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NEW CONTRIBUTIONS TO NONUNIQUE FIXED-POINT RESULTS VIA POWER TYPE CONTRACTIONS

K. KARAAĞAÇ, Ö. M. KIZANLIK AND N. TAŞ

Abstract. The geometry of fixed point set $Fix(T)$ has been studied with different approaches under the fixed-circle problem. To obtain solutions related to this open problem, some known contractive conditions have been modified on metric or generalized metric spaces. In this paper, our aim is to investigate new fixed-circle results using the power type contractions on a metric space and generalize some theorems in the literature. All the obtained theoretical results are supported by various examples. Finally, we present an application to Exponential Linear Unit (ELU) Derivative which is used in neural networks as an activation function. It is hoped this study will give information about new solutions about fixed-circle problem and will shed light on new research topics.

1. Introduction and Background

The circle with the center a_0 and the radius μ in a metric space (X, d) is defined by

$$C_{a_0,\mu} = \{a \in X : d(a, a_0) = \mu\}.$$

Let us consider $X = \mathbb{R}$ with the metrics $d_i : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined as

$$d_1(a, b) = |a - b| \text{ (usual metric)},$$

$$d_2(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases} \text{ (discrete metric)},$$

$$d_3(a, b) = \left|e^a - e^b\right|$$

and

$$d_4(a, b) = \begin{cases} 0 & \text{if } a = b \\ |a| + |b| & \text{if } a \neq b \end{cases}.$$

Hence, according to the metrics d_i, we obtain the unit circle, respectively, as follows

$$C_{0,1}^{d_1} = \{-1, 1\},\ C_{0,1}^{d_2} = \mathbb{R} - \{0\},\ C_{0,1}^{d_3} = \{\ln 2\} \text{ and } C_{0,1}^{d_4} = \{-1, 1\}.$$

We can say from here that the number of elements in the circle changes when the metric changes.

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"*Fixed-circle problem*" has occurred as a geometric approach to the fixed-point theory when the self-mapping $T : X \to X$ has more than one fixed point [8]. Now, we recall the notion of a fixed circle:

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. If $Ta = a$ for every $a \in C_{a_0,\mu}$ then $C_{a_0,\mu}$ is called as the fixed circle of T (see [8]).

Recently, some solutions were presented using different approaches, techniques and contractive conditions (see, for example [6], [7], [9], [10], [11] [13] and the references therein). Also, this problem is important in terms of having the feature of being applicable to other study areas. For example, activation functions are used to obtain an application to the fixed-circle problem. Some authors gave some various applications for this problem (see, for example [9], [10], [11], [13]).

The main goal of this paper is to obtain new solutions to the fixed-circle problem using two different auxiliary functions. The obtained existence fixed-point theorems generalize the known results in the literature (see, [8]). The uniqueness theorems are also proved for the case where the number of fixed circles is more than one. For this purpose, we use some famous contractive conditions such as Khan type contractions, Brianciari type contractions and Ćirić type contractions. We investigate some conditions exclude the possibility of identity mapping since the identity mapping fixes every circle and we show the equivalence of the obtained theorems which gives the identity map. Finally, we present an application for an exponential linear unit derivative which is used in neural networks as an activation function.

2. Main Results

In this section, we present new solutions to the fixed-circle problem using the power type contractions with two auxiliary functions on metric spaces.

In the sequel, let (X, d) be a metric space, $T : X \to X$ a self-mapping, $\mathbb{N} = \{1, 2, 3, \ldots\}$ a set of natural numbers, $C_{a_0,\mu}$ any circle on X and the fixed point set of T denoted as

$$Fix(T) = \{a \in X : Ta = a\}.$$

Also, in the examples of this section, we consider the usual metric space $(\mathbb{R}, d)$.

2.1. Existence results of a fixed circle with the function Ψ.

Let us define the mapping $\Psi : X \to [0, \infty)$ as

$$\Psi(a) = d(a, a_0), \tag{1}$$

for all $a \in X$.

Theorem 2.1. *If there exists a self-mapping T satisfying*

(1.1) $d(a, Ta) \leq [\Psi(a)]^n - [\Psi(Ta)]^n$,

(1.2) $d(Ta, a_0) \geq \mu$,

for each $a \in C_{a_0,\mu}$ and $n \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Proof. Let $a \in C_{a_0,\mu}$ be any point. We prove that $Ta = a$, that is, $a \in Fix(T)$. Using (1.1) and (1.2), we have

$$\begin{aligned} d(a, Ta) &\leq [\Psi(a)]^n - [\Psi(Ta)]^n = [d(a, a_0)]^n - [d(Ta, a_0)]^n \\ &= \mu^n - [d(Ta, a_0)]^n \leq \mu^n - \mu^n = 0 \end{aligned}$$

and so we get $a \in Fix(T)$. Consequently, we obtain

$$C_{a_0,\mu} \subset Fix(T),$$

that is, $C_{a_0,\mu}$ is a fixed circle of T. □

Remark 2.1. (1) *If we consider the conditions* (1.1) *and* (1.2) *together, we say that* $Ta \in C_{a_0,\mu}$ *for each* $a \in C_{a_0,\mu}$ *and so* $T(C_{a_0,\mu}) \subset C_{a_0,\mu}$.

(2) *Theorem 2.1 generalizes Theorem* 2.1 *given in* [8]. *Indeed, if we take* $n = 1$, *then these two theorems coincide.*

We give some illustrative examples related to Theorem 2.1.

Example 2.1. *Let us consider the circle* $C_{0,2}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} a & if \quad a < 0 \\ 2 & if \quad a \geq 0 \end{cases},$$

for all $a \in \mathbb{R}$. *Then* T *satisfies the conditions* (1.1) *and* (1.2) *and so* $C_{0,2} \subset Fix(T)$.

