On weak biharmonic submanifolds and 2-parallelity

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Abstract

It is shown that every locally Euclidean 2-parallel submanifold of a space form has harmonic curvature vector (i.e., is weak biharmonic). In four-dimensional Euclidean space, in the class of surfaces with flat connection $\overline{\nabla}$, whose one family of curvature lines consists of geodesics, a surface is weak biharmonic if and only if it is 2-parallel.

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§1. Introduction.

Let M^m be a smooth m-dimensional submanifold in a space form $N^n(c)$. For a function Φ (real or vector valued) on M^m, the operator Δ^2 can be defined by $\Delta^2\Phi = \Delta(\Delta\Phi)$, where $\Delta = g^{ij}\nabla_i\nabla_j$ and ∇ is the covariant differential operator of the Levi-Civita connection of M^m. A submanifold is said to be *biharmonic* if for the radius vector x of its point satisfies $\Delta^2 x = \Delta(\Delta x) = 0$. According to the derivation formula for the adapted to M^m orthonormal frame bundle, we have $\nabla x = dx = e_i\omega^i$; thus $\nabla_i x = e_i$, and according to the other derivation formula (see e.g. [Lu 5]) $d(\nabla_i x) = de_i$ can be expressed as $e_j\omega_i^j + h_{ij}\omega^j$. Hence $\nabla e_i = de_i - e_j\omega_i^j = h_{ij}\omega^j$, and therefore $\nabla_j(\nabla_i x) = \nabla_j e_i = h_{ij}$. Then $\Delta x = g^{ij}h_{ij} = mH$, where H is the mean curvature vector of M^m. The last formula is known as *the Beltrami equation.* It follows that a submanifold M^m in $N^n(c)$ is biharmonic if and only if its mean curvature vector H is harmonic, i.e., if $\Delta H = 0$.

There is another possibility to introduce $\Delta^D H^\alpha$ by means of the normal curvature D, and to define submanifolds with harmonic mean curvature vector (or, shortly, weak biharmonic submanifolds), as those satisfying $\Delta^D H^\alpha = 0$.

For the second fundamental tensor $h_{ij} = e_\alpha h_{ij}^\alpha$ there holds $\overline{\nabla} h_{ij}^\alpha \wedge \omega^j = 0$, where

$$\overline{\nabla} h_{ij}^\alpha = dh_{ij}^\alpha - h_{kj}^\alpha\omega_i^k - h_{ik}^\alpha\omega_j^k + h_{ij}^\beta\omega_\beta^\alpha. \tag{1.1}$$

is the covariant differential of h_{ij}^α in the Van der Waerden-Bortolotti connection $\overline{\nabla}$ (i.e., the pair ∇ and normal connection D), denoted also by $\nabla^\perp$. Therefore, due

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to Cartan's Lemma, $\overline{\nabla} h_{ij}^{\alpha} = h_{ijk}^{\alpha}\omega^k$, where $h_{ijk}^{\alpha} = h_{ikj}^{\alpha}$, or in the other notation $\overline{\nabla}_k h_{ij}^{\alpha} = \overline{\nabla}_j h_{ik}^{\alpha}$, are the components of so called (vector valued) third fundamental tensor; the last equality is the Peterson-Mainardi-Codazzi equation. If $\overline{\nabla}(\overline{\nabla}_k h_{ij}^{\alpha}) = 0$, this third fundamental tensor is said to be parallel, and the submanifold is called *2-parallel*.

The first results about 2-parallel submanifolds were obtained for curves in $\mathbf{E}^n$, i.e., for the case $m = 1$, $c = 0$. In [15] there were classified all such curves. In [19] the results have been generalized to the case $c \neq 0$.

Independently in [1] all curves with harmonic mean curvature vector in $\mathbf{E}^n$ are classified. Surprisingly, these are the same curves obtained in [15]; only the spherical Cornu spirals have not a good geometrical characterization (see [1]).

In the present paper we state and partially solve the problem for submanifolds of dimension $m > 1$. The results are summarized in Abstract; a conjecture is formulated in the final part of the paper as well.

2 Biharmonic and Weak Biharmonic Submanifolds

According to B-Y. Chen [6], a submanifold is said to be *biharmonic* if $\Delta H=0$. For such a submanifold, both the tangent and the normal components in (6) must be zero. The biharmonic submanifolds M^m have been investigated in [13], [8], [9], [14]. It is known that in many cases they reduce to minimal submanifolds, but in pseudo-Euclidean spaces they do not (see [7]).

