Tangentially cubic curves in Euclidean spaces

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Abstract. Curve design using splines is one of the most fundamental topics in CAGD. Using standard spline methods, variational curve design has been investigated in a large number of contributions. The minimizers of the L^2 norm of the second derivative have cubic segments (vanishing fourth derivative), the corresponding splines on surfaces have segments with vanishing tangential componant of the fourth derivative. Such segments are called tangentially cubic. In this paper we study with the tangentially cubic curves (i.e. $T.C$-curves) in $\mathbb{R}^n$. We give necessary and sufficient conditions for k-type curves to be $T.C$-curves. Finally, we give some examples of finite type $T.C$-curves in $\mathbb{E}^3$.

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1 Introduction

Spline curves in Euclidean spaces which minimize the L^2 norm of the second derivative and related functions are well known understood and comprise one of the basics of Geometric Modelling. The classical energy functionals are quadrics, which makes minimization easy. Geometric functionals which require nonlinear minimization techniques and their applications to curve design have been studied in [1] and see also [12]. Variational curve design has also been performed within the framework of subdivision [11].

In [13] H. Pottman and M. Hofer characterized the counterpart on surfaces, C^2 cubic splines. They give an example; whereas the minimizers of the L^2 norm of the second derivative have cubic segments (vanishing fourth derivative), the corresponding splines on surfaces have segments with vanishing tangential component of the fourth derivative. The authors call such segment "*tangentially cubic*". Hence the differential equation $\gamma^{(4)} = 0$ changes to $tpr\gamma^{(4)} = 0$ with tpr denoting the tangential projection (orthogonal projection into the corresponding tangent space of the surface S). In the same paper it has been shown that tangentially cubic curves are important for the interpolating cubic spline curves on surfaces. In view of the importance of tangentially cubic curves some explicit representations of such curves on special surfaces, namely certain cylinder surfaces.

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The notion of curves of finite type was introduced by B.Y. Chen (see for instance, [2]). In particular, for closed curves γ in a Euclidean space $\mathbb{E}^n$ (of any dimension n), the property of having finite type is equivalent to the fact that the Fourier series expansion of γ with respect to an arclength parameter is finite. In [2] , it has been shown that the circles are the only closed curves of finite type in $\mathbb{E}^2$. The only finite type curves which lie on a sphere in $\mathbb{E}^3$ are its great and small circles [5]. Moreover, in [6], closed finite type curves which lie on arbitrary quadrics in $\mathbb{E}^3$ are studied.

In the present study we consider $T.C$-curves γ of finite type. We give a necessary and sufficient conditions for k-type curves to become $T.C$-curves. Finally we give some examples of finite type $T.C$-curves in $\mathbb{E}^3$.

2 Basic concepts

Let $\gamma = \gamma(s) : I \to \mathbb{R}^m$ be a regular curve in $\mathbb{R}^m$ (i.e. $\|\gamma'\|$ is nowhere zero), where I is interval in $\mathbb{R}$. γ is called a Frenet curve of rank r ($r \in N_0$) if $\gamma'(t)$, $\gamma''(t)$,...,$\gamma^{(r)}(t)$ are linearly independent and $\gamma'(t)$, $\gamma''(t)$,...,$\gamma^{(r+1)}(t)$ are no longer linearly independent for all t in I. In this case, $Im(\gamma)$ lies in an r-dimensional Euclidean subspace of $\mathbb{R}^m$. To each Frenet curve of rank r there can be a associated an orthonormal r-frame $\{E_1, E_2, ..., E_r\}$ along γ, the Frenet r-frame, and $r-1$ functions $\kappa_1, \kappa_2, ..., \kappa_{r-1}$:$I \longrightarrow \mathbb{R}$, the Frenet curvature, such that

$$\begin{bmatrix} E_1' \\ E_2' \\ ... \\ E_r' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & & ... & 0 \\ -\kappa_1 & 0 & \kappa_2 & ... & 0 \\ ... & & & & \\ 0 & ... & 0 & -\kappa_{r-1} & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ ... \\ E_r \end{bmatrix} \tag{2.1}$$

where v is the speed of the curve.