Example 2.2. *Let us consider the circle* $C_{4,2}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} 4 & if \quad a \in C_{4,2} \\ 1 & \quad otherwise \end{cases},$$

for all $a \in \mathbb{R}$. *Then* T *satisfies the condition* (1.1) *but does not satisfy the condition* (1.2). *It can be easily seen that* $C_{4,2} \not\subseteq Fix(T)$.

Example 2.3. *Let us consider the circle* $C_{2,1}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} 0 & if \quad a \in C_{2,1} \\ 2 & \quad otherwise \end{cases},$$

for all $a \in \mathbb{R}$. *Then* T *satisfies the condition* (1.2) *but does not satisfy the condition* (1.1). *It is clear that* $C_{2,1} \not\subseteq Fix(T)$.

Remark 2.2. (1) *Example 2.1 satisfies the conditions of Theorem 2.1 and so* $C_{0,2}$ *is a fixed circle of* T. *On the other hand, the center of the circle* $C_{0,2}$ *is not fixed by* T, *that is,* $0 \notin Fix(T)$. *So even if the circle is fixed by* T, *its center need not be fixed.*

(2) *Examples 2.2 and 2.3 do not satisfy the conditions of Theorem 2.1. Hence, respectively,* $C_{4,2}$ *and* $C_{2,1}$ *are not fixed circle. But, in Example 2.3, we see that the center of* $C_{2,1}$ *is fixed. So even if the circle is not fixed, its center can be fixed.*

Theorem 2.2. *If there exists a self-mapping* T *satisfying*

(2.1) $d(a, Ta) \leq [\Psi(a)]^n + [\Psi(Ta)]^n - 2\mu^n$,

(2.2) $d(Ta, a_0) \leq \mu$,

for each $a \in C_{a_0,\mu}$ *and* $n \in \mathbb{N}$, *then* $C_{a_0,\mu} \subset Fix(T)$.

Proof. Let $a \in C_{a_0,\mu}$ be any point. We prove that $Ta = a$, that is, $a \in Fix(T)$. Using (2.1) and (2.2), we get

$$\begin{aligned} d(a, Ta) &\leq [\Psi(a)]^n + [\Psi(Ta)]^n - 2\mu^n \\ &= [d(a, a_0)]^n + [d(Ta, a_0)]^n - 2\mu^n \\ &\leq \mu^n + \mu^n - 2\mu^n = 0 \end{aligned}$$

and so we obtain $a \in Fix(T)$. Consequently, we have

$$C_{a_0,\mu} \subset Fix(T),$$

that is, $C_{a_0,\mu}$ is a fixed circle of T. □

Remark 2.3. (1) *If we consider the conditions* (2.1) *and* (2.2) *together, we say that* $Ta \in C_{a_0,\mu}$ *for each* $a \in C_{a_0,\mu}$ *and so* $T(C_{a_0,\mu}) \subset C_{a_0,\mu}$.

(2) *Theorem 2.2 generalizes Theorem* 2.2 *given in* [8]. *Indeed, if we take* $n = 1$, *then these two theorems coincide.*

We give following examples.

Example 2.4. *Let us consider the circle* $C_{0,1}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} a^3 & \text{if} \quad a \in C_{0,1} \\ 5 & \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$. *Then* T *satisfies the conditions* (2.1) *and* (2.2) *and so* $C_{0,1} \subset Fix(T)$.

Example 2.5. *Let us consider the circle* $C_{0,2}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} -5 & \text{if} \quad a = -2 \\ 3 & \text{if} \quad a = 2 \\ 10 & \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$. *Then* T *satisfies the condition* (2.1) *but does not satisfy the condition* (2.2). *It can be easily shown that* $C_{0,2} \nsubseteq Fix(T)$.

Example 2.6. *Let us consider the circle* $C_{1,3}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} 2 & \text{if} \quad a \in C_{1,3} \\ 5 & \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$. *Then* T *satisfies the condition* (2.2) *but does not satisfy the condition* (2.1). *It is clear that* $C_{1,3} \nsubseteq Fix(T)$.

Remark 2.4. (1) *Example 2.4 satisfies the conditions of Theorem 2.2 and so* $C_{0,1}$ *is a fixed circle of* T. *On the other hand, the center of the circle* $C_{0,1}$ *is not fixed by* T, *that is,* $0 \notin Fix(T)$.

(2) *Examples 2.5 and 2.6 do not satisfy the conditions of Theorem 2.2. Hence, respectively,* $C_{0,2}$ *and* $C_{1,3}$ *are not fixed circle.*

Theorem 2.3. *If there exists a self-mapping* T *satisfying*

(3.1) $d(a, Ta) \leq [\Psi(a)]^n - [\Psi(Ta)]^n$,

(3.2) $hd(a, Ta) + [d(Ta, a_0)]^n \geq \mu^n$,

for each $a \in C_{a_0,\mu}$, $n \in \mathbb{N}$ *and some* $h \in [0, 1)$, *then* $C_{a_0,\mu} \subset Fix(T)$.

Proof. Let $a \in C_{a_0,\mu}$ be any point. We prove that $Ta = a$, that is, $a \in Fix(T)$. Using (3.1) and (3.2), we have

$$\begin{aligned} d(a, Ta) &\leq [\Psi(a)]^n - [\Psi(Ta)]^n = [d(a, a_0)]^n - [d(Ta, a_0)]^n \\ &= \mu^n - [d(Ta, a_0)]^n \\ &\leq hd(a, Ta) + [d(Ta, a_0)]^n - [d(Ta, a_0)]^n = hd(a, Ta) \end{aligned}$$

and so we get $a \in Fix(T)$. Consequently, we find

$$C_{a_0,\mu} \subset Fix(T),$$

that is, $C_{a_0,\mu}$ is a fixed circle of T. □

Remark 2.5. (1) *If we consider the conditions* (3.1) *and* (3.2) *together, we say that* Ta *should be lies on or interior of the circle* $C_{a_0,\mu}$ *for each* $a \in C_{a_0,\mu}$.

