MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
ISSN: 2147–6268

14 (2) 92-105 (2026) – Research Article
https://doi.org/10.36753/mathenot.1870199

Some Fixed-Function Results on S-Metric Spaces with an Application to the Treatment Plan for Tumour Patients

Nihal Taş* and Melek Boyraz

Abstract
In this study, fixed-function results are established in the framework of S-metric spaces under both Banach-type and Reich-type contractive conditions. These results extend classical fixed-point theory to a functional setting and ensure the existence and uniqueness of fixed functions in generalized metric spaces. Moreover, the Banach-type fixed function theorem is applied to obtain the best approximation of treatment plans for tumour patients undergoing intensity modulated radiation therapy (IMRT). This application demonstrates the effectiveness of fixed-function theory in S-metric spaces for modeling and solving practical optimization problems. The proposed results contribute to both the theoretical development and applied aspects of fixed-function theory.

Keywords: *Fixed function, Fixed-point theory, S-metric space, Tumour application*

****Corresponding author***

1. Introduction and Background

Fixed-point theory constitutes one of the central and most extensively studied areas of nonlinear analysis due to its profound theoretical significance and its wide range of applications in mathematics, engineering, economics, and applied sciences. Classical fixed-point results, such as Banach's contraction principle [1], have provided powerful tools for investigating the existence and uniqueness of solutions to various functional, differential, and integral equations. Motivated by these foundational results, numerous generalizations of metric spaces and contraction conditions have been introduced in order to extend the scope of fixed-point theory and to model more complex phenomena. There are many studies in the literature on the generalizations and the application areas of fixed point theory (for example, [2–7] and the references therein).

Among these generalizations, S-metric spaces [8], introduced as a natural extension of standard metric spaces, have attracted considerable attention in recent years. An S-metric space replaces the conventional two-variable distance function with a three-variable function, allowing a richer geometric and topological structure. This framework has proven to be particularly effective in establishing fixed-point results under weaker assumptions and in settings where classical metric conditions are not sufficient. Consequently, a growing body of literature has focused on

Received : *23-01-2026,* ***Accepted :*** *05-05-2026,* ***Available online :*** *06-05-2026*

Cite as: N. Taş, M. Boyraz, Some fixed-function results on S-metric spaces with an application to the treatment plan for tumour patients, Math. Sci. Appl. E-Notes, 14(2) (2026), 92-105.

This is an open-access article distributed under the terms of the Creative Commons Attribution-NonCommercial 4.0 International License.

fixed-point theorems in S-metric spaces, including various contractive mappings and ordered structures.

Parallel to the development of generalized metric frameworks, the notion of a fixed function has emerged as an important extension of the fixed-point concept [9]. While fixed-point theory concerns the invariance of individual elements under a mapping, fixed-function theory studies the invariance of entire functions under suitable operators. This perspective is especially relevant in applications involving iterative schemes, optimization problems, and functional equations, where convergence to a stable functional profile is of primary interest rather than convergence to a single point. Fixed-function results provide a unifying framework for analyzing sequences of functions arising in applied models and offer a natural bridge between abstract fixed-point theory and practical applications.

The interplay between S-metric spaces and fixed function theory has opened new avenues for research, particularly in the analysis of convergence behaviors of function-valued iterations. Applications of these concepts can be found in areas such as image processing, signal reconstruction, optimization algorithms, and medical treatment planning, where function sequences representing different states or configurations are required to converge to a stable and optimal solution. By employing generalized distance structures and fixed-function techniques, one can obtain stronger convergence guarantees and more flexible modeling capabilities.

Motivated by these considerations, the present study aims to contribute to the existing literature by investigating fixed point and fixed function results within the framework of S-metric spaces. In addition to establishing theoretical results, we emphasize the applicability of the proposed approach by discussing relevant examples and potential applications. The obtained results not only generalize several known fixed-point theorems but also provide an effective mathematical foundation for analyzing iterative processes involving function-valued mappings in generalized metric settings.

Motivated by the work of Dhawan et al. [9] (also, [10]), we investigate the applicability of fixed-function techniques in S-metric spaces to the problem of determining the best approximation of treatment plans for tumour patients undergoing intensity modulated radiation therapy (IMRT). In this context, the treatment planning process can be naturally formulated within a functional framework, where dose distributions are modeled as elements of an appropriate function space endowed with an S-metric structure. This generalized metric setting allows the simultaneous incorporation of multiple dose interactions and provides greater flexibility than classical metric formulations.

By adopting this approach, the iterative procedures used in IMRT optimization can be interpreted as mappings acting on S-metric spaces, where convergence to an optimal dose distribution corresponds to the existence of a fixed point or fixed function. The best approximation of a treatment plan is then obtained as the limit of a sequence generated by such mappings, ensuring stability and consistency of the resulting dose distributions. The S-metric framework further enables the analysis of convergence under weaker conditions, thereby broadening the range of admissible treatment scenarios.

Consequently, inspired by the methodology of Dhawan et al. [9], our results demonstrate that S-metric space techniques provide an effective mathematical foundation for constructing and approximating clinically suitable IMRT treatment plans. This perspective not only generalizes existing fixed-function-based approaches but also offers a systematic and theoretically justified tool for addressing complex optimization problems arising in radiation therapy planning.

2. A Brief Survey of S-Metric Spaces

In this section, we recall some basic concepts related to the S-metric spaces.

Definition 2.1. [8] Let X be a nonempty set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, w \in X$:

(**S1**) $S(x, y, z) = 0 \Longleftrightarrow x = y = z$,

(**S2**) $S(x, y, z) \leq S(x, x, w) + S(y, y, w) + S(z, z, w)$.
Then S is called an S-metric on X and the pair (X, S) is called an S-metric space.

Example 2.1. [8] Let $X = \mathbb{R}^n$ and $\|.\|$ be a norm on X. Then

$$S(x, y, z) = \|y + z - 2x\| + \|y - z\|$$

is an S-metric on X.

