rajesh)

Received: 14 June 2015 Accepted: 9 July 2015
http://dx.doi.org/10.20454/jast.2015.963

2. Preliminaries

A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-open [6] if $S \subset Int(Cl^{\star}(S)) \cup Cl^{\star}(Int(S))$. The complement of a b-$\mathcal{I}$-open set is called a b-$\mathcal{I}$-closed set [6]. The intersection of all b-$\mathcal{I}$-closed sets containing S is called the b-$\mathcal{I}$-closure of S and is denoted by $b\mathcal{I}Cl(S)$. The b-$\mathcal{I}$-Interior of S is defined by the union of all b-$\mathcal{I}$-open sets contained in S and is denoted by $b\mathcal{I}Int(S)$. The family of all b-$\mathcal{I}$-open (resp. b-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $B\mathcal{I}O(X)$ (resp. $B\mathcal{I}C(X)$).The family of all b-$\mathcal{I}$-open (resp. b-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $B\mathcal{I}O(X, x)$ (resp. $B\mathcal{I}CO(X, x)$). A function $f : (X, \mathcal{I}) \to (Y, \sigma)$ is said to be slightly continuous [7] (resp. slightly γ-continuous [4] if $f^{-1}(V)$ is open (resp. γ-open) in X for every clopen set V of Y . A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ is said to be b-$\mathcal{I}$-irresolute if $f^{-1}(V) \in B\mathcal{I}O(X)$ for every $V \in B\mathcal{I}O(Y)$.

3. Slightly *b*-$\mathcal{I}$-continuous functions

Definition 3.1. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called:

(1) slightly b-$\mathcal{I}$-continuous at $x \in X$ if for each clopen subset V of Y containing $f(x)$, there exists $U \in B\mathcal{I}O(X, x)$ such that $f(U) \subset V$;

(2) slightly b-$\mathcal{I}$-continuous if it is slightly b-$\mathcal{I}$-continuous at each point of X.

Theorem 3.2. The following statements are equivalent for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$:

(1) f is slightly b-$\mathcal{I}$-continuous;

(2) for every clopen subset V of Y , $f^{-1}(V)$ is b-$\mathcal{I}$-open in X;

(3) for every clopen subset V of Y , $f^{-1}(V)$ is b-$\mathcal{I}$-closed in X;

(4) for every clopen subset V of Y , $f^{-1}(V)$ is b-$\mathcal{I}$-clopen in X.

Proof. (1) $\Rightarrow$ (2): Let V be a clopen subset of Y and let $x \in f^{-1}(V)$. Since $f(x) \in V$, by (1), there exists a b-$\mathcal{I}$-open set U_x in X containing x such that $U_x \subset f^{-1}(V)$. We obtain $f^{-1}(V) = \cup_{x\in f^{-1}(V)} U_x$. Since any union of b-$\mathcal{I}$-open subsets in b-$\mathcal{I}$-open, $f^{-1}(V)$ is b-$\mathcal{I}$-open in X.

(2) $\Rightarrow$ (3): Let V be a clopen subset of Y. Then $Y \backslash V$ is also clopen. By (2), $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is b-$\mathcal{I}$-open and hence $f^{-1}(V)$ is b-$\mathcal{I}$-closed in X. The proof of reverse implication is similar.

(3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (1): Let V be a clopen subset in Y containing $f(x)$. By (4), $f^{-1}(V)$ is b-$\mathcal{I}$-clopen in X. Set $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subset V$. Hence f is slightly b-$\mathcal{I}$-continuous. □

Proposition 3.3. Every slightly b-$\mathcal{I}$-continuous function is slightly γ-continuous.

Proof. It follows from Proposition 1(e) of [6]. The converse of Proposition 3.3 is need not be true as shown by the following example. □

Example 3.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the function $f : (X, \tau, \mathcal{I}) \to (X, \tau)$ is defined by $f(a) = c$, $f(b) = a$ and $f(c) = b$ is slightly γ-continuous but not slightly b-$\mathcal{I}$-continuous.