There is another possibility to introduce the Laplacian from the mean curvature vector H by means of the normal curvature D (or $\nabla^{\perp}$) of a submanifold. Let $H = H^{\alpha}e_{\alpha}$, where e_{α} with $\alpha \in \{m+1, ..., n\}$ be the basic vectors of the adapted frame normal to the submanifold, $H^{\alpha} = \frac{1}{m}g^{ij}h_{ij}^{\alpha}$ and let h_{ij}^{α} be the components of the second fundamental form of the submanifold, defined by $\omega_i^{\alpha} = h_{ij}^{\alpha}\omega^j$. Using exterior differentiation, the structure equations and Cartan's Lemma lead to

$$dh_{ij}^{\alpha} = h_{kj}^{\alpha}\omega_i^k + h_{ik}^{\alpha}\omega_j^k - h_{ij}^{\beta}\omega_{\beta}^{\alpha} + h_{ijk}^{\alpha}\omega^k. \tag{2.2}$$

and, reiterating the process,

$$dh_{ijk}^{\alpha} = h_{ljk}^{\alpha}\omega_i^l + h_{ilk}^{\alpha}\omega_j^l + h_{ijl}^{\alpha}\omega_k^l - h_{ijk}^{\beta}\omega_{\beta}^{\alpha} + h_{ijkl}^{\alpha}\omega^l. \tag{2.3}$$

Now for H^{α} there hold true

$$dH^{\alpha} + H^{\beta}\omega_{\beta}^{\alpha} = H_k^{\alpha}\omega_k,$$

where $H_k^{\alpha} = \frac{1}{m}g^{ij}h_{ijk}^{\alpha}$, and futher

$$dH_k^{\alpha} - H_k^{\beta}\omega_{\beta}^{\alpha} = H_l^{\alpha}\omega_k^l + \frac{1}{m}g^{ij}h_{ijkl}^{\alpha}\omega^l.$$

Here the left hand side can be interpreted as the covariant differentials DH^{α} and DH_k^{α}, correspondingly.

The covariant derivatives associated to DH_k^{α}, can be defined only if a certain tangent frame field is given on the submanifold. Then at an arbitrarily fixed point

x, where $\omega^i = 0$, we should have $de_i = 0$, whence ω_j^i must be vanish due to $\omega^k = 0$. Then we infer a linear dependency $\omega_i^j = \gamma_{ik}^j \omega^k$ and now in $DH_k^\alpha = D_l H_k^\alpha \omega^l$ one has $D_l H_k^\alpha = H_j^\alpha \gamma_{kl}^j + \frac{1}{m} g^{ij} h_{ijkl}^\alpha$. Here H_k^α can be interpreted as $D_k H^\alpha$ and we introduce

$$\Delta^D H^\alpha = g^{kl} D_l \nabla_k H^\alpha = g^{kl} H_j^\alpha \gamma_{kl}^j + \frac{1}{m} g^{ij} g^{kl} h_{ijkl}^\alpha.$$

Note that a formula for ΔH including $\Delta^D H$ was derived by B-Y. Chen [6] (see also [13]).

If $\Delta^D H^\alpha = 0$, the submanifold is said to have harmonic mean curvature vector, or more simply we will call it *weak biharmonic.*

Consider the case when the integral lines of all basic vector fields of the given orthonormal tangent frame field are geodesic lines of the submanifold. Then for every fixed value of i we should have $de_i = h_{ii}\omega^i$ for a line determined by $\omega^1 = ...\omega^{i-1} = \omega^{i+1} = ... = \omega^m = 0$, since along this line de_i may have only normal component. Thus $\gamma_{ii}^j = 0$ for all values of i.

Further, for an orthonormal frame field $g^{kl} = \delta^{kl}$, so that $g^{kl}\gamma_{kl}^j = \gamma_{11}^j + ... + \gamma_{mm}^j$ is zero in the considered case and there holds

$$\Delta^D H = \sum_{i,k=1}^{m} h_{iikk}^\alpha. \tag{2.4}$$

On the other hand it is known that a submanifold can have an orthogonal net of geodesic lines if and only if its Levi - Civita connection ∇ is flat, i.e. if it is intrisically locally Euclidean.