Lemma 2.1. *[8] Let (X, S) be an S-metric space and $x, y \in X$. Then we have*

$$S(x, x, y) = S(y, y, x).$$

Definition 2.2. [8] Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in this space.

1. A sequence $\{x_n\} \subset X$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$.
2. A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.
3. The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Lemma 2.2. *[8] Let (X, S) be an S-metric space and $\{x_n\}$, $\{y_n\}$ be two sequences in this space. If $x_n \to x$ and $y_n \to y$ then we have*

$$S(x_n, x_n, y_n) \to S(x, x, y).$$

Lemma 2.3. *[11] Let (X, S), (Y, S') be two S-metric spaces, $f : X \to Y$ be a function and $\{x_n\}$ be a sequence in this space. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

Lemma 2.4. *[12] Let (X, d) be a metric space. Then, the followings are satisfied*:

1. *$S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).*
3. *$\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, S_d).*
4. *(X, d) is complete if and only if (X, S_d) is complete.*

The metric S_d was called as the S-metric generated by d [13]. There are known instances of S-metrics that are not generated by any metric (for further information, see [12], [13] and [14]).

A following example of an S-metric is not generated by any metric:

Example 2.2. [13] Let $X = \mathbb{R}$ and the function $S : X \times X \times X \to [0, \infty)$ be defined as

$$S(x, y, z) = |x - z| + |y + z - 2x|,$$

for all $x, y, z \in \mathbb{R}$. Then S is an S-metric which is not generated by any metric and the pair (X, S) is an S-metric space.

According to Gupta, each S-metric on X defines a metric d_S on X as follows:

$$d_S(x, y) = S(x, x, y) + S(y, y, x), \tag{2.1}$$

for all $x, y \in X$ [15].

The function $d_S(x, y)$ defined in (2.1) does not always define a metric because the triangle inequality is not satisfied for all members of X everytime as seen in the following example:

Example 2.3. [13] Let $X = \{1, 2, 3\}$. If we consider the function $S : X \times X \times X \to [0, \infty)$ defined by

$$S(1, 1, 2) = S(2, 2, 1) = 5,$$

$$S(2, 2, 3) = S(3, 3, 2) = S(1, 1, 3) = S(3, 3, 1) = 2,$$

$$S(x, y, z) = 0 \text{ if } x = y = z,$$

$$S(x, y, z) = 1 \text{ otherwise},$$

for all $x, y, z \in X$, then S does not generate a metric d_S. Also, S is an S-metric on X which is not generated by any metric d.

The following theorem shows that the connection between an S-metric and a b-metric as specified in [16]:

Theorem 2.1. *[11] Let (X, S) be an S-metric space and*

$$d^S(x, y) = S(x, x, y),$$

for all $x, y \in X$. Then we have

1. *d^S is a b-metric on X,*
2. *$x_n \to x$ in (X, S) if and only if $x_n \to x$ in (X, d^S),*
3. *$\{x_n\}$ is Cauchy in (X, S) if and only if $\{x_n\}$ is Cauchy in (X, d^S).*

The metric d^S is called the b-metric generated by S.

Definition 2.3. [17] Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions:

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$,

$(G2)$ $0 < G(x, x, y)$ if $x \neq y$,

$(G3)$ $G(x, x, y) \leq G(x, y, z)$ if $y \neq z$,

$(G4)$ The symmetry on three variables

$$G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x),$$

$(G5)$ The rectangle inequality

$$G(x, y, z) \leq G(x, a, a) + G(a, y, z).$$

Then G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Remark 2.1. [18] The class of all S-metrics and the class of all G-metrics are distinct.

(1) There exists a G-metric which is not an S-metric (see Example 2.1. given in [18]).

(2) There exists an S-metric which is not a G-metric (see Example 2.2. given in [9]).

3. Main Results

In this section, we obtain some fixed-function results on S-metric spaces. Before moving on to the main results, we recall the notion of a fixed function.

Definition 3.1. [9] Let $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ be any self-mapping where $\mathcal{F}$ is a family of functions f. Then $f \in \mathcal{F}$ is called a fixed function of $\mathcal{P}$ if

$$\mathcal{P}f = f.$$

Example 3.1. [9] Let $X = [1, 2]$ and the self-mapping $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ be defined as

$$\mathcal{P}f(x) = f^2(x) - 2f(x) + 2,$$

for all $f \in \mathcal{F}$ and $x \in X$. Then, for all $x \in X$, the functions

$$f(x) = 2$$

and

$$f(x) = 1$$

are two fixed functions of $\mathcal{P}$.

Example 3.2. Let $X = [1, 5]$ and the self-mapping $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ be defined as

$$\mathcal{P}f(x) = f^2(x) - 5f(x) + 5,$$

for all $f \in \mathcal{F}$ and $x \in X$. Then, for all $x \in X$, the functions

$$f(x) = 5$$

and

$$f(x) = 1$$

are two fixed functions of $\mathcal{P}$.

Example 3.3. [9] Let $X = \mathbb{R}^+$, $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ be a self-mapping and $f \in \mathcal{F}$ be defined as

$$f(x) = \begin{cases} -1 & , \quad x \in [0, 1] \\ 0 & , \quad x \in (1, \infty) \end{cases}.$$

Then f^3 is a fixed function of $\mathcal{P}$.