Proposition 3.5. Every slightly continuous function slightly b-$\mathcal{I}$-continuous.

Proof. It follows from Proposition 1(a) of [6]. □

The converse of Proposition 3.5 is need not be true as shown by the following example.

Example 3.6. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the function $f : (X, \tau, \mathcal{I}) \to (X, \tau)$ is defined by $f(a) = c$, $f(b) = a$ and $f(c) = b$ is slightly γ-continuous but is not slightly b-$\mathcal{I}$-continuous.

Theorem 3.7. (1) *A function $f : (X, \tau, \{\varnothing\}) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous if and only if it is slightly γ-continuous.*

(2) *A function $f : (X, \tau, \mathcal{N}) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous if and only if it is slightly γ-continuous ($\mathcal{N}$ is the ideal of all nowhere dense sets).*

(3) *A function $f : (X, \tau, \mathcal{P}(X)) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous if and only if it is slightly continuous.*

Proof. It follows from Proposition 2 of [6]. □

Remark 3.8. The composition of two slightly b-$\mathcal{I}$-continuous need not be slightly b-$\mathcal{I}$-continuous.

Example 3.9. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{c\}, X\}$, $\sigma = \{\varnothing, \{a\}, \{b, c\}, X\}$, $\gamma = \{\varnothing, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ and the function $g : (X, \sigma, \mathcal{I}) \to (X, \gamma)$ defined by $g(a) = c$, $g(b) = a$ and $g(c) = b$ are slightly b-$\mathcal{I}$-continuous functions but their composition is not slightly b-$\mathcal{I}$-continuous. However, we have the following theorem

Theorem 3.10. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \gamma)$ be functions, then the following properties hold:

(1) If f is slightly b-$\mathcal{I}$-continuous and g is slightly continuous, then $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \gamma)$ is slightly b-$\mathcal{I}$-continuous.

(2) If f is b-$\mathcal{I}$-irresolute and g is slightly b-$\mathcal{J}$-continuous, then $g \circ f$ is slightly b-$\mathcal{I}$-continuous.

(3) If f is b-$\mathcal{I}$-irresolute and g is slightly continuous, then $g \circ f$ is slightly b-$\mathcal{I}$-continuous.

Proof. The proof is clear. □

Definition 3.11. A function f : $(X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be strongly b-$\mathcal{I}$-open if $f(U) \in B\mathcal{J}O(Y)$ for every $U \in B\mathcal{I}O(X)$.

Theorem 3.12. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$ be functions. Then the following properties hold:

(1) If f is strongly b-$\mathcal{I}$-open surjection and $g \circ f$ is slightly b-$\mathcal{I}$-continuous, then g is slightly b-$\mathcal{J}$-continuous.

(2) Let f be strongly b-$\mathcal{I}$-open and b-$\mathcal{I}$-irresolute surjection. Then g is slightly b-$\mathcal{J}$-continuous if and only if $g \circ f$ is slightly b-$\mathcal{I}$-continuous.

Proof. The proof is clear. □

Definition 3.13. [3] Let A and X_0 be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset X_0 \subset X$. Then $(X_0, \tau_{|X_0}, \mathcal{I}_{|X_0})$ is an ideal topological space with an ideal $\mathcal{I}_{|X_0} = \{\mathcal{I} \in \mathcal{I} | \mathcal{I} \subset X_0\} = \{\mathcal{I} \cap X_0 | \mathcal{I} \in \mathcal{I}\}$.

Theorem 3.14. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a slightly b-$\mathcal{I}$-continuous function and $U \in \tau$. Then the restriction $f_{|U} : (U, \tau_{|U}, \mathcal{I}_{|U}) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous.

Proof. It follows from Theorem 3.15 of [9]. □

Theorem 3.15. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function and $\{U_\lambda : \lambda \in \triangle\}$ be an open cover of X. If the restriction function $f_{|U_\lambda} : (U_\lambda, \tau_{|U_\lambda}, \mathcal{I}_{|U_\lambda})$ is $\mathcal{I}$-continuous for each $\lambda \in \triangle$, then f is slightly b-$\mathcal{I}$-continuous.

