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ON THE TRIGONOMETRIC APPROXIMATION IN WEIGHTED LORENTZ SPACES

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Abstract

In this study, we investigate the approximation properties of Nörlund, Riesz and matrix means of trigonometric Fourier series in weighted Lorentz spaces with Muckenhoupt weights.

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1 Introduction and Main Results

Let $T = [-\pi, \pi]$. A measurable $2\pi-$periodic function $\omega : T \to [0, \infty]$ is called a weight function if the set $\omega^{-1}(\{0, \infty\})$ has the Lebesgue measure zero. Given a weight function ω and a measurable set e we put

$$\omega(e) = \int_e \omega(x)dx. \tag{1.1}$$

We define the decreasing rearrangement $f^*_\omega(t)$ of $f : T \to \mathbb{R}$ with respect to the Borel measure (1.1) by

$$f^*_\omega(t) = inf\{\tau \geq 0 : \omega(\{x \in T : |f(x)| > \tau\} \leq t)\}.$$

Let $t > 0$. Then the avarage function of f is defined as follows:

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*_\omega(t)dt.$$

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Let $1 < p, q < \infty$ and $f : T \to \mathbb{R}$ be a 2π-periodic measurable function. Then the weighted Lorentz spaces $L^{pq}_\omega(T)$ is defined [5, p.20], [1, p.219] as the set of all measurable functions f such that $\|f\|_{pq,\omega} < \infty$ where

$$\|f\|_{L^{pq}_\omega(T)} = \left\{ f : \|f\|_{pq,\omega} = \left(\int_T (f^{**}(t))^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} \right\}.$$

If $p = q$, $L^{pq}_\omega(T)$ turns into weighted Lebesgue space $L^p_\omega(T)$ [5, p.20].

By $E_n(f)_{L^{pq}_\omega}$ we denote the best approximation of $f \in L^{pq}_\omega(T)$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L^{pq}_\omega} = \inf \|f - T_k\|_{pq,\omega},$$

where the infimum is taken with respect to all trigonometric polynomials of degree $k \leq n$.

The weight functions ω used in the paper belong to the Muckenhoupt class $A_p(T)$ [11] which is defined by

$$\sup \frac{1}{|I|} \int_I \omega(x)dx \left(\frac{1}{|I|} \int_I \omega^{1-p^{'}}(x)dx \right)^{p-1} < \infty, p^{'} = \frac{p}{p-1}, 1 < p < \infty,$$

where the supremum is taken with respect to all the intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I.

The modulus of continuity of the function $f \in L_\omega^{pq}(T)$ is defined [8] as

$$\Omega(f,\delta)_{L_\omega^{pq}} = \sup_{|h|\leqslant\delta} \|A_h f\|_{pq,\omega}, \quad \delta > 0,$$

where

$$(A_h f)(x) := \frac{1}{h} \int\limits_0^h |f(x+t) - f(x)|\, dt$$

is the Steklov operator.

The modulus of continuity $\Omega(f,\delta)_{L_\omega^{pq}}$ is defined in this way, because the space $L_\omega^{pq}(T)$ is noninvariant, in general, under the usual shift $f(x) \to f(x+h)$. Whenever $\omega \in A_p(T)$, $1 < p, q < \infty$, the Hardy Littlewood maximal operator of every $f \in L_\omega^{pq}(T)$ is bounded in $L_\omega^{pq}(T)$ [3, Theorem 3]. Therefore the Steklov operator $A_h f$ belongs to $L_\omega^{pq}(T)$. Thus, $\Omega(f,\delta)_{L_\omega^{pq}}$ makes sense for every $\omega \in A_p(T)$. Moreover the modulus of continuity $\Omega(f,\delta)_{L_\omega^{pq}}$ is non-decreasing, non-negative, continuous function satisfying the conditions

$$\lim_{\partial\to 0} \Omega(f,\delta)_{L_\omega^{pq}} = 0, \Omega(f_1+f_2,\delta)_{L_\omega^{pq}} \leq \Omega(f_1,\delta)_{L_\omega^{pq}} + \Omega(f_2,\delta)_{L_\omega^{pq}}.$$