In fact, to obtain $E_1, E_2, ..., E_r$ it is sufficient to apply the Gram-Schmidt orthonormalization procees to $\gamma'(t)$, $\gamma''(t)$,...,$\gamma^{(r)}(t)$. Moreover, the functions $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ are easily obtained as by-product during this calculation. More precisely, $E_1, E_2, ..., E_r$ and $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ are determined by the following formulas:

$$v_1(t) \quad : \quad = \gamma'(t); E_1 := \frac{v_1}{\|v_1(t)\|} \tag{2.2}$$

$$v_r(t) \quad : \quad = \gamma^{(k)}(t) - \sum_{i=1}^{k-1} \langle \gamma^k(t), v_i(t) \rangle \frac{v_i(t)}{\|v_i(t)\|}; \tag{2.3}$$

$$\kappa_{r-1}(t) \quad : \quad = \frac{\|v_k(t)\|}{\|v_{k-1}(t)\| \, \|v_1(t)\|} \tag{2.4}$$

$$E_r \quad : \quad = \frac{v_k}{\|v_k(t)\|} \tag{2.5}$$

where $k \in \{2, 3, ..., r\}$. It is natural and convenient to define Frenet curvatures $\kappa_r = \kappa_{r+1} = ... = \kappa_{m-1} = 0$. It is clear that $E_1, E_2, ..., E_r$ and $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ can be defined for any regular curve (not necessarily a Frenet curve) in the neighborhood of a point s_0 for which $\gamma'(s_0)$, $\gamma''(s_0)$,...,$\gamma^{(r)}(s_0)$ are linearly independent [9].

3 Tangentially cubic curves

Let S be a regular surface imbedded in $\mathbb{R}^m$. Moreover a sequence of the points $P_i \in$S, $i = 1, 2, ..., N$ and any real numbers $u_1 < ... < u_n$ are given. We are seeking a curve $\gamma(u)$ on S, which interpolates the given data, $\gamma(u_i) = P_i$, and minimizes some energy functional.

Recall that a cubic C^2 spline curve arises as minimizer of the energy

$$E_2(x) = \int_{u_1}^{u_N} \gamma''(s)^2 du \tag{3.1}$$

under interpolation condition $\gamma(u_i) = P_i$ (see, [10]). The case where the admissible curves $\gamma(u)$ are restricted to the given surface S has recently been studied in [13].

Let $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{R}^m$ be a unit speed curve in $\mathbb{R}^m$. Denoting the Frenet frame of γ by $\nu_1(s), \nu_2(s), ..., \nu_m(s)$ and the curvatures by $\kappa_1, \kappa_2, ..., \kappa_{m-1}$ we have

$$\gamma'(s) = \nu_1, \tag{3.2}$$

$$\gamma''(s) = \kappa_1(s)\nu_2(s), \tag{3.3}$$

$$\gamma'''(s) = -\kappa_1^2(s)\nu_1(s) + \kappa_1'(s)\nu_2(s) + \kappa_1(s)\kappa_2(s)\nu_3(s), \tag{3.4}$$

$$\gamma'^v(s) = -3\kappa_1(s)\kappa_1'(s)\nu_1(s) + (-\kappa_1^3 + \kappa_1'' - \kappa_1\kappa_2^2)\nu_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')\nu_3 + \kappa_1\kappa_2\kappa_3\nu_4. \tag{3.5}$$

Definition 3.1. *Let γ be a regular curve in $\mathbb{R}^m$. If the fourth derivative $\gamma'^v(s)$ of γ is orthogonal to $\gamma'(s)$, then γ is called a tangentially cubic curve (T.C -curve) of $\mathbb{R}^m$.*

Hence the condition for a tangentially cubic parametrization becomes

$$0 = \langle \gamma'^v(s), \gamma'(s) \rangle = -3\kappa_1(s)\kappa_1'(s). \tag{3.6}$$

So, by (3.6) we can say that the arclength parametrization of a curve γ in $\mathbb{R}^m$ is tangentially cubic if and only if the curve possesses constant curvature κ_1 [13].

In the plane we get only circles and straight lines. However, in $\mathbb{R}^m$ this family is relatively rich, since the torsion $\tau = \kappa_2$ can be arbitrary.