Proof. It follows from Theorem 3.15 of [9]. □

Remark 3.16. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-open if and only if for all $x \in A$,* there exists $A_x \in B\mathcal{I}O(X)$ such that $x \in A_x \subset A$.

Theorem 3.17. *A function$f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous if and only if the graph function* $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$ is slightly b-$\mathcal{I}$-continuous.

Proof. Let $x \in X$ and let W be a clopen subset of $X \times Y$ containing $g(x)$. Then $W \cap (\{x\} \times Y)$ is clopen in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to Y . Hence $\{y \in Y | (x, y) \in W\}$ is a clopen subset of Y. Since f is slightly b-$\mathcal{I}$-continuous, $\cup\{f^{-1}(y) | (x, y) \in W\}$ is a b-$\mathcal{I}$-open subset of $(X, \tau, \mathcal{I})$. Further, $x \in \cup\{f^{-1}(y) | (x, y) \in W\} \subset g^{-1}(W)$. Hence $g^{-1}(W)$ is b-$\mathcal{I}$-open. Then g is slightly b-$\mathcal{I}$-continuous.

Conversely, Let F be a clopen subset of Y . Then $X \times F$ is a clopen subset of $X \times Y$. Since g is slightly b-$\mathcal{I}$-continuous, $g^{-1}(X \times F)$ is a b-$\mathcal{I}$-open subset of X. Also, $g^{-1}(X \times F) = f^{-1}(F)$. Hence f is slightly b-$\mathcal{I}$-continuous. □

Definition 3.18. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-connected [9] if X is not the union of two disjoint nonempty b-$\mathcal{I}$-open sets of X.

Theorem 3.19. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous surjection and $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-connected, then Y is connected.

Proof. Suppose Y is not connected. Then there exist nonempty disjoint clopen subsets U and V of Y such that $Y = U \cup V$. Since f is slightly b-$\mathcal{I}$-continuous, we have $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty disjoint b-$\mathcal{I}$-closed and b-$\mathcal{I}$-open sets in X. Moreover, $f^{-1}(U) \cup f^{-1}(V) = X$. This shows that X is not b-$\mathcal{I}$-connected. This is a contradiction and hence Y is connected. □

Theorem 3.20. If f is a slightly b-$\mathcal{I}$-continuous function from a b-$\mathcal{I}$-connected space $(X, \tau, \mathcal{I})$ onto space (Y, σ), then Y is not a discrete space.

Proof. Suppose that Y is a discrete space. Let A be a proper nonempty open subset of Y. Then $f^{-1}(A)$ is any proper nonempty b-$\mathcal{I}$-open subset of $(X, \tau, \mathcal{I})$, which is a contradiction to the assumption that $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-connected. □

Theorem 3.21. An ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-connected if for every slightly b-$\mathcal{I}$-continuous function from a space $(X, \tau, \mathcal{I})$ into any T_0-space Y is constant.

Proof. Straightforward. □

Let $\{X_\alpha : \alpha \in \Lambda\}$ and $\{Y_\alpha : \alpha \in \Lambda\}$ be two families of topological spaces with the same index set Λ. The product space of $\{X_\alpha : \alpha \in \Lambda\}$ is denoted by $\prod\{X_\alpha : \alpha \in \Lambda\}$ (or simply $\prod X_\alpha$). Let $f_\alpha : X_\alpha \to Y_\alpha$ be a function for each $\alpha \in \Lambda$. The product function $f : \prod X_\alpha \to \prod Y_\alpha$ is defined by $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\} \in \prod X_\alpha$.

Theorem 3.22. If a function $f : (X, \tau, \mathcal{I}) \to \prod Y_\alpha$ is slightly b-$\mathcal{I}$-continuous, then $P_\alpha \circ f : (X, \tau, \mathcal{I}) \to Y_\alpha$ is slightly b-$\mathcal{I}$-continuous for each $\alpha \in \Lambda$, where P_α is the projection of $\prod Y_\alpha$ onto Y_α.