Definition 3.2. Let (X, S) be an S-metric space, $\mathcal{F}$ be the collection of all bounded functions defined on X and $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ be a self-mapping. Then $\mathcal{P}$ is called a $\mathcal{P}_S$-contraction mapping on X if for any real number $\lambda \in [0, 1)$ and for all $f, g \in \mathcal{F}$, we have

$$S^*(\mathcal{P}f, \mathcal{P}f, \mathcal{P}g) \leq \lambda S^*(f, f, g),$$

where

$$\begin{aligned} S^*(f, g, h) &= \sup\{S(f, g, h) : x, y, z \in X\} \\ &= \sup\{|f(x) - h(z)| + |f(x) + h(z) - 2g(y)| : x, y, z \in X\}. \end{aligned} \tag{3.1}$$

Remark 3.1. The function $S^* : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \to [0, \infty)$ is an S-metric on $\mathcal{F}$. Indeed, for $x, y, z \in X$ and $f, g, h \in \mathcal{F}$, we have

$(S1)$

$$\begin{aligned} S^*(f, g, h) &= 0 \Leftrightarrow \sup\{S(f, g, h) : x, y, z \in X\} = 0 \\ &\Leftrightarrow \sup\{|f(x) - h(z)| + |f(x) + h(z) - 2g(y)| : x, y, z \in X\} = 0 \\ &\Leftrightarrow f \sim g \sim h. \end{aligned}$$

$(S2)$

$$\begin{aligned} & S^*(f, g, h) \\ = \ & \sup\{S(f, g, h) : x, y, z \in X\} \\ = \ & \sup\{|f(x) - h(z)| + |f(x) + h(z) - 2g(y)| : x, y, z \in X\} \\ = \ & \sup\{|f(x) - h(z)| + |f(x) - g(y) + h(z) - g(y)| : x, y, z \in X\} \\ = \ & \sup\left\{ \begin{array}{c} |f(x) - p(t) + p(t) - h(z)| \\ + |f(x) - p(t) + p(t) - g(y) + h(z) - p(t) + p(t) - g(y)| : x, y, z, t \in X \end{array} \right\} \\ \leq \ & \sup\left\{ \begin{array}{c} |f(x) - p(t)| + |p(t) - h(z)| + |f(x) - p(t)| + |p(t) - g(y)| \\ + |h(z) - p(t)| + |p(t) - g(y)| : x, y, z, t \in X \end{array} \right\} \\ = \ & \sup\{2|f(x) - p(t)| + 2|g(y) - p(t)| + 2|h(z) - p(t)| : x, y, z, t \in X\} \\ = \ & \sup\{S(f, f, p) + S(g, g, p) + S(h, h, p) : x, y, z, t \in X\} \\ \leq \ & \sup\{S(f, f, p) : x, t \in X\} + \sup\{S(g, g, p) : y, t \in X\} + \sup\{S(h, h, p) : z, t \in X\} \\ = \ & S^*(f, f, p) + S^*(g, g, p) + S^*(h, h, p). \end{aligned}$$

Therefore, the function S^* is an S-metric. Also, S^* is not generated by any metric d. To show that, on the contrary, we assume that the S-metric S^* is generated by any metric d, that is, for all $f, g, h \in \mathcal{F}$, there exists a metric d such that

$$S^*(f, g, h) = d(f, h) + d(g, h).$$

Then, we get

$$\begin{array}{rcl} S^{*}(f,f,h) & = & 2d(f,h), \\ S^{*}(f,f,h) & = & 2\sup\{|f(x)-h(z)| : x,z\in X\} \\ & \Longrightarrow & d(f,h)=\sup\{|f(x)-h(z)| : x,z\in X\} \end{array}$$

and

$$\begin{array}{rcl} S^{*}(g,g,h) & = & 2d(g,h), \\ S^{*}(g,g,h) & = & 2\sup\{|g(y)-h(z)| : y,z\in X\} \\ & \Longrightarrow & d(g,h)=\sup\{|g(y)-h(z)| : y,z\in X\}. \end{array}$$

Hence, we obtain

$$\begin{array}{rl} & \sup\{|f(x)-h(z)|+|f(x)+h(z)-2g(y)| : x,y,z\in X\} \\ = & \sup\{|f(x)-h(z)| : x,z\in X\}+\sup\{|g(y)-h(z)| : y,z\in X\}, \end{array}$$

which is a contradiction. Indeed, if we consider the following constant functions

$$f(x)=5,\, g(y)=0 \text{ and } h(z)=1,$$

so we get

$$10\neq 5.$$

Consequently, the S-metric S^{*} is not generated by any metric d.

Theorem 3.1. (*Banach Type Fixed-Function Theorem*) *Let (X,S) be a complete S-metric space with the S-metric S defined as*

$$S(x,y,z)=|x-z|+|x+z-2y|,$$

for all $x,y,z\in X$ [13]. Let $\mathcal{F}$ be the collection of all bounded functions defined on X with S-metric S^{*} defined as in (3.1). If $\mathcal{P}$ is a $\mathcal{P}_S$-contraction mapping and continuity on X, then there exists a unique fixed function $f\in\mathcal{F}$, that is, there exists some $f\in\mathcal{F}$ such that

$$\mathcal{P}f=f.$$

Proof. Let $f,g,h\in\mathcal{F}$. Since $\mathcal{P}$ is a $\mathcal{P}_S$-contraction mapping on X, there exists a real number $\lambda\in[0,1)$ such that

$$S^{*}(\mathcal{P}f,\mathcal{P}f,\mathcal{P}g)\leq\lambda S^{*}(f,f,g),$$

for all $f,g\in\mathcal{F}$ where

$$S^{*}(f,g,h)=\sup\{|f(x)-h(z)|+|f(x)+h(z)-2g(y)| : x,y,z\in X\}.$$

Then we have

$$\begin{array}{rcl} S^{*}\left(\mathcal{P}^{2}f,\mathcal{P}^{2}f,\mathcal{P}^{2}g\right) & \leq & \lambda S^{*}(\mathcal{P}f,\mathcal{P}f,\mathcal{P}g) \\ & \leq & \lambda^{2}S^{*}(f,f,g) \end{array}$$

and so, similarly, we obtain

$$S^{*}(\mathcal{P}^{n}f,\mathcal{P}^{n}f,\mathcal{P}^{n}g)\leq\lambda^{n}S^{*}(f,f,g).$$