Parametric representations of special space curves with constant curvature κ_1 have been given by E. Salkowski [14], whose formulae for $n \neq \frac{1}{2}$, $m \in \mathbb{R} \setminus \{0\}$:

$$\begin{aligned}
x(t) &= \frac{-1}{\sqrt{1+m^2}} \left(\frac{1-n}{4(1+2n)} \sin(1+2n)t + \frac{1+n}{4(1-2n)} \sin(1-2n)t + \frac{1}{2}\sin t \right), \\
y(t) &= \frac{1}{\sqrt{1+m^2}} \left(\frac{1-n}{4(1+2n)} \cos(1+2n)t + \frac{1+n}{4(1-2n)} \cos(1-2n)t + \frac{1}{2}\cos t \right), \\
z(t) &= \frac{1}{4m\sqrt{1+m^2}} \cos 2nt.
\end{aligned}$$

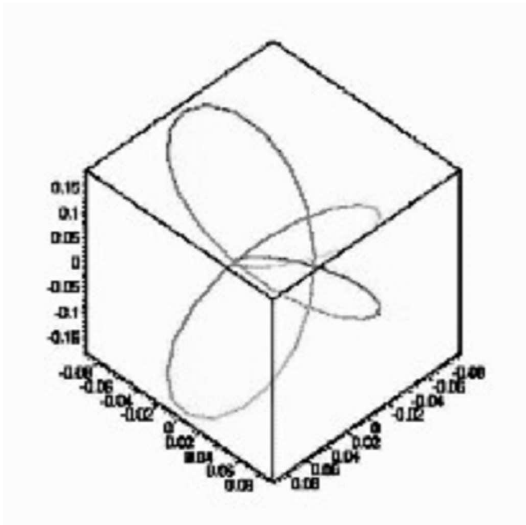

Figure 1: $T.C$-curve in $\mathbb{R}^3$

Fig. 1 shows an example of $T.C$-curve with $m = 1, n = 2$;

For a tangentially cubic spline curve

$$\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \subset S \tag{3.7}$$

the fourth derivative $\gamma^{(4)}(s)$ must be orthogonal to the surface S and thus also orthogonal to the corresponding ruling of S. This requires a vanishing third component $\gamma_3^{(4)}(s) = 0$ and shows that the function γ_3 is cubic

$$\gamma_3(s) = a_0 + a_1 s + ... + a_3 s^3 \tag{3.8}$$

The other two coordinate functions $(\gamma_1(s), \gamma_2(s))$ have its fourth derivative orthogonal to a parametrization $(\gamma_1(s), \gamma_2(s), 0)$ of the (planar) orthogonal cross section of S. This is therefore a tangentially cubic parametrization of the cross-section curve. Choosing the data points for a spline on a cross section, it is possible to find a result on energy minimizing parametrization of a given curve [13].

Example 3.2. *A special families of T.C-curves can be written down explicitly on the unit cylinder $x_1^2 + x_2^2 = 1$ are the curves derived from [13]);*

a) a scaled arclength parametrization

$$\gamma(t) = (\cos(at + b), \sin(at + b), a_0 + a_1 t + a_2 t^2 + a_3 t^3), \tag{3.9}$$

b) the circle parametrization,

$$\gamma(t) = (\cos(\ln|C + t| + D), \sin(\ln|C + t| + D), a_0 + a_1 t + a_2 t^2 + a_3 t^3) \tag{3.10}$$

(see, Fig. 2).

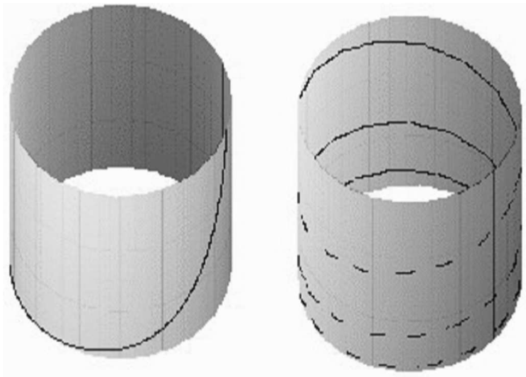

Figure 2: $T.C$-curves on cylinders

Definition 3.3. *Polynomial spiral is a plane curve whose curvature function is a polynomial function of arclength parameter. So, every Polynomial spiral can be given by the following parametrization [7];*

$$\beta(s) = (\int_0^s \cos(P_k(t)dt, \int_0^s \sin(P_k(t)dt)) \tag{3.11}$$

where the curvature function $\kappa_\gamma(s) = P_k^{'}(t)$ is a polynomial function of degree (k-1).