Proof. Let V_α be any clopen set of Y_α. Then, $P_\alpha^{-1}(V_\alpha)$ is clopen in $\prod Y_\alpha$ and hence $(P_\alpha \circ f)^{-1}(V_\alpha) = f^{-1}(P_\alpha^{-1}(V_\alpha))$ is b-$\mathcal{I}$-open in X. Therefore, $P_\alpha \circ f$ is slightly b-$\mathcal{I}$-continuous. □

Lemma 3.23. [10] For any function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, $f(\mathcal{I})$ is an ideal on Y.

Now, we recall the following definitions.

Definition 3.24. A collection $\{G_\alpha : \alpha \in \triangle\}$ is called a b-$\mathcal{I}$-open cover of a subset A of an ideal topological space $(X, \tau, \mathcal{I})$ if $A \subset \cup\{G_\alpha : X \backslash G_\alpha \in B\mathcal{I}O(X), \alpha \in \triangle\}$.

Definition 3.25. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-compact (resp. b-$\mathcal{I}$-Lindel$\ddot{o}f$) if for every b-$\mathcal{I}$-clopen (resp. b-$\mathcal{I}$-open) cover $\{W_\alpha : \alpha \in \triangle\}$ on X, there exists a finite (resp. countable) subset $\triangle_0$ of $\triangle$ such that $X \backslash \cup \{W_\alpha : \alpha \in \triangle_0\} \in \mathcal{I}$.

Definition 3.26. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be mildly b-$\mathcal{I}$-compact (resp. mildly b-$\mathcal{I}$-Lindel$\ddot{o}f$) if for every b-$\mathcal{I}$-clopen (resp. b-$\mathcal{I}$-open) cover $\{W_\alpha : \alpha \in \triangle\}$ on X, there exists a finite (resp. countable) subset $\triangle_0$ of $\triangle$ such that $X \backslash \cup \{W_\alpha : \alpha \in \triangle_0\} \in \mathcal{I}$.

Definition 3.27. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be mildly $\mathcal{I}$-compact (resp. mildly $\mathcal{I}$-Lindelöf) if for every clopen (resp. clopen) cover $\{W_\alpha : \alpha \in \triangle\}$ on X, there exists a finite (resp. countable) subset $\triangle_0$ of $\triangle$ such that $X \setminus \cup \{W_\alpha : \alpha \in \triangle_0\} \in \mathcal{I}$.

Theorem 3.28. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous surjection and $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-compact, then Y is mildly $f(\mathcal{I})$-compact.

Proof. Let $\{V_\alpha : \alpha \in \triangle\}$ be an clopen cover of Y. Since f is slightly b-$\mathcal{I}$-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in \triangle\}$ is a b-$\mathcal{I}$-open cover of X. Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-compact, there exists a finite subset $\triangle_0$ of $\triangle$ such that $X \setminus \cup \{f^{-1}(V_\alpha) : \alpha \in \triangle_0\} \in \mathcal{I}$. Thus, $Y \setminus \cup \{V_\alpha : \alpha \in \triangle_0\} \in \mathcal{I}$ and hence Y is mildly $f(\mathcal{I})$-compact. □

Theorem 3.29. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a slightly b-$\mathcal{I}$-continuous surjection. If $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-Lindelöf, then (Y, σ) is mildly $f(\mathcal{I})$-Lindelöf.