At first, we show that $\{f_n\}$ $(n\in\mathbb{N})$ is a Cauchy sequence. To do this, let $f_0\in\mathcal{F}$ and let us define the sequence $\{f_n\}$ by

$$f_1=\mathcal{P}f_0,$$

$$f_2=\mathcal{P}f_1=\mathcal{P}^{2}f_0$$

$$\vdots$$

$$f_n=\mathcal{P}f_{n-1}=\mathcal{P}^{n}f_0.$$

Let $n, m \in \mathbb{N}$ be some positive integers with $n > m$ such that

$$n = m + t,$$

where $t \geq 1$. Then, we get

$$\begin{aligned} S^*(f_m, f_m, f_n) &= S^*(f_m, f_m, f_{m+t}) \\ &\leq 2S^*(f_m, f_m, f_{m+1}) + S^*(f_{m+t}, f_{m+t}, f_{m+1}) \\ &\vdots \\ &\leq 2\sum_{i=m}^{m+t-2} S^*(f_i, f_i, f_{i+1}) + S^*(f_{m+t-1}, f_{m+t-1}, f_{m+t}) \\ &\leq 2\lambda^m S^*(f_0, f_0, f_1)\left[1 + \lambda + \lambda^2 + \cdots\right] \\ &\leq \frac{2\lambda^m}{1-\lambda} S^*(f_0, f_0, f_1). \end{aligned}$$

Since $\mathcal{F}$ is the collection of all bounded functions, we have

$$S^*(f_m, f_m, f_n) \to 0,$$

as $n, m \to \infty$. Therefore, $\{f_n\}$ is Cauchy. Using the completeness hypothesis, the sequence $\{f_n\}$ is convergent in X, that is, there exists $f \in \mathcal{F}$ such that

$$\lim_{n\to\infty} f_n = f.$$

By the continuity of $\mathcal{P}$, we have

$$\lim_{n\to\infty} \mathcal{P}f_n = \mathcal{P}f.$$

Also, we get

$$\lim_{n\to\infty} f_{n+1} = f.$$

Hence, using the uniqueness of the limit, we obtain

$$\mathcal{P}f = f,$$

which shows that f is a fixed function of $\mathcal{P}$. Finally, to show the uniqueness of the fixed function f, we assume that g is another fixed function of $\mathcal{P}$, that is,

$$\mathcal{P}g = g.$$

Using the hypothesis, we get

$$0 \leq S^*(f, f, g) = S^*(\mathcal{P}f, \mathcal{P}f, \mathcal{P}g) \leq \lambda S^*(f, f, g) < S^*(f, f, g),$$

a contradiction. Consequently, f is a unique fixed function of $\mathcal{P}$. □

Remark 3.2. 1) The completeness of $(\mathcal{F}, S^*)$ follows from considering together the definition of the function S^*, the completeness of the space (X, S) and the notions of Cauchy sequences, convergence, and completeness. This can be easily seen by using Lemma 2.4.

2) This theorem provides a natural generalization of the classical Banach fixed-point theorem, using the concept of a fixed function on S-metric spaces.

3) This theorem generalizes Theorem 2.6 proved in [9].

4) Since the S-metric used in the statement of Theorem 3.1 is an S-metric that cannot be generated by a metric, this further increases the importance of working in S-metric spaces. Thus, instead of working with an arbitrary S-metric, we are dealing with one that is not induced by any metric.

5) Since every S-metric generates a b-metric, then Theorem 3.1 can be also considered on b-metric spaces. In other words, by working on an S-metric space, we automatically generalize the same study on a b-metric space.

Example 3.4. Let $X = [0, 1]$ and the function $S : X \times X \times X \to [0, \infty)$ be an S-metric defined as in Example 2.2. We assume that $\mathcal{F} = C[0, 1]$ is the set of all real-valued continuous functions defined on $[0, 1]$ and the function $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ is defined by

$$\mathcal{P}f(x) = \frac{4}{5} f(x),$$

for all $f \in \mathcal{F}$ and $x \in [0,1]$. Then (X,S) is a complete S-metric space and $\mathcal{F} = C[0,1]$ is the collection of all real-valued continuous, bounded functions defined on X. Let

$$f_n(x) = \frac{x^n}{n},$$

for all $x \in [0,1]$. Therefore, $\{f_n(x)\}_{x\in[0,1]}$ is Cauchy. Also, the function $\mathcal{P}$ is a $\mathcal{P}_S$-contraction mapping with $\lambda = \frac{4}{5}$. Indeed, we have

$$\begin{aligned} S^*(\mathcal{P}f, \mathcal{P}f, \mathcal{P}g) &= S^*\left(\frac{4}{5}f, \frac{4}{5}f, \frac{4}{5}g\right) \\ &= \sup\left\{\left|\frac{4}{5}f(x) - \frac{4}{5}g(y)\right| + \left|\frac{4}{5}f(x) + \frac{4}{5}g(y) - \frac{8}{5}f(x)\right| : x, y \in [0,1]\right\} \\ &= \frac{4}{5}\sup\{|f(x) - g(y)| + |f(x) + g(y) - 2f(x)| : x, y \in [0,1]\} \\ &= \frac{4}{5}S^*(f,f,g). \end{aligned}$$

Consequently, null function is a unique fixed function.

Definition 3.3. Let (X,S) be an S-metric space, $\mathcal{F}$ be the collection of all bounded functions defined on X, $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ be a self-mapping and the function S^* be defined as in (3.1). Then $\mathcal{P}$ is called a Reich type $\mathcal{P}_S$-contraction mapping on X if we have

$$S^*(\mathcal{P}f, \mathcal{P}f, \mathcal{P}g) \leq \alpha S^*(f, f, \mathcal{P}f) + \beta S^*(g, g, \mathcal{P}g) + \gamma S^*(f, f, g),$$

for all $f, g \in \mathcal{F}$ and α, β, γ nonnegative with $\alpha + \beta + \gamma < 1$.