Let us consider a cylinder surface S parametrization of the form

$$S := x(s,v) = ((\int_0^s \cos(P_k(t)dt, \int_0^s \sin(P_k(t)dt), v). \tag{3.12}$$

If the orthogonal cross-section $c(t)$ of is a $T.C$-curve then by definition

$$\kappa_\gamma^{'}(s) = P_k^{''}(t) = 0. \tag{3.13}$$

So, $P_k(t) = at + b$. Therefore, S becomes a right circular cylinder and the $T.C$-curves on S will be of the form

$$\gamma(s) = (\int_0^s \cos(at+b)dt, \int_0^s \sin(at+b)dt, a_0 + a_1 t + a_2 t^2 + a_3 t^3). \tag{3.14}$$

Example 3.4. *A (p,q) torus curve α with $p,q \neq 0$ on a standard torus of revolution with radii b and $1, 0 < b < 1$ is parametrized by*

$$\alpha(t) = ((1 + b\cos(qt))\cos(pt), (1 + b\cos(qt))\sin(pt), b\sin(qt)). \tag{3.15}$$

In [8] it has been shown that if $b = \frac{p^2}{p^2+q^2}$ then α has points of zero curvature . Hence α is a $T.C$-curve.

Fig.3 shows the torus curve with $p = 1, q = 2$.

A nice explicit representations of tangentially cubic curves on cylinders arise as follows (see [13]);

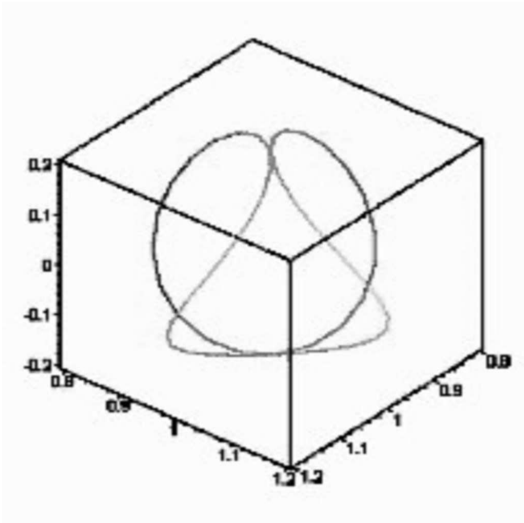

Figure 3: Torus curve

Example 3.5. *On a cylinder surface with the hypercycloid*

$$h(s) = cosh\sqrt{3}s \tag{3.16}$$

as cross section, the curves

$$\gamma(s) = \begin{pmatrix} cosh\sqrt{3}s \sin s + \sqrt{3} \sin h\sqrt{3}s \cos s \\ -cosh\sqrt{3}s \cos s + \sqrt{3} \sin h\sqrt{3}s \sin s \\ a_0 + a_1 s + ... + a_3 s^3 \end{pmatrix} \tag{3.17}$$

are tangentially cubic.

4 Curves of finite type

Let $f(s)$ be a periodic continuous function with period $2\Pi r$. Then it is well-known that $f(s)$ has a Fourier series expansion given by

$$f(s) = \frac{a_0}{2} + a_1 \cos\frac{s}{r} + a_2 \cos\frac{2s}{r} + ... + b_1 \sin\frac{s}{r} + b_2 \sin\frac{2s}{r} + ..., \tag{4.1}$$

where a_k and b_k are the Fourier coefficients defined by

$$a_k = \frac{1}{\Pi r}\int_{-\Pi r}^{\Pi r} f(s) \cos\frac{ks}{r} ds, k = 0, 1, 2, ... \tag{4.2}$$

$$b_k = \frac{1}{\Pi r}\int_{-\Pi r}^{\Pi r} f(s) \sin\frac{ks}{r} ds, k = 1, 2, ... \tag{4.3}$$