Proof. Let $\{V_\alpha : \alpha \in \triangle\}$ be an clopen cover of Y. Since f is slightly b-$\mathcal{I}$-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in \triangle\}$ is a b-$\mathcal{I}$-open cover of X. Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-Lindelöf, there exists a countable subset $\triangle_0$ of $\triangle$ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \triangle_0\}$. Thus, $Y = \cup\{V_\alpha : \alpha \in \triangle_0\}$ and hence (Y, σ) is mildly $f(\mathcal{I})$-Lindelöf. □

4. Separation axioms

Definition 4.1. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be:

(1) b-$\mathcal{I}$-T_1 [2] if for each pair of distinct points x and y of X, there exist β-$\mathcal{I}$-open sets U and V of X such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.
(2) b-$\mathcal{I}$-T_2 [2] if for each pair of distinct points x and y in X, there exists disjoint b-$\mathcal{I}$-open sets U and V in X such that $x \in U$ and $y \in V$.
(3) clopen-T_1 [11] if for each pair of distinct points x and y of X, there exist clopen sets U and V of X such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.
(4) clopen-T_2 [11] if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 4.2. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a slightly b-$\mathcal{I}$-continuous injection and Y is a clopen-T_1 space, then $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-T_1 space.

Proof. Suppose that Y is clopen-T_1. For any two distinct points x and y in X, there exist clopen sets V and W of Y such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is slightly b-$\mathcal{I}$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are b-$\mathcal{I}$-open subsets of $(X, \tau, \mathcal{I})$ such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-T_1 space. □

Theorem 4.3. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a slightly b-$\mathcal{I}$-continuous injection and Y is a clopen-T_2 space, then $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-T_2 space.

Proof. For any pair of distinct points x and y in X, there exist disjoint clopen sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is slightly b-$\mathcal{I}$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are b-$\mathcal{I}$-open sets in$(X, \tau, \mathcal{I})$ containing x and y, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \varnothing$ because $U \cap V = \varnothing$. This shows that the space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2. □

Definition 4.4. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-regular if for each closed set F and each point $x \notin F$, there exist disjoint b-$\mathcal{I}$-open sets U and V of X such that $F \subset U$ and $x \in V$.

Definition 4.5. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-normal [9] if for any pair of disjoint closed subsets F_1 and F_2 of X, there exist disjoint b-$\mathcal{I}$-open sets U and V of X such that $F_1 \subset U$ and $F_2 \subset V$.

Definition 4.6. *A* topological space (X, τ) is said to be:

(1) ultra Hausdorff [11] if every two distinct points of X can be separated by disjoint clopen sets.
(2) ultra regular [11] if each pair of a point and a closed set not containing the point can be separated by disjoint clopen sets.
(3) ultra normal [11] if every two disjoint closed sets of X can be separated by clopen sets.

Theorem 4.7. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a slightly b-$\mathcal{I}$-continuous injection. Then

(1) if (Y, σ) is ultra Hausdorff, then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2,
(2) if (Y, σ) is ultra regular and f is open or closed, then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-regular,
(3) if (Y, σ) is ultra normal and f is closed, then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-normal.

Proof. (1) Let x_1, x_2 be two distinct points of X. Then since f is injective and Y is ultra Hausdorff, there exist clopen sets V_1 and V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$, and $V_1 \cap V_2 = \varnothing$. By Theorem 3.2, $x_i \in f^{-1}(V_i) \in B\mathcal{I}O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$. Thus, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2.

(2) (i) Suppose that f is open. Let $x \in X$ and U be an open set containing x. Then $f(U)$ is an open set of Y containing $f(x)$. Since Y is ultra regular, there exists a clopen set V such that $f(x) \in V \subset f(U)$. Since f is a slightly b-$\mathcal{I}$-continuous injection, by Definition 3.1 $x \in f^{-1}(V) \subset U$ and $f^{-1}(V)$ is b-$\mathcal{I}$-clopen in X. Therefore, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-regular. (ii) Suppose that f is closed. Let $x \in X$ and F be any closed set of X not containing x. Since f is injective and closed, $f(x) \notin f(F)$ and $f(F)$ is closed in Y. By the ultra regularity of Y, there exists a clopen set V such that $f(x) \in V \subset Y \backslash f(F)$. Therefore, $x \in f^{-1}(V)$ and $F \subset X \backslash f^{-1}(V)$. By Theorem 3.2, $f^{-1}(V)$ is an b-$\mathcal{I}$-clopen set in $(X, \tau, \mathcal{I})$. Thus, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-regular.