In weighted Lorentz spaces, Lipschitz class $Lip(\alpha, L_\omega^{pq})$ is defined as

$$Lip(\alpha, L_\omega^{pq}) := \left\{ f \in L_\omega^{pq}(T) : \Omega(f,\delta)_{L_\omega^{pq}} = O(\delta^\alpha),\ 0 < \alpha \leq 1 \right\}.$$

Since $L^{pq}_{\omega}(T) \subset L^{1}(T)$ when $\omega \in A_p(T)$, $1 < p, q < \infty$ [5, the proof of Prop. 3.3], the Fourier series and the conjugate Fourier series of $f \in L^{pq}_{\omega}(T)$ are given as

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(a_k(f)\cos kx + b_k(f)\sin kx\right), \tag{1.2}$$

$$\tilde{f}(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(a_k(f)\sin kx - b_k(f)\cos kx\right).$$

Here $a_0(f), a_k(f), b_k(f), k = 1, 2, ...,$ are Fourier coefficient of f. Let $S_n(f, x)$, $(n = 0, 1, 2, ...)$ be the nth partial sum of the series (1.2) at the point x, that is,

$$S_n(f, x) = \sum_{k=0}^{n} U_k(f)(x),$$

where

$$U_0(f)(x) := \frac{a_0(f)}{2},$$

$$U_k(f)(x) := a_k(f)\cos kx + b_k(f)\sin kx, k = 1, 2, ...$$

Let $(p_n)_0^{\infty}$ be a sequence of positive numbers. We consider Nörlund and Riesz means of the series (1.2) defined by

$$N_n(f, x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m} S_m(f, x) \tag{1.3}$$

and

$$R_n(f, x) = \frac{1}{P_n}\sum_{m=0}^{n} p_n S_m(f, x)$$

where $P_n = \sum_{m=0}^{n} p_m$, $p_{-1} = P_{-1} = 0$. In the case $p_n = 1, n \geq 0$, both of $N_n(f, x)$ and $R_n(f, x)$ yield the Cesàro mean of the series (1.2)

$$\sigma_n(f, x) = \frac{1}{n+1}\sum_{m=0}^{n} S_m(f, x).$$

Let $A = (a_{nk})$ be a lower triangular regular matrix with nonnegative entries and row sums t_n. The operator Δ is defined by $\Delta_k a_{nk} = a_{nk} - a_{n,k+1}$. Such a matrix A is said to have monotone rows if, for each n, $\{a_{nk}\}$ is either nonincreasing or nondecreasing in $k, 0 \leq k \leq n$. We define

$$\tau_n(f,x) = \sum_{k=0}^{n} a_{nk} S_k(f,x),\ n = 0, 1, 2, ...$$

Note that, in the case of $p_n = 0,\ n \geqslant 0,\ N_n(f,x)$ and $R_n(f,x)$ yield $\sigma_n(f,x)$. Furthermore $\tau_n(f,x)$ is a generalization of $N_n(f,x)$ and $R_n(f,x)$.

Quade [12] investigated the approximation properties of the σ_n Cesàro mean in Lebesgue spaces. Similar results were studied by many researchers [2, 4, 9, 10]. In [2], Chandra gave some conditions on the sequence $(p_n)_0^\infty$ and investigated approximation problems of N_n mean and R_n mean to approximate f function in the Lebesgue spaces. In [9], Leindler weakened the conditions given by Chandra on the sequence $(p_n)_0^\infty$ and investigated same approximation properties in Lebesgue spaces. In [4], Guven obtained the generalizations of Chandra's [2] results for weighted Lebesgue spaces. Mittal et al. in [10] have generalized the results obtained by Chandra [2] to more general classes of triangular matrix methods.

In this work, we generalize the results obtained by Chandra [2] and Mittal et al. [10] to weighted Lorentz spaces. Our main results are the following.