Theorem 3.2. (*Reich Type Fixed-Function Theorem*) *Let (X,S) be a complete S-metric space with the S-metric S defined as in Theorem 3.1. Let $\mathcal{F}$ be the collection of all bounded functions defined on X with S-metric S^* defined as in (3.1). If $\mathcal{P}$ is a Reich type $\mathcal{P}_S$-contraction mapping and continuity on X, then there exists a unique fixed function $f \in \mathcal{F}$, that is, there exists some $f \in \mathcal{F}$ such that*

$$\mathcal{P}f = f.$$

Proof. Let us define the sequence $\{f_n\}$ $(n \in \mathbb{N})$ by

$$f_n = \mathcal{P}f_{n-1} = \mathcal{P}^n f_0.$$

At first, we show that $\{f_n\}$ is a Cauchy sequence. For $f_1, f_2 \in \mathcal{F}$, we have

$$\begin{aligned} S^*(f_1, f_1, f_2) &= S^*(\mathcal{P}f_0, \mathcal{P}f_0, \mathcal{P}f_1) \\ &\leq \alpha S^*(f_0, f_0, \mathcal{P}f_0) + \beta S^*(f_1, f_1, \mathcal{P}f_1) + \gamma S^*(f_0, f_0, f_1) \\ &= \alpha S^*(f_0, f_0, f_1) + \beta S^*(f_1, f_1, f_2) + \gamma S^*(f_0, f_0, f_1) \\ &= (\alpha + \gamma) S^*(f_0, f_0, f_1) + \beta S^*(f_1, f_1, f_2) \end{aligned}$$

$$\Longrightarrow (1-\beta) S^*(f_1, f_1, f_2) \leq (\alpha + \gamma) S^*(f_0, f_0, f_1)$$

$$\Longrightarrow S^*(f_1, f_1, f_2) \leq \frac{\alpha + \gamma}{1 - \beta} S^*(f_0, f_0, f_1)$$

and

$$\begin{aligned} S^*(f_2, f_2, f_3) &\leq \frac{\alpha + \gamma}{1 - \beta} S^*(f_1, f_1, f_2) \\ &\leq \left(\frac{\alpha + \gamma}{1 - \beta}\right)^2 S^*(f_0, f_0, f_1). \end{aligned}$$

If we continue this process in this manner, then we see that $\{f_n\}$ is a Cauchy sequence since $\frac{\alpha+\gamma}{1-\beta} < 1$ and $f_0, f_1 \in \mathcal{F}$ are bounded. By the hypothesis, the sequence $\{f_n\}$ is convergent to $f \in \mathcal{F}$. Now we show that f is a fixed function

of $\mathcal{P}$. To do this, let t be any arbitrary positive integer. So, using the triangle inequality and the property of the Reich type $\mathcal{P}_S$-contraction mapping, we obtain

$$\begin{aligned} S^*(f,f,\mathcal{P}f) &\leq S^*(f,f,f_t)+S^*(f,f,f_t)+S^*(\mathcal{P}f,\mathcal{P}f,f_t) \\ &= 2S^*(f,f,f_t)+S^*(\mathcal{P}f,\mathcal{P}f,f_t) \\ &= 2S^*(f,f,f_t)+S^*(\mathcal{P}f,\mathcal{P}f,\mathcal{P}f_{t-1}) \\ &\leq 2S^*(f,f,f_t)+\alpha S^*(f,f,\mathcal{P}f)+\beta S^*(f_{t-1},f_{t-1},\mathcal{P}f_{t-1}) \\ &\quad +\gamma S^*(f,f,f_{t-1}) \end{aligned}$$

$$\Longrightarrow (1-\alpha)S^*(f,f,\mathcal{P}f) \leq 2S^*(f,f,f_t)+\beta S^*(f_{t-1},f_{t-1},f_t)+\gamma S^*(f,f,f_{t-1}).$$

By taking t sufficiently large, we get

$$0 \leq S^*(f,f,\mathcal{P}f) < \varepsilon,$$

which implies

$$S^*(f,f,\mathcal{P}f)=0,$$

that is,

$$\mathcal{P}f=f.$$

Finally, we prove that f is a unique fixed function of $\mathcal{P}$ For this purpose, let g be another fixed function of $\mathcal{P}$ such that

$$f \nsim g.$$

Using the hypothesis, we get

$$\begin{aligned} S^*(f,f,g) &= S^*(\mathcal{P}f,\mathcal{P}f,\mathcal{P}g) \\ &\leq \alpha S^*(f,f,\mathcal{P}f)+\beta S^*(g,g,\mathcal{P}g)+\gamma S^*(f,f,g) \\ &= \alpha S^*(f,f,f)+\beta S^*(g,g,g)+\gamma S^*(f,f,g) \\ &= \gamma S^*(f,f,g) \\ &\Longrightarrow (1-\gamma)S^*(f,f,g) \leq 0 \\ &\Longrightarrow S^*(f,f,g) \leq 0, \end{aligned}$$

which is a contradiction. Consequently, f is a unique fixed function of $\mathcal{P}$. □

Remark 3.3. 1) The completeness of $(\mathcal{F},S^*)$ follows from considering together the definition of the function S^*, the completeness of the space (X,S) and the notions of Cauchy sequences, convergence, and completeness. This can be easily seen by using Lemma 2.4.

2) This theorem provides a natural generalization of the classical Reich fixed-point theorem, using the concept of a fixed function on S-metric spaces.

3) This theorem generalizes Theorem 2.9 proved in [9].