We introduce in the following the 2-parallel submanifolds. The formula (3) can be written as $\overline{\nabla} h_{ijk}^\alpha = h_{ijkl}^\alpha \omega^l$, where we recall that $\overline{\nabla}$ is the covariant differential operator of the van der Waerden-Bortolotti connection. According to F. Dillen ([11]), a submanifold is called *2-parallel* if $\overline{\nabla}^2 h = 0$, or component-wise, $\overline{\nabla}_l \overline{\nabla}_k h_{ij}^\alpha = \overline{\nabla}_l h_{ijk}^\alpha = 0$, or equivalently, $h_{ijkl}^\alpha = 0$.

Theorem 1. *A locally Euclidean 2-parallel submanifold M^m in $N^n(c)$ is weak biharmonic.*

Proof. Really, from $h_{ijkl}^\alpha = 0$ and (4) it follows directly that $\Delta^D H = 0$. □

Corollary. *Each 2-parallel curve M^1 in $N^n(c)$ is biharmonic.*

As well, the converse is true. Indeed, for a curve the following Bartels-Frenet formulae hold[1]:

$$dx = e_1 ds,\ de_1 = (-cx + \kappa_1 e_2)ds,\ de_2 = (-\kappa_1 e_1 + \kappa_2 e_3)ds,\ de_3 = (-\kappa_2 e_2 + \kappa_3 e_4)ds,$$

etc; here $i = 1$, $\alpha \in \{2, 3, ..., n\}$. Therefore $\omega_1^1 = 0$, $h_{11}^2 = \kappa_1$, $h_{11}^3 = ... = h_{11}^n = 0$, $\omega_2^3 = -\omega_3^2 = \kappa_2 ds$. Now $\overline{\nabla} h_{11}^2 = d\kappa_1$, so $h_{111}^2 = \kappa_1$, $\overline{\nabla} h_{11}^3 = \kappa_1\kappa_2 ds$, so $h_{111}^3 = \kappa_1\kappa_2$;

[1] Recent investigations ([18]) had shown that the famous Frenet formulae, given for $n = 3$, $c = 0$ by F. Frenet in 1847 and then by J. Serret in 1851, were published already in 1831 (in their preliminary version, covariantly) and belong actually to Martin Bartels (1769-1836).

in the same way we can deduce that $h^4_{111} = ... = h^n_{111} = 0$. Hence the 2-parallelity condition gives a system

$$\kappa_1'' - \kappa_1\kappa_2^2 = 0,\ (\kappa_1\kappa_2)' + \kappa_1'\kappa_2 = 0,\ \kappa_1\kappa_2\kappa_3 = 0,$$

(cf. [19], sect.18), which coincides with the system characterizing in [1] (for the case $c = 0$) the curve with harmonic mean curvature vector or, as we have called it, weak biharmonic curve. This gives the following

Proposition 1. *A curve M^1 in $N^n(c)$ is weak biharmonic, if and only if it is 2-parallel.*

We examine the problem for dimension $m > 1$. In one direction, we have

Proposition 2. *All two and three-dimensional, as well as all normally flat m-dimensional 2-parallel submanifolds in $\boldsymbol{E}^n$ are weak biharmonic.*

These submanifolds have been classified in correspondingly, [15], [16], [17] (see also [19]), where it is shown that all of them are locally Euclidean, and hence Theorem 1 applies.

Conversely, arises the question if every locally Euclidean weak biharmonic submanifold in $N^n(c)$ is 2-parallel.

We consider this problem in the Euclidean space $\mathbf{E}^4$ (i.e., for $c = 0$) for surfaces with flat $\overline{\nabla}$ - i.e., for both ∇ and D being flat.

3 Surfaces with Flat $\overline{\nabla}$ in $\mathbf{E}^4$

It is known (due to É. Cartan) that in the case of a normally flat submanifold all h^α_{ij} (i.e., for every value of α) can be diagonalized by an orthogonal transformation in the tangent space (see [4]). For surfaces this means that we can set $h^\alpha_{12} = 0$. The vectors $h^\alpha_{11}e_\alpha$ and $h^\alpha_{22}e_\alpha$ are called then the principal curvature vectors and are denoted by k_1 and k_2 correspondingly. It is known that the curvature 2-forms of ∇ are now $\Omega^j_i = -\langle k_i, k_j\rangle\,\omega^i \wedge \omega^j$ (see [19], Sect. 12). For a surface the only non-zero here can be $\Omega^2_1 = -\Omega^1_2 = -K\omega^1 \wedge \omega^2$. In the case of flat ∇, when $K = 0$, the vectors k_1 and k_2 are therefore orthogonal and can be used for a further adapting of the frame, if they are, of course, non-zero. Let e_3 and e_4 be taken collinear to them, so that $k_1 = k^3e_3$, $k_2 = k^4e_4$, $k^3k^4 \neq 0$. This implies $h^3_{11} = k^3$, $h^4_{22} = k^4$ and all other h^α_{ij} being zero. The system (2) gives now