Theorem 1. *Let* $1 < p, q < \infty$, $\omega \in A_p(\mathbb{T}), 0 < \alpha \leq 1$ *and let* $(p_n)_0^\infty$ *be a monotonic sequence of positive numbers such that*

$$(n+1)p_n = O(P_n). \tag{1.4}$$

then

$$\|f - N_n(f)\|_{pq,\omega} = O(n^{-\alpha}), n = 1, 2, ... \tag{1.5}$$

Theorem 2. *Let* $1 < p, q < \infty, \omega \in A_p(\mathbb{T}), 0 < \alpha \leq 1$ *and let* $(p_n)_0^\infty$ *be a sequence of positive real numbers satisfying the relation*

$$\sum_{m=0}^{n-1} \left| \Delta \left(\frac{P_m}{m+1} \right) \right| = O\left(\frac{P_n}{n+1} \right), \tag{1.6}$$

then

$$\|f-R_n(f)\|_{pq,\omega}=O(n^{-\alpha}),n=1,2,... \tag{1.7}$$

where

$$\Delta\left(\frac{P_m}{m+1}\right)=\frac{P_m}{m+1}-\frac{P_{m+1}}{m+2}.$$

Theorem 3. *Let* $f\in Lip(\alpha,L_{\omega}^{pq})$ *and let* A *have monotone rows and satisfy*

$$|t_n-1|=O(n^{-\alpha}). \tag{1.8}$$

(i) If $1<p,q<\infty, 0<\alpha<1,$ *and* A *also satisfies*

$$(n+1)\max\{a_{n0},\ a_{nr}\}=O(1), \tag{1.9}$$

where $r:=\left[\frac{n}{2}\right]$ *, then*

$$\|f-\tau_n(f)\|_{pq,\omega}=O(n^{-\alpha}). \tag{1.10}$$

(ii) If $1<p,q<\infty, \alpha=1,$ *then the estimate* (1.10) *is satisfied.*

Lemma 1. *Let* $f\in Lip(1,L_{\omega}^{pq})$*. Then for* $n=1,2,...$ *the estimate*

$$\|\sigma_n(f)-S_n(f)\|_{pq,\omega}=O(n^{-1})$$

holds.

Proof. If $f\in Lip(\alpha,L_{\omega}^{pq})$, from Lemma 4.6 of [7] it can be deduced that f is absolutely continuous and $f^{'}\in L_{\omega}^{pq}$. If the Fourier series of f is

$$f(x)\sim\sum_{k=0}^{n}U_k(f)(x),$$

then the conjugate function $\tilde{f}'(x)$ has the Fourier series

$$\tilde{f}'(x)\sim\sum_{k=0}^{n}kU_k(f)(x).$$

On the other hand,

$$\begin{aligned} S_n(f)(x) - \sigma_n(f)(x) &= \sum_{k=1}^{n} \frac{k}{n+1} U_k(f)(x) \\ &= \frac{1}{n+1} S_n(\tilde{f}')(x). \end{aligned}$$

Since the partial sums and the conjugate operator is uniform bounded in the space $L^{pq}_{\omega}(\mathbb{T})$ (see [7]), we get that

$$\left\| S_n(f) - \sigma_n(f) \right\|_{pq,\omega} = O(n^{-1})$$

for n=1,2,....

Lemma 2. *Let* $0 < \alpha \leq 1, 1 < p, q < \infty$ *and* $f \in$ *Lip*$(\alpha, L^{pq}_{\omega})$, $\omega \in A_p(\mathbb{T})$. *Then*

$$\left\| f - S_n(f) \right\|_{pq,\omega} = O(n^{-\alpha}) \text{ , } n = 1, 2, ...$$

Proof. Let $t^*_n (n = 0, 1, ...)$ be a trigonometric polynomial of best approximation to f, that is

$$\left\| f - t^*_n \right\|_{pq,\omega} = \inf \left\| f - t_n \right\|_{pq,\omega} ,$$

where the infimum is taken over all trigonometric polynomials t_n of degree at most n. From Lemma 2.3 of [13], we have

$$\left\| f - t^*_n \right\|_{pq,\omega} = O(\Omega(f, 1/n)_{L^{pq}_{\omega}})$$

and hence

$$\left\| f - t^*_n \right\|_{pq,\omega} = O(n^{-\alpha}).$$

By the uniform boundedness of the partial sums $S_n(f)$ in the space L^{pq}_{ω}(see [7, Prop. 3.4],[6, Th. 6.6.2],[14, Chapter VI]), we get

$$\begin{aligned} \left\| f - S_n(f) \right\|_{pq,\omega} &\leq \left\| f - t^*_n \right\|_{pq,\omega} + \left\| t^*_n - S_n(f) \right\|_{pq,\omega} \\ &= \left\| f - t^*_n \right\|_{pq,\omega} + \left\| S_n(t^*_n - f) \right\|_{pq,\omega} \\ &= O(\left\| f - t^*_n \right\|_{pq,\omega}) \\ &= O(n^{-\alpha}). \end{aligned}$$