(2) *Theorem 2.3 generalizes Theorem* 2.3 *given in* [8]. *Indeed, if we take* $n = 1$, *then these two theorems coincide.*

(3) *If we consider Example 2.1, then* T *satisfies the conditions of Theorem 2.3 and so* $C_{0,2}$ *is a fixed circle of* T. *Also, if we take the self-mapping* T *defined in Example 2.2, then* T *satisfies the condition* (3.1) *but does not satisfy the condition* (3.2) *and so* $C_{4,2}$ *is not a fixed circle of* T. *Similarly, we consider Example 2.3, then* T *satisfies the condition* (1.2) *but does not satisfy the condition* (1.1) *and so* $C_{2,1}$ *is not a fixed circle of* T.

Theorem 2.4. *If there exists a self-mapping* T *satisfying*

(4.1) $d(a, Ta) \leq \left[\frac{\Psi(Ta)}{\Psi(a)}\right]^n - 1$,

(4.2) $d(Ta, a_0) \leq \mu$,

for each $a \in C_{a_0,\mu}$ *and* $n \in \mathbb{N}$, *then* $C_{a_0,\mu} \subset Fix(T)$.

Proof. Let $a \in C_{a_0,\mu}$ be any point. We prove that $Ta = a$, that is, $a \in Fix(T)$. Using (4.1) and (4.2), we have

$$\begin{aligned} d(a, Ta) &\leq \left[\frac{\Psi(Ta)}{\Psi(a)}\right]^n - 1 = \left[\frac{d(Ta, a_0)}{d(a, a_0)}\right]^n - 1 \\ &= \left[\frac{d(Ta, a_0)}{\mu}\right]^n - 1 \leq 1 - 1 = 0 \end{aligned}$$

and so we get $a \in Fix(T)$. Consequently, we get

$$C_{a_0,\mu} \subset Fix(T),$$

that is, $C_{a_0,\mu}$ is a fixed circle of T. □

Remark 2.6. (1) *If we consider the conditions* (4.1) *and* (4.2) *together, we say that* $Ta \in C_{a_0,\mu}$ *for each* $a \in C_{a_0,\mu}$ *and so* $T(C_{a_0,\mu}) \subset C_{a_0,\mu}$.

(2) *If we take* $n = 1$, *then the condition* (4.1) *is reduced as follows*:

$$d(a, Ta) \leq \frac{\Psi(Ta) - \Psi(a)}{\Psi(a)}.$$

Example 2.7. *Let us consider the circle* $C_{0,3}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} \frac{a+6}{3} & \text{if} \quad a \geq 0 \\ a & \text{if} \quad a < 0 \end{cases},$$

for all $a \in \mathbb{R}$. *Then* T *satisfies the conditions* (4.1) *and* (4.2) *and so* $C_{0,3} \subset Fix(T)$.

Example 2.8. *Let us consider the circle* $C_{3,1}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = (a-1)^2,$$

for all $a \in \mathbb{R}$. *Then* T *satisfies the condition* (4.1) *but does not satisfy the condition* (4.2). *It can be easily said that* $C_{3,1} \nsubseteq Fix(T)$.

Example 2.9. *Let us consider the circle* $C_{2,4}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} \frac{a}{2} & \text{if} \quad a \in C_{2,4} \\ 7 & \quad \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$*. Then* T *satisfies the condition* (4.2) *but does not satisfy the condition* (4.1)*. It can be easily seen that* $C_{2,4} \nsubseteq Fix(T)$*.*

Remark 2.7. (1) *Example 2.7 satisfies the conditions of Theorem 2.4 and so* $C_{0,3}$ *is a fixed circle of* T*. On the other hand, the center of the circle* $C_{0,3}$ *is not fixed by* T*, that is,* $0 \notin Fix(T)$*.*

(2) *Examples 2.8 and 2.9 do not satisfy the conditions of Theorem 2.4. Hence, respectively,* $C_{3,1}$ *and* $C_{2,4}$ *are not fixed circle.*

Theorem 2.5. *If there exists a self-mapping* T *satisfying*
(5.1) $d(a,Ta) \leq \max\{[\Psi(a)]^n, [\Psi(Ta)]^n\} - \mu^n$*,*
(5.2) $d(Ta, a_0) \leq \mu$*,*
for each $a \in C_{a_0,\mu}$ *and* $n \in \mathbb{N}$*, then* $C_{a_0,\mu} \subset Fix(T)$*.*

Proof. Let $a \in C_{a_0,\mu}$ be any point. We prove that $Ta = a$, that is, $a \in Fix(T)$. Using (5.1) and (5.2), we have

$$\begin{aligned} d(a,Ta) &\leq \max\{[\Psi(a)]^n, [\Psi(Ta)]^n\} - \mu^n \\ &= \max\{[d(a,a_0)]^n, [d(Ta,a_0)]^n\} - \mu^n \\ &\leq \mu^n - \mu^n = 0 \end{aligned}$$

and so we get $a \in Fix(T)$. Consequently, we get

$$C_{a_0,\mu} \subset Fix(T),$$

that is, $C_{a_0,\mu}$ is a fixed circle of T. □

Remark 2.8. (1) *If we consider the conditions* (5.2)*, we say that* Ta *should be lies on or interior of the circle* $C_{a_0,\mu}$ *for each* $a \in C_{a_0,\mu}$ *and so* $T(C_{a_0,\mu}) \subset C_{a_0,\mu}$*.*

(2) *If we take* $n = 1$*, then the condition* (5.1) *is reduced as follows*:

$$d(a,Ta) \leq \max\{\Psi(a), \Psi(Ta)\} - \mu.$$

Also, the S*-metric version of this inequality was used to obtain a fixed-circle result in* [4]*.*

Example 2.10. *Let us consider the circle* $C_{0,1}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} sgna & if \quad a \in \{-1,1\} \\ 0 & otherwise \end{cases},$$

for all $a \in \mathbb{R}$*. Then* T *satisfies the conditions* (5.1) *and* (5.2) *and so* $C_{0,1} \subset Fix(T)$*.*