4) Since the S-metric used in the statement of Theorem 3.2 is an S-metric that cannot be generated by a metric, this further increases the importance of working in S-metric spaces. Thus, instead of working with an arbitrary S-metric, we are dealing with one that is not induced by any metric.

5) Since every S-metric generates a b-metric, then Theorem 3.2 can be also considered on b-metric spaces. In other words, by working on an S-metric space, we automatically generalize the same study on a b-metric space.

4. An Application to the Treatment Plan for Tumour Patients

The application discussed in this section focuses on determining an optimal treatment strategy for tumour patients undergoing intensity modulated radiation therapy (IMRT). Within this framework, an appropriate truncation of the dose deposition coefficient (DDC) matrix is employed in order to enhance the precision of the computed dose distributions. A notable contribution in this direction was made by Tian et al. [19], who proposed a fluence map optimization (FMO) model in which the DDC matrix is decomposed into two distinct parts according to a prescribed intensity threshold.

Motivated by this approach, one can construct a sequence of functions derived from the obtained theoretical results, where each function corresponds to the dose distribution of a particular patient. This sequence is shown to converge to a fixed function, which represents a common and suitable dose distribution applicable to multiple tumour patients simultaneously. In addition, related algorithmic techniques addressing similar challenges in tomotherapy have been investigated by Bortfeld [20] and Shepard et al. [21].

In practical implementations of these methods, the DDC matrix is typically calculated to approximate the dose delivered to each voxel within the region of interest by individual beamlets of unit intensity. However, such computations generally produce extremely large datasets, leading to significant demands on memory usage and computational resources. To mitigate these issues, it is common practice to discard small-magnitude entries in the DDC matrix. While this truncation strategy improves computational efficiency, it may also have a non-negligible impact on the overall quality of the resulting treatment plan.

The problem under consideration can be efficiently addressed by employing a fixed point iterative scheme, which has proven to be both robust and computational advantageous. Within this framework, an appropriate truncation strategy for the dose deposition coefficient (DDC) matrix is incorporated, leading to a substantial enhancement in the accuracy of the numerical outcomes. In the fluence map optimization model proposed by Tian et al. [19], this strategy is realized by decomposing the DDC matrix into two distinct submatrices, denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$, according to a prescribed threshold parameter.

• The matrix $\mathcal{P}_1$, referred to as the major component of the DDC matrix, is defined as the collection of all entries whose magnitudes exceed the prescribed threshold value.

• The minor component $\mathcal{P}_2$ consists of the remaining entries of the DDC matrix that are not included in the major component $\mathcal{P}_1$.

In fact, the component $\mathcal{P}_1$ primarily captures the dose contributions associated with voxels located within the tumour region, whereas $\mathcal{P}_2$ accounts for scattered dose contributions propagating over relatively large spatial distances.

The problem can be interpreted as:

$$x^{(k+1)} = \arg\min \left\| \mathcal{P}_1 x + \delta^{(k)} - T \right\| \tag{4.1}$$

and

$$\delta^{(k+1)} = \mathcal{P}_2 x^{(k+1)}. \tag{4.2}$$

Equation (4.1) describes an inner iterative scheme that can be efficiently solved by applying a suitable numerical algorithm to compute the dose vector $\delta^{(k)}$, which corresponds to the contribution associated with the minor component $\mathcal{P}_2$. Owing to the fact that the matrix $\mathcal{P}_1$ contains a significantly smaller number of nonzero entries in comparison with the full DDC matrix, the corresponding inner iteration exhibits a faster convergence behavior than that obtained from the original formulation.

The outer iteration, governed by equation (4.2), subsequently updates the dose vector $\delta^{(k+1)}$ by incorporating the effect of the minor matrix $\mathcal{P}_2$. Here, the operator T denotes the prescribed dose levels for voxels within the planned target volume, as well as the admissible threshold doses for voxels associated with organs at risk. This iterative mapping naturally generates a sequence

$$\left\{ x^{(0)}, x^{(1)}, x^{(2)}, \ldots \right\}$$

of dose distributions corresponding to an individual patient.

Under the assumption that the entries of $\mathcal{P}_2$ are sufficiently small or negligible and that there exists a constant λ ensuring the satisfaction of a suitable contraction condition, the existence of an appropriate dose distribution for a patient is guaranteed. Building upon this framework, our results enable the construction of effective treatment plans that can simultaneously accommodate multiple patients. Overall, the proposed methodology offers a computational efficient and straightforward approach for estimating clinically acceptable treatment plans.

In the present study, two tumour patients with distinct tumour severity levels are taken into consideration. Let X denote the collection of threshold intensity values, measured in Gy, prescribed for administration on specified days and during designated treatment sessions. Each patient undergoes therapy twice daily, where the indices F and G represent the set of treatment days and the corresponding treatment sessions, respectively:

$$X = \begin{cases} (1, F_1G_1), \left(\frac{1}{2}, F_1G_2\right), (1, F_2G_1), \left(\frac{1}{2}, F_2G_2\right) & , \quad Patient - I \\ (1, F_1G_1), (2, F_1G_2), (1, F_2G_1), (2, F_2G_2) & , \quad Patient - II \end{cases}.$$

It is worth noting that the set X is complete, as it constitutes a closed and bounded subset of $\mathbb{R}^2$. Let $\mathcal{F} = \{f_1, f_2\}$ denote the family of dose functions, where each function characterizes a distinct dose distribution over the tumour region corresponding to different patients undergoing IMRT as follows:

$$f_1(x) = \begin{cases} 2x & , \quad Patient - I \\ x & , \quad Patient - II \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{x}{3} & , \quad Patient - I \\ \frac{2x}{3} & , \quad Patient - II \end{cases}.$$

It should be noted that $\mathcal{F}$ constitutes a family of bounded functions. Let the mapping $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ be defined by

$$\mathcal{P}f = f^2 - 2f + 2,$$

for all $f \in \mathcal{F}$. Now we show that $\mathcal{P}$ is a $\mathcal{P}_S$-contraction mapping. To do this, we consider the following cases:

For Patient-I	
Case I:	$x = y = 1 \Longrightarrow \lvert\mathcal{P}f_1 - \mathcal{P}f_2\rvert = \frac{5}{9}$ and $\lvert f_1 - f_2\rvert = \frac{5}{3}$
Case II:	$x = y = \frac{1}{2} \Longrightarrow \lvert\mathcal{P}f_1 - \mathcal{P}f_2\rvert = \frac{25}{36}$ and $\lvert f_1 - f_2\rvert = \frac{5}{6}$
Case III:	$x = 1, y = \frac{1}{2} \Longrightarrow \lvert\mathcal{P}f_1 - \mathcal{P}f_2\rvert = \frac{11}{36}$ and $\lvert f_1 - f_2\rvert = \frac{11}{6}$
Case IV:	$x = \frac{1}{2}, y = 1 \Longrightarrow \lvert\mathcal{P}f_1 - \mathcal{P}f_2\rvert = \frac{4}{9}$ and $\lvert f_1 - f_2\rvert = \frac{2}{3}$

Using the above cases for Patient-I, we get

$$\begin{aligned} S^*(\mathcal{P}f_1, \mathcal{P}f_1, \mathcal{P}f_2) &= \sup\{|\mathcal{P}f_1 - \mathcal{P}f_2| + |\mathcal{P}f_1 + \mathcal{P}f_2 - 2\mathcal{P}f_1|\} \\ &= 2\sup\{|\mathcal{P}f_1 - \mathcal{P}f_2|\} = 2.\frac{25}{36} = \frac{25}{18} \\ &\leq \frac{2}{3}.\frac{11}{3} = \frac{2}{3}\left(2.\frac{11}{6}\right) \\ &= \frac{2}{3}[2\sup\{|f_1 - f_2|\}] \\ &= \frac{2}{3}[\sup\{2|f_1 - f_2|\}] \\ &= \frac{2}{3}\sup\{|f_1 - f_2| + |f_1 + f_2 - 2f_1|\} \\ &= \frac{2}{3}S^*(f_1, f_1, f_2) \end{aligned}$$

and so $\mathcal{P}$ is a $\mathcal{P}_S$-contraction mapping with $\lambda = \frac{2}{3}$.

For Patient-II	
Case I:	$x = y = 1 \Longrightarrow \lvert\mathcal{P}f_1 - \mathcal{P}f_2\rvert = \frac{1}{9}$ and $\lvert f_1 - f_2\rvert = \frac{1}{3}$
Case II:	$x = y = 2 \Longrightarrow \lvert\mathcal{P}f_1 - \mathcal{P}f_2\rvert = \frac{8}{9}$ and $\lvert f_1 - f_2\rvert = \frac{2}{3}$
Case III:	$x = 1, y = 2 \Longrightarrow \lvert\mathcal{P}f_1 - \mathcal{P}f_2\rvert = \frac{1}{9}$ and $\lvert f_1 - f_2\rvert = \frac{1}{3}$
Case IV:	$x = 2, y = 1 \Longrightarrow \lvert\mathcal{P}f_1 - \mathcal{P}f_2\rvert = \frac{8}{9}$ and $\lvert f_1 - f_2\rvert = \frac{4}{3}$

Using the above cases for Patient-II, we get

$$\begin{aligned} S^*(\mathcal{P}f_1, \mathcal{P}f_1, \mathcal{P}f_2) &= \sup\{|\mathcal{P}f_1 - \mathcal{P}f_2| + |\mathcal{P}f_1 + \mathcal{P}f_2 - 2\mathcal{P}f_1|\} \\ &= 2\sup\{|\mathcal{P}f_1 - \mathcal{P}f_2|\} = 2.\frac{8}{9} = \frac{16}{9} \\ &\leq \frac{2}{3}.\frac{8}{3} = \frac{2}{3}\left(2.\frac{4}{3}\right) \\ &= \frac{2}{3}[2\sup\{|f_1 - f_2|\}] \\ &= \frac{2}{3}[\sup\{2|f_1 - f_2|\}] \\ &= \frac{2}{3}\sup\{|f_1 - f_2| + |f_1 + f_2 - 2f_1|\} \\ &= \frac{2}{3}S^*(f_1, f_1, f_2) \end{aligned}$$

and so $\mathcal{P}$ is a $\mathcal{P}_S$-contraction mapping with $\lambda = \frac{2}{3}$.

Consequently, all the conditions of Theorem 3.1 are satisfied from which it follows that there exists a unique fixed function f_1 of $\mathcal{P}$ which ensures the simultaneous determination of suitable dose levels for two patients.

5. Conclusion and Future Works

In this paper, we have developed fixed-function results in the framework of S-metric spaces under Banach-type and Reich-type contractive conditions and demonstrated the applicability of the Banach-type result to the problem of determining the best approximation of treatment plans for tumour patients undergoing intensity modulated radiation therapy (IMRT). The obtained results extend classical fixed-point theory to a functional level and provide a rigorous convergence mechanism for function-valued iterative processes arising in applied models.

Beyond the established fixed-function theorems, the present work highlights the significance of S-metric spaces as a genuinely distinct generalization of metric structures. In particular, it is well known in the literature that there exist S-metrics which cannot be generated by any metric, indicating that S-metric spaces are not merely reformulations of metric spaces but possess intrinsically richer geometric features. Moreover, despite certain superficial similarities, S-metric and G-metric spaces are mutually independent in general, as neither structure can be derived from the other without imposing additional restrictive assumptions. This independence underscores the necessity of studying fixed-point and fixed-function problems directly within the S-metric framework.