Lemma 3. [2] *Let* (p_n) *be a non-increasing sequence of positive numbers. Then*

$$\sum_{m=1}^{n} m^{-\alpha} p_{n-m} = O(n^{-\alpha} P_n)$$

for $0 < \alpha < 1$.

Lemma 4. [11] *Let* A *have monotone rows and satisfy*

$$(n+1)\max\{a_{n0},\ a_{nr}\} = O(1).$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=1}^{n} a_{nk}(k+1)^{-\alpha} = O(n^{-\alpha}).$$

2 Proof of Main Theorems

Proof of Theorem 1. *Case I.* Let $0 < \alpha < 1$.
By (1.3), we have

$$N_n(f)(x) - f(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}\left\{S_m(f,x) - f(x)\right\}.$$

By (1.4), Lemma 2 and Lemma 3,

$$\begin{aligned}
\|f - N_n(f)\|_{pq,\omega} &\leq \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}\|f - S_m(f)\|_{pq,\omega} \\
&= \frac{1}{P_n}\sum_{m=1}^{n} p_{n-m} O(m^{-\alpha}) + \frac{p_n}{P_n}\|f - S_0(f)\|_{pq,\omega} \\
&= \frac{1}{P_n} O(n^{-\alpha} P_n) + O(\frac{1}{n+1}) \\
&= O(n^{-\alpha}).
\end{aligned}$$

Case II. Let $\alpha = 1$.

Since

$$N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} U_m(f)(x),$$

from Abel's transformation,

$$\begin{aligned} S_n(f)(x) - N_n(f)(x) &= \frac{1}{P_n} \sum_{m=1}^{n} (P_n - P_{n-m})\, U_m(f)(x) \\ &= \frac{1}{P_n} \sum_{m=1}^{n} \Delta_m \left(\frac{P_n - P_{n-m}}{m} \right) \sum_{k=1}^{m} k U_k(f)(x) \\ &\quad + \frac{1}{n+1} \sum_{k=1}^{n} k U_k(f)(x). \end{aligned}$$

Hence,

$$\begin{aligned} \|S_n(f) - N_n(f)\|_{pq,\omega} = \frac{1}{P_n} \sum_{m=1}^{n} \left| \Delta_m \left(\frac{P_n - P_{n-m}}{m} \right) \right| \left\| \sum_{k=1}^{m} k U_k(f)(x) \right\|_{pq,\omega} \\ + \frac{1}{n+1} \left\| \sum_{k=1}^{n} k U_k(f)(x) \right\|_{pq,\omega}. \end{aligned} \tag{2.1}$$

Since

$$\sigma_n(f)(x) - S_n(f)(x) = \frac{1}{n+1} \sum_{k=1}^{n} k U_k(f)(x), \tag{2.2}$$

By Lemma 1, we get

$$\left\| \sum_{k=1}^{n} k U_k(f)(x) \right\|_{pq,\omega} = (n+1) \left\| \sigma_n(f) - S_n(f) \right\|_{pq,\omega} = O(1). \tag{2.3}$$

Combining (2.1) and (2.2), we have

$$\|S_n(f) - N_n(f)\|_{pq,\omega} = O\left(\frac{1}{P_n} \right) \sum_{m=1}^{n} \left| \Delta_m \left(\frac{P_n - P_{n-m}}{m} \right) \right| + O(n^{-1}). \tag{2.4}$$