Let γ be a closed curve of length $2\Pi r$. If an isometric immersion $x : M \rightarrow \mathbb{R}^m$ then

$$x^{(j)} = \frac{d^j x}{ds^j}. \tag{4.4}$$

Because $\Delta = -\frac{d^2}{ds^2}$, we have

$$\Delta^j H = (-1)^j x^{(2j+2)} j = 0, 1, 2, ... \tag{4.5}$$

If x is of finite type, each coordinate function x_i satisfies the following homogeneous ordinary differential equation with constant coefficients;

$$x_i^{(2k+2)} + c_1 x_i^{(2k)} + ... + c_{k-1} x_i^{(4)} + c_k x_i^{(2)} = 0, i = 1, 2, ..., m \tag{4.6}$$

for some integer $k \geq 1$ and constant $c_1, ..., c_k$.Because our solutions x_i of above differential equation are periodic solutions with period $2\Pi r$, each x_i is a finite linear combination of the following particular solutions ;

$$1, \cos(\frac{n_i s}{r}), \qquad \sin(\frac{m_i s}{r}), \qquad n_i, m_i \in Z. \tag{4.7}$$

Therefore, each x_i is of the following form

$$x_i = c_i + \sum_{p_A}^{q_A} (a_A(t) \cos \frac{ts}{r} + b_A(t) \sin \frac{ts}{r}) \tag{4.8}$$

for some suitable constant $c_i, a_A(t), b_A(t) \ \ (A = 1, ..., n)$ and integers p_A, q_A . Thus each x_i has a Fourier series expansion of finite sum. Similarly, if each x_i has a Fourier series expansion of finite sum then x is finite type [3].

Theorem 4.1. *[4]Let γ be a closed curve of length $2\Pi r$ in $\mathbb{R}^m$.Then isometric immersion $x : \gamma \rightarrow \mathbb{R}^m$ is of finite type if and only if the Fourier series expansion of each coordinate function of γ,*

$$\gamma(s) = a_0 + \sum_{t=1}^{\infty} (a_t \cos \frac{ts}{r} + b_t \sin \frac{ts}{r}) \tag{4.9}$$

has only finite nonzero terms.

Thus using the above theorems, we have the following corollaries.

Corollary 4.2. *Every closed* k-type *curve γ in $\mathbb{R}^m$ can be written in the following form*

$$\gamma(s) = a_0 + \sum_{i=1}^{k} (a_i \cos \lambda_{t_i} s + b_i \sin \lambda_{t_i} s), \tag{4.10}$$

where $T(x) = \{t_1, t_2, ...t_k\}$ is the order of the curve and $a_0, a_1, ..., a_k, b_1, ..., b_k$ are vectors in $\mathbb{R}^m$ such that for any i in $\{1, 2, ..., k\}$, a_i and b_i are not simultaneously zero. Moreover, if $q = t_k$ is the upper order of γ , then $|a_q| = |b_q| \neq 0$.

Corollary 4.3. *Every* null k-type *curve* γ *in* $\mathbb{R}^m$ *can be written in the following form*

$$\gamma(s) = a_0 + b_0 s + \sum_{t=1}^{k}(a_i \cos\lambda_{t_i} s + b_i \sin\lambda_{t_i} s), \tag{4.11}$$

where $a_0, b_0, a_1, ..., a_k, b_1, ..., b_k$ *are vectors in* $\mathbb{R}^m$ *such that* $b_0 \neq 0$ *and for any* i *in* $\{1, 2, ..., k\}$, a_i *and* b_i *are not simultaneously zero. Moreover,* $|a_q| = |b_q| \neq 0$, *where* q *is the upper order of the curve* γ.

Furthermore from (4.10) and (4.11) we obtain the following:

Corollary 4.4. 1) *Every* k-type *curve of* $\mathbb{R}^m$ *lies in an affine* $2k$*-subspace* $\mathbb{R}^{2k}$ *of* $\mathbb{R}^m$.