Consequently, the results presented in this study contribute not only to the theory of fixed functions in generalized metric spaces but also to the broader understanding of S-metric spaces as an essential and flexible analytical setting. The combination of theoretical developments with a concrete application to IMRT treatment planning further demonstrates that S-metric-based fixed function theory provides a powerful mathematical tool for addressing complex optimization problems in applied sciences.

Future research may focus on extending the fixed-function results in S-metric spaces to more general contractive conditions, including interpolative and rational-type contractions. Another direction is to investigate fixed-function theory in hybrid frameworks combining S-metric spaces with additional structures such as partial orders or graph-based settings. Moreover, a comparative study between S-metric and other generalized metric spaces, may provide further insights into their theoretical and applied differences. Finally, the application to IMRT treatment planning can be enhanced by incorporating more realistic clinical constraints and numerical implementations.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions. This work is supported by 2209-A - Research Project Support Programme for Undergraduate Students of TUBITAK under the project number 1919B012409507.

Author Contributions: Conceptualization: NT, MB; Methodology: NT, MB; Software: NT, MB; Validation: NT, MB; Formal analysis: NT, MB; Investigation: NT, MB; Resources: NT, MB; Data curation: NT, MB; Writing – original

draft: NT, MB; Writing – review & editing: NT, MB; Visualization: NT, MB; Supervision: NT; Project administration: MB.

Artificial Intelligence Statements: The authors confirm that this manuscript was prepared without the use of any generative artificial intelligence (AI) tools or AI-assisted technologies. All data analysis, results, and written content are entirely the original work of the authors.

Conflict of Interest Disclosure: No potential conflict of interest was declared by authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundam. Math., **3** (1922), 133-181.

[2] S. Reich, *Some remarks concerning fixed points*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Sci. Fis. Nat., **52**(8) (1971), 689-697.

[3] İ. Demir, R. Bozyikit, *Some fixed soft point theorems on a new soft topology related to a self soft mapping*, Ital. J. Pure Appl. Math., **42** (2019), 36-50.

[4] İ. Demir, *Fixed point theorems in complex valued fuzzy b-metric spaces with application to integral equations*, Miskolc Math. Notes, **22**(1) (2021), 153-171. https://doi.org/10.18514/MMN.2021.3173

[5] İ. Demir, *Common fixed soft element results in soft complex valued b-metric spaces*, TWMS J. Appl. Eng. Math., **11**(1) (2021),134-150.

[6] M. Younis, M. Öztürk, *Some novel proximal point results and applications*, Univers. J. Math. Appl., **8**(1) (2025), 8-20 . https://doi.org/10.32323/ujma.1597874

[7] M. Younis, H. Ahmad, M. Öztürk, et al., *Unveiling fractional-order dynamics: a new method for analyzing Rössler chaos*, J. Comput. Appl. Math., **468** (2025), 116639. https://doi.org/10.1016/j.cam.2025.116639

[8] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesn., **64**(3) (2012), 258-266.

[9] P. Dhawan, J. Kaur, V. Gupta, *Novel results on a fixed function and their application based on the best approximation of the treatment plan for tumour patients getting intensity modulated radiation therapy (IMRT)*, Proc. Est. Acad. Sci., **68**(3) (2019). https://doi.org/10.3176/PROC.2019.3.01

[10] B. S. Radha, *Application to medical science using G-metric spaces*, Eur. J. Mol. Clin. Med.,**8**(3) (2021), 2070-2083.

[11] S. Sedghi, N. V. Dung, *Fixed point theorems on S-metric spaces*, Mat. Vesn., **66**(1) (2014), 113-124.

[12] N. T. Hieu, N. T. Ly, N. V. Dung, *A generalization of Ciric quasi-contractions for maps on S-metric spaces*, Thai J. Math., **13**(2) (2015), 369-380.

[13] N. Y. Özgür, N. Taş, *Some new contractive mappings on S-metric spaces and their relationships with the mapping (S25)*, Math. Sci., **11**(1) (2017), 7-16. https://doi.org/10.1007/s40096-016-0199-4

[14] N. Özgür, N. Taş, *On S-metric spaces with some topological aspects*, Electron. J. Math. Anal. Appl., **11**(2) (2023), 1-8. https://doi.org/10.21608/ejmaa.2023.206319.1029

[15] A. Gupta, *Cyclic contraction on S-metric space*, Int. J. Anal. Appl., **3**(2) (2013), 119-130.

[16] I. A. Bakhtin, *The contraction principle in quasimetric spaces*, Func. An. Ulianowsk Gos. Ped. Ins., **30** (1989), 26-37.

[17] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289-297.

[18] N. V. Dung, N. T. Hieu, S. Radojević, *Fixed point theorems for g-monotone maps on partially ordered S-metric spaces*, Filomat, **28**(9) (2014), 1885-1898. https://doi.org/10.2298/FIL1409885D

[19] Z. Tian, M. Zarepisheh, X. Jia, et al., *The fixed-point iteration method for IMRT optimization with truncated dose deposition coefficient matrix*, arXiv: 1303.3504 (2013) [physics.med-ph]. https://doi.org/10.48550/arXiv.1303.3504

[20] T. Bortfeld, *Optimized planning using physical objectives and constraints*, Semin. Radiat. Oncol., **9** (1999), 20-34.

[21] D. M. Shepard, G. H. Olivera, P. J. Reckwerdt, et al., *Iterative approaches to dose optimization in tomotherapy*, Phys. Med. Biol., **45** (2000), 69-90.

Affiliations

NIHAL TAŞ
ADDRESS: Balıkesir University, Dept. of Mathematics, 10145, Balıkesir-Türkiye
E-MAIL: nihaltas@balikesir.edu.tr
ORCID ID:0000-0002-4535-4019

MELEK BOYRAZ
ADDRESS: Balıkesir University, Dept. of Mathematics, 10145, Balıkesir-Türkiye
E-MAIL: melekboyraz@outlook.com
ORCID ID: 0009-0007-4998-3166