(3). Similar to the proof of (2). □

The graph of a function $f : (X, \tau) \to (Y, \sigma)$, denoted by $G(f)$ is the set $\{(x, f(x)) : x \in X\} \subset X \times Y$.

Definition 4.8. *A graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be strongly b-$\mathcal{I}$-co-closed if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in B\mathcal{I}C(X, x)$ and a clopen set V of Y containing y such that $(U \times V) \cap G(f) = \varnothing$.*

Lemma 4.9. *A graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly b-$\mathcal{I}$-co-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in B\mathcal{I}C(X, x)$ and a clopen set V of Y containing y such that $f(U) \cap V = \varnothing$.*

Proof. It is an immediate consequence of Definition 4.8. □

Theorem 4.10. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous and Y is clopen-T_1, then $G(f)$ is strongly b-$\mathcal{I}$-closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) \backslash G(f)$, then $f(x) \neq y$ and there exists a clopen set V of Y such that $f(x) \in V$ and $y \notin V$. Since f is slightly b-$\mathcal{I}$-continuous, $f^{-1}(V) \in B\mathcal{I}C(X, x)$. Take $U = f^{-1}(V)$. We have $f(U) \subset V$. Therefore, we obtain $f(U) \cap (Y \backslash V) = \varnothing$ and $Y \backslash V$ is clopen in Y. This shows that, $G(f)$ is strongly b-$\mathcal{I}$-co-closed in $X \times Y$. □

Theorem 4.11. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is slightly b-$\mathcal{I}$-continuous and Y is clopen-T_2, then $G(f)$ is strongly b-$\mathcal{I}$-co-closed in $X \times Y$.

Proof. Similar to that of Theorem 4.10. □

Theorem 4.12. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ has a strongly b-$\mathcal{I}$-co-closed graph $G(f)$. If f is injective, then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1.

Proof. Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in (X \times Y) \backslash G(f)$. By Lemma 4.9, there exists a b-$\mathcal{I}$-clopen set U of X and a clopen set V of Y such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \varnothing$. Hence, $U \cap f^{-1}(V) = \varnothing$ and $y \notin U$. This implies that $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1. □

Theorem 4.13. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ has a strongly b-$\mathcal{I}$-co-closed graph $G(f)$. If f is surjective strongly b-$\mathcal{I}$-open, then $(Y, \sigma, \mathcal{J})$ is b-$\mathcal{J}$-T_2 space.

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y)\backslash G(f)$. By Definition 4.8, there exists a b-$\mathcal{I}$-clopen set U of X and a clopen set V of Y such that $(x, y_2) \in U \times V$ and $(U \times V) \cap G(f) = \varnothing$. Then, we have $f(U) \cap V = \varnothing$. Since f is strongly b-$\mathcal{I}$-open, then $f(U)$ is b-$\mathcal{J}$-open such that $f(x) = y_1 \in f(U)$. This implies that $(Y, \sigma, \mathcal{J})$ is b-$\mathcal{J}$-T_2. □

Definition 4.14. An ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-connected between subsets A and B if there exists no b-$\mathcal{I}$-clopen set K for which $A \subset K$ and $K \cap B = \varnothing$.

Definition 4.15. *A* function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is b-$\mathcal{I}$-set-connected if whenever X is b-$\mathcal{I}$-connected between A and B, then $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to relative topology on $f(X)$.

Theorem 4.16. *A* function $f : X \to Y$ is b-$\mathcal{I}$-set-connected if and only if $f^{-1}(K)$ is b-$\mathcal{I}$-clopen for every clopen subset K of $f(X)$ with respect to relative topology on $f(X)$.