2) *Every null* null k-type *curve of* $\mathbb{R}^m$ *lies in an affine* $(2k-1)$*-subspace* $\mathbb{R}^{2k-1}$ *of* $\mathbb{R}^m$.

5 Results

In this part we give some results on $T.C$-curve of finite type.

Theorem 5.1. *Let* $\gamma \subset \mathbb{R}^m$ *be a closed* k-type *curve. Then* γ *in* $\mathbb{R}^m$ *is tangentially cubic curves if and only if*

1) $|a_i| = |b_i|$,
2) $\langle a_i, b_j\rangle = 0$, *and*
3) $\langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0,\ \ i \neq j,\ 1 \leq i, j \leq k.$

Proof. Let $\gamma \subset \mathbb{R}^m$ be a closed *k-type* curve. Then differentiating (4.10) and using (3.6) we get

$$\begin{aligned} 0 &= \langle \gamma'^{v}(s), \gamma'(s)\rangle \\ &= \langle \sum_{i=1}^{k}(a_i\lambda_{t_i}^4 \cos\lambda_{t_i} s + b_i\lambda_{t_i}^4 \sin\lambda_{t_i} s), \sum_{i=1}^{k}(-a_i\lambda_{t_i} \sin\lambda_{t_i} s + b_i\lambda_{t_i} \cos\lambda_{t_i} s)\rangle \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k}\lambda_{t_i}^4\lambda_{t_j} \left\{ \begin{array}{c} \langle a_i, -a_j\rangle \cos\lambda_{t_i} s \sin\lambda_{t_j} s + \langle a_i, b_j\rangle \cos\lambda_{t_i} s \cos\lambda_{t_j} s + \\ +\langle b_i, -a_j\rangle \sin\lambda_{t_i} s \sin\lambda_{t_j} s + \langle b_i, b_j\rangle \sin\lambda_{t_i} s \cos\lambda_{t_j} s \end{array} \right\}. \end{aligned}$$

If $i = j$ then

$$0 = \langle \gamma'^{v}(s), \gamma'(s)\rangle = \sum_{i=1}^{k} \left\{ \begin{array}{c} \lambda_{t_i}^4 \{\cos\lambda_{t_i} s\langle a_i, b_0\rangle + \sin\lambda_{t_i} s\langle b_i, b_0\rangle\} \\ +\lambda_{t_i}^5 \left\{\frac{\sin 2\lambda_{t_i} s}{2}(|b_i|^2 - |a_i|^2) + \cos 2\lambda_{t_i} s\langle a_i, b_i\rangle\right\} \end{array} \right\}.$$

Therefore we obtain $|a_i| = |b_i|$ and $\langle a_i, b_i\rangle = 0$. The converse statement is trivial. Hence our theorem is proved.□

Theorem 5.2. *Let* $\gamma \subset \mathbb{R}^m$ *be a* null k-type *curve. Then* γ *in* $\mathbb{R}^m$ *is tangentially cubic curves if and only if*

1) $|a_i| = |b_i|$,
2) $\langle a_i, b_j\rangle = 0$,
3) $\langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0,\ \ i \neq j$,
4) $\langle a_i, b_0\rangle = \langle b_i, b_0\rangle = 0,\ \ 1 \leq i, j \leq k.$

Proof. Let $\gamma \subset \mathbb{R}^m$ be a null *k-type* curve then differentiating (4.11) and using (3.6) we get

$$\begin{aligned} 0 &= \langle \gamma'^{v}(s), \gamma'(s) \rangle \\ &= \langle \sum_{i=1}^{k} (a_i \lambda_{t_i}^4 \cos \lambda_{t_i} s + b_i \lambda_{t_i}^4 \sin \lambda_{t_i} s), b_0 + \sum_{i=1}^{k} (-a_i \lambda_{t_i} \sin \lambda_{t_i} s + b_i \lambda_{t_i} \cos \lambda_{t_i} s) \rangle \\ &= \sum_{i=1}^{k} \lambda_{t_i}^4 \left\{ \cos \lambda_{t_i} s \langle a_i, b_0 \rangle + \sin \lambda_{t_i} s \langle b_i, b_0 \rangle \right\} + \\ &\quad + \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_{t_i}^4 \lambda_{t_j} \left\{ \begin{array}{c} \langle a_i, -a_j \rangle \cos \lambda_{t_i} s \sin \lambda_{t_j} s + \langle a_i, b_j \rangle \cos \lambda_{t_i} s \cos \lambda_{t_j} s + \\ + \langle b_i, -a_j \rangle \sin \lambda_{t_i} s \sin \lambda_{t_j} s + \langle b_i, b_j \rangle \sin \lambda_{t_i} s \cos \lambda_{t_j} s \end{array} \right\}. \end{aligned}$$