Example 2.11. *Let us consider the circle* $C_{2,4}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} -|a| & if \quad a \in C_{2,4} \\ 0 & otherwise \end{cases},$$

for all $a \in \mathbb{R}$*. Then* T *does not satisfy the condition* (5.2)*. It is obvious that* $C_{2,4} \nsubseteq Fix(T)$*.*

Example 2.12. *Let us consider the circle* $C_{0,1}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} a + sgna & if \quad a \in \{-1,1\} \\ 1 & otherwise \end{cases},$$

for all $a \in \mathbb{R}$*. Then* T *satisfies the condition* (5.1) *but does not satisfy the condition* (5.2)*. It is clear that* $C_{0,1} \nsubseteq Fix(T)$*.*

Example 2.13. *Let us consider the circle $C_{2,\frac{1}{2}}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Ta = \begin{cases} [|a|] + \frac{3}{2} & if \quad a = \frac{3}{2} \\ [|a|] - \frac{1}{2} & if \quad a = \frac{5}{2} \\ 0 & otherwise \end{cases},$$

for all $a \in \mathbb{R}$. Then T satisfies the condition (5.2) but does not satisfy the condition (5.1). It can be easily seen that $C_{2,\frac{1}{2}} \nsubseteq Fix(T)$.

Remark 2.9. (1) *Example 2.10 satisfies the conditions of Theorem 2.5 and so $C_{0,1}$ is a fixed circle of T. On the other hand, the center of the circle $C_{0,3}$ is fixed by T, that is, $0 \in Fix(T)$.*

(2) *Examples 2.11, 2.12 and 2.13 do not satisfy the conditions of Theorem 2.5. Hence, respectively, $C_{2,4}$, $C_{0,1}$ and $C_{2,\frac{1}{2}}$ are not fixed circle.*

Theorem 2.6. *If there exists a self-mapping T satisfying*

(6.1) $d(a, Ta) \leq \min \{[\Psi(a)]^n, [\Psi(Ta)]^n\} - \mu^n$,

(6.2) $d(Ta, a_0) \geq \mu$,

for each $a \in C_{a_0,\mu}$ and $n \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Proof. Let $a \in C_{a_0,\mu}$ be any point. We prove that $Ta = a$, that is, $a \in Fix(T)$. Using (6.1) and (6.2), we find

$$\begin{aligned} d(a, Ta) &\leq \min \{[\Psi(a)]^n, [\Psi(Ta)]^n\} - \mu^n \\ &= \min \{[d(a, a_0)]^n, [d(Ta, a_0)]^n\} - \mu^n \\ &\leq \mu^n - \mu^n = 0 \end{aligned}$$

and so we get $a \in Fix(T)$. Consequently, we have

$$C_{a_0,\mu} \subset Fix(T),$$

that is, $C_{a_0,\mu}$ is a fixed circle of T. □

Remark 2.10. (1) *If we consider the conditions (6.2), we say that Ta should be lies on or exterior of the circle $C_{a_0,\mu}$ for each $a \in C_{a_0,\mu}$ and so $T(C_{a_0,\mu}) \subset C_{a_0,\mu}$.*

(2) *If we take $n = 1$, then the condition (6.1) is reduced as follows:*

$$d(a, Ta) \leq \min \{\Psi(a), \Psi(Ta)\} - \mu.$$

Example 2.14. *Let us consider the circle $C_{0,1}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Ta = \begin{cases} \frac{1}{a} & if \quad a \in \{-1, 1\} \\ 0 & otherwise \end{cases},$$

for all $a \in \mathbb{R}$. Then T satisfies the conditions (6.1) and (6.2) and so $C_{0,1} \subset Fix(T)$.

Example 2.15. *Let us consider the circle $C_{0,2}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Ta = \begin{cases} a^2 - 2a - 1 & if \quad a = 2 \\ a^2 + 2a - 1 & if \quad a = -2 \\ 0 & otherwise \end{cases},$$

for all $a \in \mathbb{R}$. Then T does not satisfy the conditions (6.1) and (6.2). It is obvious that $C_{0,2} \nsubseteq Fix(T)$.

Remark 2.11. (1) *Example 2.14 satisfies the conditions of Theorem 2.6 and so* $C_{0,1}$ *is a fixed circle of* T*. On the other hand, the center of the circle* $C_{0,3}$ *is fixed by* T*, that is,* $0 \in Fix(T)$*.*

(2) *Example 2.15 does not satisfy the conditions of Theorem 2.6. Hence, respectively,* $C_{0,2}$ *is not fixed circle.*

Theorem 2.7. *If there exists a self-mapping* T *satisfying*

(7.1) $d(a,Ta) \leq [\Psi(a)]^n [\Psi(Ta)]^n - \mu^{2n}$,

(7.2) $d(Ta,a_0) \leq \mu$,

for each $a \in C_{a_0,\mu}$ *and* $n \in \mathbb{N}$*, then* $C_{a_0,\mu} \subset Fix(T)$*.*

Proof. Let $a \in C_{a_0,\mu}$ be any point. We prove that $Ta = a$, that is, $a \in Fix(T)$. Using (7.1) and (7.2), we get

$$\begin{aligned} d(a,Ta) &\leq [\Psi(a)]^n [\Psi(Ta)]^n - \mu^{2n} \\ &= [d(a,a_0)]^n [d(Ta,a_0)]^n - \mu^{2n} \leq 0 \end{aligned}$$

and so we get $a \in Fix(T)$. Consequently, we have

$$C_{a_0,\mu} \subset Fix(T),$$

that is, $C_{a_0,\mu}$ is a fixed circle of T. □

Remark 2.12. (1) *If we consider the conditions* (7.1) *and* (7.2) *together, we say that* $Ta \in C_{a_0,\mu}$ *for each* $a \in C_{a_0,\mu}$ *and so* $T(C_{a_0,\mu}) \subset C_{a_0,\mu}$*.*