In the other hand,

$$\begin{aligned}\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right) &= \frac{P_{n-m-1}-P_{n-m}}{m}+\frac{P_n-P_{n-m-1}}{m(m+1)}\\ &= \frac{P_n-P_{n-m-1}}{m(m+1)}-\frac{p_{n-m}}{m}\\ &= \frac{1}{m(m+1)}\left\{(P_n-P_{n-m-1})-mp_{n-m}\right\}\\ &= \frac{1}{m(m+1)}\left\{\sum_{k=n-m+1}^{n}p_k-mp_{n-m}\right\}.\end{aligned}$$

This equality implies that

$$\left\{\frac{P_n-P_{n-m}}{m}\right\}_{m=1}^{n+1}$$

is non-increasing whenever (p_n) is non-decreasing and non-decreasing whenever (p_n) is non-increasing. This implies that

$$\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)\right|=\left|p_n-\frac{P_n}{n+1}\right|=\frac{1}{n+1}=O\left(P_n\right), \tag{2.5}$$

by using convention $P_{-1}=0$. Using (2.5) and (1.4) in (2.4), we obtain

$$\|S_n(f)-N_n(f)\|_{pq,\omega}=O(n^{-1}). \tag{2.6}$$

Finally, by using (2.6) and Lemma 2, we get

$$\|f-N_n(f)\|_{pq,\omega}=O(n^{-\alpha})$$

with $\alpha=1$.

Proof of Theorem 2. *Case I.* Let $0<\alpha<1$.

We have

$$f(x)-R_n(f)(x)=\frac{1}{P_n}\sum_{m=0}^{n}p_m\left\{f(x)-S_m(f,x)\right\}.$$

By Lemma 2,

$$\|f-R_n(f)\|_{pq,\omega}\leq\frac{1}{P_n}\sum_{m=0}^{n}p_m\,\|f-S_m(f)\|_{pq,\omega}$$

$$= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n} m^{-\alpha}p_m, \tag{2.7}$$

By Abel's transformation

$$\sum_{m=1}^{n} m^{-\alpha}p_m = \sum_{m=1}^{n-1} P_m[m^{-\alpha} - (m+1)^{-\alpha} + n^{-\alpha}P_n \tag{2.8}$$

$$\leq \sum_{m=1}^{n-1} m^{-\alpha}\frac{P_m}{m+1} + n^{-\alpha}P_n,$$

by (1.6)

$$\sum_{m=1}^{n-1} m^{-\alpha}\frac{P_m}{m+1} = \sum_{m=1}^{n-1}\left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right)\left(\sum_{k=1}^{m} k^{-\alpha}\right) + \frac{P_n}{n+1}\sum_{m=1}^{n-1} m^{-\alpha}.$$

This implies

$$\sum_{m=1}^{n} m^{-\alpha}p_m = O\left(n^{-\alpha}P_n\right).$$

From (2.7) and (2.8), we obtain

$$\|f - R_n(f)\|_{pq,\omega} = O(n^{-\alpha}).$$

Case II. Let $\alpha = 1$. By Abel's transformation,

$$\begin{aligned} R_n(f)(x) &= \frac{1}{P_n}\sum_{m=0}^{n-1}\left\{P_m\left(S_m(f)(x) - S_{m+1}(f)(x)\right) + P_nS_n(f)(x)\right\} \\ &= \frac{1}{P_n}\sum_{m=0}^{n-1} P_n\left(-U_{m+1}(f)(x)\right) + S_n(f)(x) \end{aligned}$$

and so

$$R_n(f)(x) - S_n(f)(x) = -\frac{1}{P_n}\sum_{m=0}^{n-1} P_mU_{m+1}(f)(x).$$

Once again, by Abel's transformation, we get

$$\begin{aligned}\sum_{m=0}^{n-1} P_m U_{m+1}(f)(x) &= \sum_{m=0}^{n-1} \Delta\left(\frac{P_m}{m+1}\right) \sum_{k=0}^{m} (k+1) U_{m+1}(f)(x) \\ &= \sum_{m=0}^{n-1} \left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right) \left(\sum_{k=0}^{m} (k+1) U_{m+1}(f)(x)\right) \\ &\quad + \frac{P_n}{n+1} \sum_{k=0}^{n-1} (k+1)\, U_{m+1}(f)(x)\end{aligned}$$