Proof. Let K be any clopen subset of $f(X)$ with respect to the relative topology on $f(X)$. Suppose that $f^{-1}(K)$ is not b-$\mathcal{I}$-closed in X. Then there exists $x \in X\backslash f^{-1}(K)$ such that for every b-$\mathcal{I}$-open set U with $x \in U$, $U \cap f^{-1}(K) \neq \varnothing$. Suppose that there exists a b-$\mathcal{I}$-clopen set A such that $f^{-1}(K) \subset A$ and $x \notin A$. Then $x \in X\backslash A \subset X\backslash f^{-1}(K)$ and $X\backslash A$ is a b-$\mathcal{I}$-open set containing x and disjoint from $f^{-1}(K)$. This contradiction implies that X is b-$\mathcal{I}$-set-connected between x and $f^{-1}(K)$. Since f is b-$\mathcal{I}$-set-connected, $f(X)$ is connected between $f(x)$ and $f(f^{-1}(K))$. So, $f(f^{-1}(K)) \subset K$ and $f(x) \notin K$, is a contradiction. Hence, $f^{-1}(K)$ is $\mathcal{I}$-closed in X. By using the complements, we obtain that $f^{-1}(K)$ is b-$\mathcal{I}$-open. Conversely, suppose that there exist subsets A and B of X for which $f(X)$ is not connected between $f(A)$ and $f(B)$ in relative topology on $f(X)$ such that $f(A) \subset K$ and $K \cap f(B) = \varnothing$. Then $A \subset f^{-1}(K)$, $B \cap f^{-1}(K) = \varnothing$ and $f^{-1}(K)$ is b-$\mathcal{I}$-clopen, which implies that X is not b-$\mathcal{I}$-connected between A and B. We obtain that f is b-$\mathcal{I}$-set-connected. □

Theorem 4.17. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function. If f is b-$\mathcal{I}$-set-connected, then it is slightly b-$\mathcal{I}$-continuous.

Proof. Let F be a clopen subset of Y. Then $F \cap f(X)$ is clopen in the relative topology on $f(X)$. Since f is b-$\mathcal{I}$-set-connected, it follows that $f^{-1}(F) = f^{-1}(F \cap f(X))$ is b-$\mathcal{I}$-clopen in X. □

Theorem 4.18. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function. If f is slightly b-$\mathcal{I}$-continuous surjection, then it is b-$\mathcal{I}$-set-connected.

Proof. It follows from Theorem 4.16. □

5. Acknowledgment

I am very thankful to my supervisor Dr. N. Rajesh for his helps, advises, suggestions and criticisms during the preparation of this paper.

References

[1] D. Andrijevic, *On b-open sets*, Math. Vesnik **48** (1996), 59–64.
[2] R. Balaji and N. Rajesh, *New separation axioms via b-$\mathcal{I}$-open sets*, submitted.
[3] J. Dontchev, *On hausdoroff spaces via topological ideals and $\mathcal{I}$-irresolute functions*, Annals of the New York Academy of Sciences **767** (1995), 28–38.
[4] E. Ekici and M. Caldas, *Slightly γ- continuous functions*, Bol. Soc. Paran. Mat., **22** (2004), no. 2, 63–74.
[5] A. A. El-Atik, *A study of some types of mappings on topological spaces*, Master's Thesis, Tanta University, **8** (1995), 67–74.

[6] A. Caksu Guler and G. Aslim, *b-$\mathcal{I}$-open sets and decomposition of continuity via idealization*, Proc. Inst. Math. Mech., **22** (2005), 27–32.
[7] R. C. Jain, *The role of regularly open sets in general topology*, Ph.D. Thesis, Meerut Univ., (1980).
[8] K. Kuratowski, *Topology*, Academic Press, New York, (1966).
[9] M.Akdag, *On b-$\mathcal{I}$-open sets and b-$\mathcal{I}$-continuous functions*, Inter. J. Math. Math. Sci. **22** (2007), 27–32.
[10] R. L. Newcomb, *Topologies which are compact modulo an ideal*, Ph.D. Thesis, University of California, USA (1967).
[11] R. Staum, *The algebra of bounded continuous functions into a nonarchimedianfiled*, Pacic J. Math., **50** (1974), 169–185.
[12] R. Vaidyanathaswamy, *The localisation theory in set topology*, Proc. Indian Acad. Sci., **20** (1945), 51–61.