(2) *If we take* $n = 1$*, then the condition* (7.1) *is reduced as follows*:

$$d(a,Ta) \leq \Psi(a)\Psi(Ta) - \mu^2.$$

Example 2.16. *Let us consider the circle* $C_{1,1}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} [|a|] & \text{if} \quad a \in \{0,2\} \\ 0 & \quad \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$*. Then* T *satisfies the conditions* (7.1) *and* (7.2) *and so* $C_{1,1} \subset Fix(T)$*.*

Example 2.17. *Let us consider the circle* $C_{1,2}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} -3 & \text{if} \quad a = -1 \\ 2 & \text{if} \quad a = 3 \\ 0 & \quad \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$*. Then* T *satisfies the condition* (7.1) *but does not satisfy the condition* (7.2) *for* $a = 3$ *and* $Ta = 2$*. It is clear that* $C_{1,2} \nsubseteq Fix(T)$*.*

Example 2.18. *Let us consider the circle* $C_{0,1}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Ta = \begin{cases} a^2 - a - 1 & \text{if} \quad a \in C_{0,1} \\ 0 & \quad \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$*. Then* T *satisfies the condition* (7.2) *but does not satisfy the condition* (7.1)*. It is obvious that* $C_{0,1} \nsubseteq Fix(T)$*.*

Remark 2.13. (1) *Example 2.16 satisfies the conditions of Theorem 2.7 and so* $C_{1,1}$ *is a fixed circle of* T*. On the other hand, the center of the circle* $C_{1,1}$ *is not fixed by* T*, that is,* $1 \notin Fix(T)$*.*

(2) *Examples 2.17 and 2.18 do not satisfy the conditions of Theorem 2.7. Hence, respectively,* $C_{1,2}$ *and* $C_{0,1}$ *are not fixed circle. But, in Example 2.18, we see that the center of* $C_{0,1}$ *is fixed.*

2.2. **Existence results of a fixed circle with the function Ω.** Let us define the mapping $\Omega : X \to [0,\infty)$ as

$$\Omega(a) = \begin{cases} d(a,a_0) & \text{if} \quad 0<\mu<1 \\ m(a,a_0) & \text{if} \quad \mu \geq 1 \end{cases}, \tag{2}$$

for all $a \in X$ where

$$m(a,a_0) = \frac{d(a,a_0)}{1+d(a,a_0)}.$$

Theorem 2.8. *If there exists a self-mapping T satisfying*
(8.1) $d(a,Ta) \leq [\Omega(a)]^p [\Omega(Ta)]^q$,
(8.2) $d(Ta,a_0) = \mu$,
for each $a \in C_{a_0,\mu}$ and $p,q \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Proof. Let $a \in C_{a_0,\mu}$ be any point. We prove that $Ta = a$, that is, $a \in Fix(T)$. Let us consider the following cases:

Case 1: Let $0 < \mu < 1$. Using (8.1) and (8.2), we obtain

$$\begin{aligned} d(a,Ta) &\leq [\Omega(a)]^p [\Omega(Ta)]^q = [d(a,a_0)]^p [d(Ta,a_0)]^q \\ &= \mu^p \mu^q = \mu^{p+q} = \mu^n \to 0, \end{aligned}$$

as $n \to \infty$. Then we get $a \in Fix(T)$.

Case 2: Let $\mu \geq 1$. Using (8.1) and (8.2), we find

$$\begin{aligned} d(a,Ta) &\leq [\Omega(a)]^p [\Omega(Ta)]^q = [m(a,a_0)]^p [m(Ta,a_0)]^q \\ &= \left[\frac{d(a,a_0)}{1+d(a,a_0)}\right]^p \left[\frac{d(Ta,a_0)}{1+d(Ta,a_0)}\right]^q \\ &= \left(\frac{\mu}{1+\mu}\right)^p \left(\frac{\mu}{1+\mu}\right)^q = \left(\frac{\mu}{1+\mu}\right)^{p+q} = \left(\frac{\mu}{1+\mu}\right)^n \to 0, \end{aligned}$$

as $n \to \infty$. So we have $a \in Fix(T)$.

Consequently, we get

$$C_{a_0,\mu} \subset Fix(T),$$

that is, $C_{a_0,\mu}$ is a fixed circle of T. □

Now we give the following illustrative examples:

Example 2.19. *Let us consider the circle $C_{0,\frac{1}{4}}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Ta = \begin{cases} a & \text{if} \quad a \in C_{0,\frac{1}{4}} \\ 3 & \quad \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$. Then T satisfies the conditions (8.1) and (8.2) and so $C_{0,\frac{1}{4}} \subset Fix(T)$.

Example 2.20. *Let us consider the circle $C_{0,1}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Ta = a^3,$$

for all $a \in \mathbb{R}$. Then T satisfies the conditions (8.1) and (8.2) and so $C_{0,1} \subset Fix(T)$.

Example 2.21. *Let us consider the circle $C_{1,\frac{2}{3}}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Ta = \begin{cases} 0 & \text{if} \quad a = \frac{1}{3} \\ 2 & \text{if} \quad a = \frac{5}{3} \\ 1 & \quad \text{otherwise} \end{cases},$$

for all $a \in \mathbb{R}$. Then T satisfies the condition (8.1) *but does not satisfy the condition* (8.2). *It can be easily seen that $C_{1,\frac{2}{3}} \nsubseteq Fix(T)$.*

Example 2.22. *Let us consider the circle $C_{2,5}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Ta = \begin{cases} 7 & if \quad a = -3 \\ -3 & if \quad a = 7 \\ 5 & otherwise \end{cases},$$

for all $a \in \mathbb{R}$. Then T satisfies the condition (8.2) *but does not satisfy the condition* (8.1). *It is clear that $C_{2,5} \nsubseteq Fix(T)$.*

If we use the auxiliary function Ω with the similar approaches given in the previous subsection, we get the following corollaries:

Corollary 2.1. *If there exists a self-mapping T satisfying*
$(C.1.1)$ $d(a, Ta) \leq [\Omega(a)]^n - [\Omega(Ta)]^n$,
$(C.1.2)$ $d(Ta, a_0) = \mu$,
for each $a \in C_{a_0,\mu}$ and $n \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Corollary 2.2. *If there exists a self-mapping T satisfying*
$(C.2.1)$ $d(a, Ta) \leq [\Omega(a)]^n + [\Omega(Ta)]^n - 2\mu^n$,
$(C.2.2)$ $d(Ta, a_0) = \mu$,
for each $a \in C_{a_0,\mu}$ and $n \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Corollary 2.3. *If there exists a self-mapping T satisfying*
$(C.3.1)$ $d(a, Ta) \leq [\Omega(a)]^n - [\Omega(Ta)]^n$,
$(C.3.2)$ $hd(a, Ta) + [d(Ta, a_0)]^n = \mu^n$,
for each $a \in C_{a_0,\mu}$, $n \in \mathbb{N}$ and some $h \in [0, 1)$, then $C_{a_0,\mu} \subset Fix(T)$.

Corollary 2.4. *If there exists a self-mapping T satisfying*
$(C.4.1)$ $d(a, Ta) \leq \left[\frac{\Omega(Ta)}{\Omega(a)}\right]^n - 1$,
$(C.4.2)$ $d(Ta, a_0) = \mu$,
for each $a \in C_{a_0,\mu}$ and $n \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Corollary 2.5. *If there exists a self-mapping T satisfying*
$(C.5.1)$ $d(a, Ta) \leq \max\{[\Omega(a)]^n, [\Omega(Ta)]^n\} - \mu^n$,
$(C.5.2)$ $d(Ta, a_0) = \mu$,
for each $a \in C_{a_0,\mu}$ and $n \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Corollary 2.6. *If there exists a self-mapping T satisfying*
$(C.6.1)$ $d(a, Ta) \leq \min\{[\Omega(a)]^n, [\Omega(Ta)]^n\} - \mu^n$,
$(C.6.2)$ $d(Ta, a_0) = \mu$,
for each $a \in C_{a_0,\mu}$ and $n \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Corollary 2.7. *If there exists a self-mapping T satisfying*
$(C.7.1)$ $d(a, Ta) \leq [\Omega(a)]^n [\Omega(Ta)]^n - \mu^{2n}$,
$(C.7.2)$ $d(Ta, a_0) = \mu$,
for each $a \in C_{a_0,\mu}$ and $n \in \mathbb{N}$, then $C_{a_0,\mu} \subset Fix(T)$.

Remark 2.14. (1) *If we consider Theorem 2.8 and Corollaries 2.1-2.2, then we have $Ta \in C_{a_0,\mu}$ for all $a \in C_{a_0,\mu}$.*

(2) *In Example 2.19, the radius of the circle is $\mu = \frac{1}{4}$, that is, $0 < \mu < 1$ and in Example 2.20, the radius of the circle is $\mu = 1$, that is, $\mu \geq 1$.*

2.3. **Exclude the identity map and equivalence of conditions.** In this subsection, we investigate some conditions to exclude the identity map $I_X : X \to X$ defined as $I_X(a) = a$ for all $a \in X$ from the obtained existence fixed-circle results. For this purpose, we recall the family of functions φ (see [5] and the references therein) as follows:

Let $\varphi : [0, \infty) \to [0, \infty)$ satisfies the following properties:

- φ is continuous and strictly increasing,
- $\varphi(t) = 0 \Leftrightarrow t = 0$,
- $\varphi(t) \geq ct^n$ for all $t > 0$, where $c > 0$ and $n > 0$ are constant.

The set of functions φ is denoted by Φ.

Theorem 2.9. *Let $T : X \to X$ a self-mapping having a fixed circle $C_{a_0,\mu}$, Ω defined as in (2) and $\varphi \in \Phi$. T satisfies the condition*

$$(I_1) \quad \varphi\left(d(a, Ta)\right) \leq c\varphi\left[\Omega(a) - \Omega(Ta)\right],$$

for all $a \in X$ and some $c \in (0, 1)$ if and only if $T = I_X$.

Proof. Let $a \in X$ be any point such that $a \notin Fix(T)$. Let us consider the following cases:

Case 1: Let $0 < \mu < 1$. By (I_1) and the triangle inequality, we get

$$\begin{aligned} \varphi\left(d(a, Ta)\right) &\leq c\varphi\left[\Omega(a) - \Omega(Ta)\right] = c\varphi\left[d(a, a_0) - d(Ta, a_0)\right] \\ &\leq c\varphi\left[d(a, Ta) + d(Ta, a_0) - d(Ta, a_0)\right] \\ &= c\varphi\left[d(a, Ta)\right], \end{aligned}$$

a contradiction since $c \in (0, 1)$.

Case 2: Let $\mu \geq 1$. Then we get

$$\begin{aligned} \varphi\left(d(a, Ta)\right) &\leq c\varphi\left[\Omega(a) - \Omega(Ta)\right] = c\varphi\left[m(a, a_0) - m(Ta, a_0)\right] \\ &\leq c\varphi\left[m(a, Ta) + m(Ta, a_0) - m(Ta, a_0)\right] \\ &= c\varphi\left[m(a, Ta)\right] = c\varphi\left[\frac{d(a, Ta)}{1 + d(a, Ta)}\right] \leq c\varphi\left[d(a, Ta)\right], \end{aligned}$$

a contradiction.

Under the above cases, we obtain $a \in Fix(T)$ for all $a \in X$ and so $T = I_X$.

The converse statement is clear, that is, the identity map I_X satisfies the condition (I_1). □

Now, using the technique given in [1], we prove a following theorem.

Theorem 2.10. *Let $T : X \to X$ a self-mapping having a fixed circle $C_{a_0,\mu}$ and Ω defined as in (2). T satisfies the condition*

$$(I_2) \qquad \int_0^{d(a,Ta)} \varpi(t)dt \leq c\left[\int_0^{\Omega(a)} \varpi(t)dt - \int_0^{\Omega(Ta)} \varpi(t)dt\right],$$

for all $a \in X$ and some $c \in (0, 1)$, where $\varpi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (that is, with finite integral) on each compact subset of $[0, \infty)$, and such that

$$\int_0^{\varepsilon} \varpi(t)dt > 0$$

for each $\varepsilon > 0$ *if and only if* $T = I_X$.