If we take $i = j$, the final equation becomes

$$0 = \langle \gamma'^{v}(s), \gamma'(s) \rangle = \sum_{i=1}^{k} \left\{ \begin{array}{c} \lambda_{t_i}^4 \left\{ \cos \lambda_{t_i} s \langle a_i, b_0 \rangle + \sin \lambda_{t_i} s \langle b_i, b_0 \rangle \right\} \\ + \lambda_{t_i}^5 \left\{ \frac{\sin 2\lambda_{t_i} s}{2} (|b_i|^2 - |a_i|^2) + \cos 2\lambda_{t_i} s \langle a_i, b_i \rangle \right\} \end{array} \right\}.$$

Therefore we obtain $|a_i| = |b_i|$ and $\langle a_i, b_i \rangle = 0$ and $\langle a_i, b_0 \rangle = \langle b_i, b_0 \rangle = 0$. The converse statement is trivial. Hence our theorem is proved. □

We obtain the following results:

Corollary 5.3. *The curve given by the parametrization (3.7) can be written of the form*

$$\gamma(t) = (0, 0, \frac{1}{4m\sqrt{1+m^2}}) \cos 2nt + \sum_{i=1}^{3} (a_i \cos \lambda_{t_i} t + b_i \sin \lambda_{t_i} t) \tag{5.1}$$

where $\lambda_{t_1} = 1 + 2n$, $\lambda_{t_2} = 1 - 2n$, $\lambda_{t_3} = 1$ *and*

(5.2)

$$\begin{array}{lll} a_1 = (0, \frac{1-n}{4(1+2n)\sqrt{1+m^2}}, 0), & a_2 = (0, \frac{1+n}{4(1-2n)\sqrt{1+m^2}}, 0), & a_3 = (0, \frac{1}{2\sqrt{1+m^2}}, 0), \\ b_1 = (-\frac{1-n}{4(1+2n)\sqrt{1+m^2}}, 0, 0), & b_2 = (-\frac{1+n}{4(1-2n)\sqrt{1+m^2}}, 0, 0), & b_3 = (-\frac{1}{2\sqrt{1+m^2}}, 0, 0). \end{array}$$

Corollary 5.4. *The curve given by the parametrization (5.1) is*

i) $1-$*type if* $n = 0$ *and* $a_0 = (0, 0, \frac{1}{4m\sqrt{1+m^2}})$ *, or*

ii) $3-$*type if* $n = \frac{1}{4}$ *and* $a_0 = (0, 0, 0)$*, (See Fig. 4 for* $m = 2$*).*

The curve given by the parametrization (5.1) can not be of $2-$*type.*

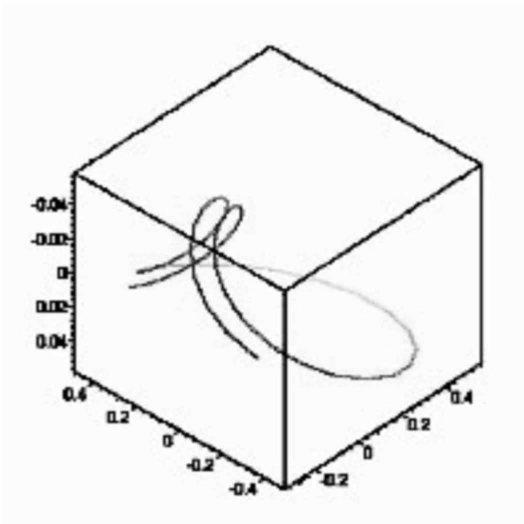

Figure 4: $T.C$-curve of 3-type.

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