and so

$$\begin{aligned}\left\|\sum_{m=0}^{n-1} P_m U_{m+1}(f)\right\|_{pq,\omega} &\leq \sum_{m=0}^{n-1} \left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right| + \left\|\sum_{k=0}^{m} (k+1) U_{m+1}(f)\right\|_{pq,\omega} \\ &\quad + \frac{P_n}{n+1} \left\|\sum_{k=0}^{n-1} (k+1)\, U_{m+1}(f)\right\|_{pq,\omega} \\ &= \sum_{m=0}^{n-1} \left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right| (m+2) \left\|S_{m+1}(f) - \sigma_{m+1}(f)\right\|_{pq,\omega} \\ &\quad + P_n \left\|S_n(f) - \sigma_n(f)\right\|_{pq,\omega} \\ &= O(1) \sum_{m=0}^{n-1} \left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right| + O\left(\frac{P_n}{n+1}\right).\end{aligned}$$

Therefore

$$\left\|R_n(f) - S_n(f)\right\|_{pq,\omega} = O\left(n^{-1}\right). \tag{2.9}$$

Applying (2.9) and Lemma 2 to

$$\left\|f - R_n(f)\right\|_{pq,\omega} = \left\|f - S_n(f)\right\|_{pq,\omega} + \left\|S_n(f) - R_n(f)\right\|_{pq,\omega},$$

we get

$$\left\|f - R_n(f)\right\|_{pq,\omega} = O(n^{-\alpha})$$

for $\alpha = 1$.

Proof of Theorem 3. Case I. Let $0 < \alpha < 1$.

$$\begin{aligned} \tau_n(f) - f &= \sum_{k=0}^{n} a_{n,k} s_k(f) - t_n f + (t_n - 1) f \\ &= \sum_{k=0}^{n} a_{n,k} (s_k(f) - f) + (t_n - 1) f. \end{aligned}$$

From (1.8), Lemma 2 and Lemma 4,

$$\begin{aligned} \|\tau_n(f) - f\|_{pq,\omega} &\leq \sum_{k=0}^{n} a_{n,k} \|s_k(f) - f\|_{pq,\omega} + |t_n - 1| \, \|f\|_{pq,\omega} \\ &= \sum_{k=1}^{n} a_{n,k} O((k+1)^{-\alpha}) + O(n^{-\alpha}) \\ &= O(n^{-\alpha}). \end{aligned}$$

Case II. Let $\alpha = 1$.

$$\|\tau_n(f) - f\|_{pq,\omega} \leq \|\tau_n(f) - S_n(f)\|_{pq,\omega} + \|S_n(f) - f\|_{pq,\omega} .$$

by Lemma 2,

$$\|f - S_n(f)\|_{pq,\omega} = O(n^{-1}).$$

Hence it remains to prove that

$$\|\tau_n(f) - S_n(f)\|_{pq,\omega} = O(n^{-1}).$$

If we define $A_{nk} := \sum_{i=k}^{n} a_{ni}$, and use the fact that $A_{n0} = t_n$, then we have

$$\tau_n(f) = \sum_{k=0}^{n} a_{nk} S_k(f) = \sum_{k=0}^{n} a_{nk} \sum_{i=0}^{k} U_i(f)(x) = \sum_{k=0}^{n} A_{nk} U_k(f)(x).$$

Also

$$\begin{aligned} S_n(f) &= \sum_{k=0}^{n} U_k(f)(x) = \sum_{k=0}^{n} A_{n0} U_k(f)(x) + \sum_{k=0}^{n} (1 - A_{n0}) U_k(f)(x) \\ &= \sum_{k=0}^{n} A_{n0} U_k(f)(x) + (1 - t_n) \sum_{k=0}^{n} U_k(f)(x) \\ &= \sum_{k=0}^{n} A_{n0} U_k(f)(x) + (1 - t_n) S_n(f). \end{aligned}$$

Hence

$$\|\tau_n(f) - S_n(f)\|_{pq,\omega} \leq \left\| \sum_{k=1}^{n} (A_{nk} - A_{n0}) U_k(f)(x) \right\|_{pq,\omega} + |1 - t_n| \, \|f\|_{pq,\omega}.$$