Proof. Let $a \in X$ be any point such that $a \notin Fix(T)$. Let us consider the following cases:

Case 1: Let $0 < \mu < 1$. By (I_2) and the triangle inequality, we get

$$\begin{aligned}\int_0^{d(a,Ta)} \varpi(t)dt &\leq c\left[\int_0^{\Omega(a)} \varpi(t)dt - \int_0^{\Omega(Ta)} \varpi(t)dt\right] \\ &= c\left[\int_0^{d(a,a_0)} \varpi(t)dt - \int_0^{d(Ta,a_0)} \varpi(t)dt\right] \leq c\left[\int_0^{d(a,Ta)} \varpi(t)dt\right],\end{aligned}$$

a contradiction.

Case 2: Let $\mu \geq 1$. Then we get

$$\begin{aligned}\int_0^{d(a,Ta)} \varpi(t)dt &\leq c\left[\int_0^{\Omega(a)} \varpi(t)dt - \int_0^{\Omega(Ta)} \varpi(t)dt\right] \\ &= c\left[\int_0^{m(a,a_0)} \varpi(t)dt - \int_0^{m(Ta,a_0)} \varpi(t)dt\right] \\ &\leq c\left[\int_0^{m(a,Ta)} \varpi(t)dt\right] \leq c\left[\int_0^{d(a,Ta)} \varpi(t)dt\right],\end{aligned}$$

a contradiction.

Under the above cases, we obtain $a \in Fix(T)$ for all $a \in X$ and so $T = I_X$.

The converse statement can be easily proved, that is, the identity map I_X satisfies the condition (I_2). □

In the following theorem, we see that the equivalence of Theorems 2.9 and 2.10.

Theorem 2.11. *Let* $T : X \to X$ *a self-mapping having a fixed circle* C *and* Ω *defined as in (2).* T *satisfies the condition* (I_1) *if and only if* T *satisfies the condition* (I_2).

Proof. If T satisfies the condition (I_1), then by Theorem 2.9, we get $T = I_X$ and so from the converse statement of Theorem 2.10, T satisfies the condition (I_2). Similarly, the converse of this theorem can be easily seen. □

Remark 2.15. (1) *Theorems 2.9, 2.10 and 2.11 can be also considered with the function* Ψ *defined as in (1).*

(2) *If* T *satisfies the conditions of existence theorems given in previous subsections and does not satisfy the condition* (I_1) *or* (I_2)*, then* T *can not be identity map and* T *has a fixed circle.*

2.4. **Uniqueness results of a fixed circle.** If we consider Example 2.10, the self-mapping T fixes the circle $C_{0,1}$, $C_{-\frac{1}{2},\frac{1}{2}}$ and $C_{\frac{1}{2},\frac{1}{2}}$, that is, the number of fixed circles of T is more than one. For this reason, we investigate some uniqueness conditions of a fixed circle using different techniques.

Theorem 2.12. *Let $C_{a_0,\mu}$ be a fixed circle of T and $\varphi \in \Phi$. If T satisfies the condition*

$$(U_1) \quad \varphi\left(d(Ta,Tb)\right) \leq c\varphi\left(d(a,b)\right),$$

for all $a \in C_{a_0,\mu}$, $b \in X - C_{a_0,\mu}$ and some $c \in (0,1)$, then $C_{a_0,\mu}$ is the unique fixed circle of T.

Proof. Assume that there exist two fixed circles $C_{a_0,\mu}$ and C_{a_1,μ^*} of T. Let $a \in C_{a_0,\mu}$, $b \in C_{a_1,\mu^*}$ and $a \neq b$. Using the condition (U_1), we obtain

$$\varphi\left(d(Ta,Tb)\right) = \varphi\left(d(a,b)\right) \leq c\varphi\left(d(a,b)\right),$$

a contradiction. Consequently, it should be $a = b$ for all $a \in C_{a_0,\mu}$, $b \in C_{a_1,\mu^*}$ and so $C_{a_0,\mu}$ is the unique fixed circle of T. □

Theorem 2.13. *Let $C_{a_0,\mu}$ be a fixed circle of T. If T satisfies the condition*

$$(U_2) \quad \int_0^{d(Ta,Tb)} \varpi(t)dt \leq c \int_0^{d(a,b)} \varpi(t)dt,$$

for all $a \in C_{a_0,\mu}$, $b \in X - C_{a_0,\mu}$ and some $c \in (0,1)$, where $\varpi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping which is summable (that is, with finite integral) on each compact subset of $[0,\infty)$, and such that

$$\int_0^{\varepsilon} \varpi(t)dt > 0$$

for each $\varepsilon > 0$ then $C_{a_0,\mu}$ is the unique fixed circle of T.

Proof. Assume that there exist two fixed circles $C_{a_0,\mu}$ and C_{a_1,μ^*} of T. Let $a \in C_{a_0,\mu}$, $b \in C_{a_1,\mu^*}$ and $a \neq b$. Using the condition (U_2), we obtain

$$\int_0^{d(Ta,Tb)} \varpi(t)dt = \int_0^{d(a,b)} \varpi(t)dt \leq c \int_0^{d(a,b)} \varpi(t)dt,$$

a contradiction. Consequently, it should be $a = b$ for all $a \in C_{a_0,\mu}$, $b \in C_{a_1,\mu^*}$ and so $C_{a_0,\mu}$ is the unique fixed circle of T. □

Theorem 2.14. *Let $C_{a_0,\mu}$ be a fixed circle of T. If T satisfies the condition*

$$(U_3) \quad d(Ta,Tb) \leq qd(a,b) + rd(a,Ta) + sd(b,Tb) + t\left[d(a,Tb) + d(b,Ta)\right],$$

for all $a \in C_{a_0,\mu}$, $b \in X - C_{a_0,\mu}$ with $q + r + s + 2t < 1$, then $C_{a_0,\mu}$ is the unique fixed circle of T.

Proof. Assume that there exist two fixed circles $C_{a_0,\mu}$ and C_{a_1,μ^*} of T. Let $a \in C_{a_0,\mu}$, $b \in C_{a_1,\mu^*}$ and $a \neq b$. Using the condition (U_3), we obtain

$$\begin{aligned} d(Ta,Tb) &= d(a,b) \\ &\leq qd(a,b) + rd(a,Ta) + sd(b,Tb) + t\left[d(a,Tb) + d(b,Ta)\right] \\ &= qd(a,b) + t\left[d(a,b) + d(b,a)\right] = (q+2t)d(a,b), \end{aligned}$$

a contradiction. Consequently, it should be $a = b$ for all $a \in C_{a_0,\mu}$, $b \in C_{a_1,\mu^*}$ and so $C_{a_0,\mu}$ is the unique fixed circle of T. □

Remark 2.16. (1) *As seen in Theorems 2.12, 2.13 and 2.14, the uniqueness conditions do not have to be uniform. For example, the following conditions can be used to obtain new uniqueness results*:

$$(U_4) \quad d(Ta, Tb) \leq c\,[d(a, Tb) + d(a, Ta)]$$

and

$$(U_5) \quad d(Ta, Tb) \leq c\,[d(Ta, b) + d(a, Ta)]\,,$$

for all $a \in C_{a_0,\mu}$, $b \in X - C_{a_0,\mu}$ *and some* $c \in (0, 1)$.

(2) *The condition* (U_1) *is Khan type contractive condition* [5], *the condition* (U_2) *is Brianciari type contractive condition* [1] *and the condition* (U_3) *is Ćirić type contractive condition* [2].

3. An Application to Exponential Linear Unit Derivative

In this section, we obtain an application to exponential linear unit derivative for activation functions to show the applicable of the obtained fixed-circle results.

Activation functions are mathematical functions used in the neural networks. They are used for constructing a neural network to learn and make sense of something and there are a lot of activation functions used in the neural networks. One of them is "*Exponential linear unit activation function* (ELU)" (see [3], [12] and the references therein). This function tends to converge cost to zero faster and produce more accurate results. ELU has an extra alpha constant which should be positive number different from other activation functions as follows:

$$Ta = ELU(a) = \begin{cases} \alpha\,(e^a - 1) & ; \quad a < 0 \\ a & ; \quad a \geq 0 \end{cases},$$

where $\alpha \geq 0$.

On the other hand, the derivative of activation function is important in neural networks because it is fed to backprogapagation algorithm during learning. The derivative of ELU is as follows:

$$T'a = \begin{cases} 1 & \text{if} \quad a = 0 \text{ and } \alpha = 1 \\ 1 & \text{if} \quad a > 0 \\ \alpha e^a & \text{if} \quad a < 0 \end{cases}.$$

To show the significance of the activation functions for the fixed-circle problem, we give an application to derivative of exponential linear unit activation functions as follows:

Let us take $X = \mathbb{R} - \{-1\}$ and $\alpha = e$. Then we get

$$T'a = \begin{cases} 1 & ; \quad a \geq 0 \\ e^{a+1} & ; \quad a < 0 \end{cases},$$

for all $a \in \mathbb{R} - \{-1\}$ as seen in the following figure.

Then T' satisfies the conditions of existence theorems for the unit circle $C_{0,1}$. Consequently, T fixes the circle $C_{0,1} = \{1\}$.

4. Conclusion

In this study, we investigate new fixed-circle results using the power type contractions with necessary examples on a metric space. To do this, we use two different auxiliary functions. The importance of the obtained results is to generalize some

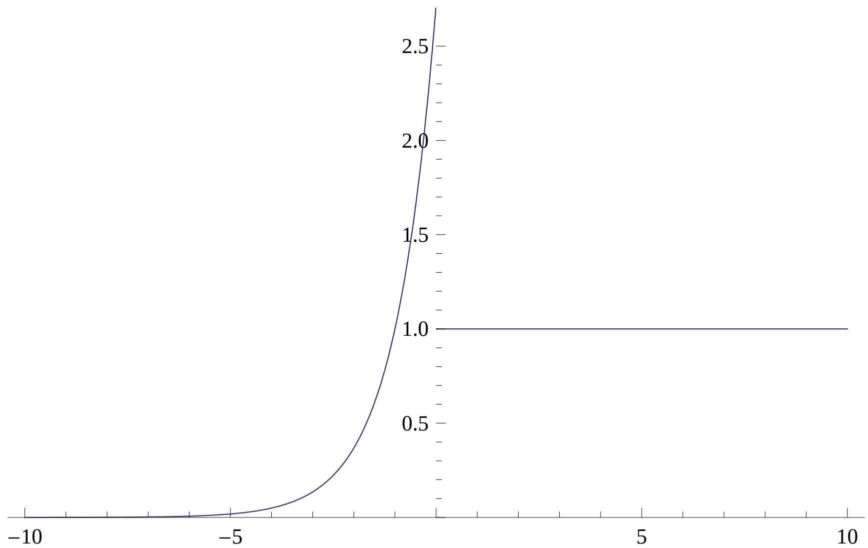

Figure 1. The derivative of ELU for $\alpha = -e$.

theorems in the literature. Finally, we present an application to Exponential Linear Unit (ELU) Derivative which is used in neural networks as an activation function.

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K. Karaağaç
Balikesir University, Department of Mathematics, 10145 Balikesir, Türkiye
Email address: kubra.mergen@gmail.com

Ö. M. Kizanlik
Balikesir University, Department of Mathematics, 10145 Balikesir, Türkiye
Email address: ozlemmorali@outlook.com

N. Taş
Balikesir University, Department of Mathematics, 10145 Balikesir, Türkiye
Email address: nihaltas@balikesir.edu.tr