We define

$$b_{nk} = \frac{A_{nk} - A_{n0}}{k}$$

for each $1 \leq k \leq n$. If we use summation by parts, then we get

$$\begin{aligned} \sum_{k=1}^{n} \left(A_{nk} - A_{n0} \right) A_k(f)(x) &= \sum_{k=1}^{n} \left(\frac{A_{nk} - A_{n0}}{k} \right) k U_k(f)(x) \\ &= \sum_{k=1}^{n} b_{nk} \left[\sum_{j=0}^{k} j U_j(f)(x) - \sum_{j=0}^{k-1} j U_j(f)(x) \right] \\ &= \sum_{k=1}^{n} b_{nk} \sum_{j=0}^{k} j U_j(f)(x) - \sum_{k=1}^{n} b_{nk} \sum_{j=0}^{k-1} j U_j(f)(x) \\ &= b_{nn} \sum_{j=1}^{n} j U_j(f)(x) + \sum_{k=1}^{n-1} \triangle_k b_{nk} \sum_{j=0}^{k} j U_j(f)(x). \end{aligned}$$

Hence

$$\|\tau_n(f) - f\|_{pq,\omega} \leq \left\| b_{nn} \sum_{j=1}^{n} j U_j(f) \right\|_{pq,\omega} + \left\| \sum_{k=1}^{n-1} \triangle_k b_{nk} \sum_{j=0}^{k} j U_j(f) \right\|_{pq,\omega} + O(n^{-1}).$$

Also

$$\begin{aligned}\sigma_n(f) &= \frac{1}{n+1}\sum_{k=0}^{n} S_k(f) = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{k} U_j(f)(x)\\ &= \frac{1}{n+1}\sum_{j=0}^{n} U_j(f)(x)\sum_{k=j}^{n} 1 = \frac{1}{n+1}\sum_{j=0}^{n}(n-j+1)U_j(f)(x)\\ &= \sum_{j=0}^{n} U_j(f)(x) - \frac{1}{n+1}\sum_{j=0}^{n} jU_j(f)(x).\end{aligned}$$

By Lemma 4,

$$\begin{aligned}\left\|\sum_{j=1}^{n} jU_j(f)(x)\right\|_{pq,\omega} &= \|(n+1)(S_n(f)-\sigma_n(f))+S_n(f)\|_{pq,\omega}\\ &= (n+1)O(n^{-1}) + \|f\|_{pq,\omega} = O(1).\end{aligned}$$

Note that

$$|b_{nn}| = (n+1)^{-1}|A_{n0}-A_{nn}| = (n+1)^{-1}|t_n - a_{nn}| = (n+1)^{-1}O(1).$$

Therefore

$$\left\|b_{nn}\sum_{j=1}^{n} jU_j(f)(x)\right\|_{pq,\omega} = O(n^{-1}).$$

We can write

$$\begin{aligned}\triangle_k b_{nk} &= \frac{1}{k}\triangle_k\left(A_{nk}-A_{n0}\right) + \frac{A_{n,k+1}-A_{n0}}{k(k+1)}\\ &= \frac{1}{k(k+1)}\left[(k+1)\triangle_k A_{nk} + \sum_{r=k+1}^{n} a_{nr} - \sum_{r=0}^{n} a_{nr}\right]\\ &= \frac{1}{k(k+1)}\left[(k+1)a_{nk} - \sum_{r=0}^{k} a_{nr}\right].\end{aligned}$$

If $\{a_{nk}\}$ is nonincreasing in k, then $\triangle_k b_{nk} \le 0$, and if $\{a_{nk}\}$ is nondecreasing in k

then $\triangle_k b_{nk} \geqslant 0$, so that

$$\begin{aligned}\sum_{k=1}^{n-1} |\triangle_k b_{nk}| &= |b_{n1} - b_{nn}| = \left| A_{n1} - A_{n0} - \frac{A_{nn} - A_{n0}}{n} \right| \leq |a_{n0}| + \left| \frac{a_{nn} - t_n}{n} \right| \\ &= O(n^{-1}) + \frac{O(1)}{n} = O(n^{-1}),\end{aligned}$$

and (1.10) is satisfies.

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