$$\begin{array}{rcl} dk^3 = h^3_{111}\omega^1 + h^3_{112}\omega^2, & \quad & k^3\omega^4_3 = h^4_{111}\omega^1 + h^4_{112}\omega^2, \\ k^3\omega^2_1 = h^3_{112}\omega^1 + h^3_{122}\omega^2, & & -k^4\omega^2_1 = h^4_{112}\omega^1 + h^4_{122}\omega^2, \\ -k^4\omega^4_3 = h^3_{122}\omega^1 + h^3_{222}\omega^2, & & dk^4 = h^4_{122}\omega^1 + h^4_{222}\omega^2. \end{array}$$

Thus

$$k^4h^3_{112} + k^3h^4_{112} = 0,\ k^3h^3_{112} + k^4h^4_{111} = 0,$$

$$k^4h^3_{122} + k^3h^4_{122} = 0,\ k^3h^3_{222} + k^4h^4_{112} = 0.$$

From the relations of the first column it follows that there exist some functions λ and μ on the surface such that

$$h^3_{112} = \lambda k^3, h^4_{112} = -\lambda k^4,\ h^3_{122} = \mu k^3, h^4_{122} = -\mu k^4. \tag{3.5}$$

After substitution into the equations of the second column, they give

$$h^4_{111} = -\mu(k^3)^2(k^4)^{-1},\ h^3_{222} = -\lambda(k^3)^{-1}(k^4)^2. \tag{3.6}$$

We simplify the notations taking $k^3 = a$, $k^4 = b$, $h^3_{111} = \varphi$, $h^4_{222} = \psi$; then

$$\omega^2_1 = \lambda\omega^1 + \mu\omega^2, \quad da = \varphi\omega^1 + \lambda a\omega^2, \tag{3.7}$$

$$db = \mu b\omega^1 + \psi\omega^2, \quad -\omega^4_3 = ab(\mu b^{-2}\omega^1 + \lambda a^{-2}\omega^2). \tag{3.8}$$

Exterior differentiation leads to the following covariant equations

$$\begin{gathered}(d\lambda - \lambda^2\omega^2)\wedge\omega^1 + (d\mu + \mu^2\omega^1)\wedge\omega^2 = 0,\\ (d\varphi - \lambda\mu a\omega^2)\wedge\omega^1 + (d(\lambda a) + \lambda\varphi\omega^1)\wedge\omega^2 = 0,\\ -(d(\mu b) + \mu\psi\omega^2)\wedge\omega^1 + (d\psi - \lambda\mu b\omega^1)\wedge\omega^2 = 0,\\ \left[ab^{-1}d\mu - \phi\omega^2\right]\wedge\omega^1 + \left[a^{-1}bd\lambda + \phi\omega^1\right]\wedge\omega^2 = 0,\end{gathered}$$

where $\phi = \frac{1}{2}(\varphi ba^{-2}\lambda + \psi ab^{-2}\mu)$.

Due to Cartan's Lemma from the first equation we get

$$d\lambda = \rho\omega^1 + (\sigma + \lambda^2)\omega^2,\ d\mu = (\sigma - \mu^2)\omega^1 + \tau\omega^2. \tag{3.9}$$

Now the second and third ones reduce to

$$\begin{gathered}[d\varphi - (a\lambda\mu + a\rho 2\lambda\varphi)\omega^2]\wedge\omega^1 = 0,\\ [d\psi - (b\lambda\mu - b\tau - 2\mu\psi)\omega^1]\wedge\omega^2 = 0,\end{gathered}$$

and give

$$d\varphi = \tilde{\varphi}\omega^1 + (a\lambda\mu + a\rho + 2\lambda\varphi)\omega^2, \tag{3.10}$$

$$d\psi = (b\lambda\mu - b\tau - 2\mu\psi)\omega^1 + \tilde{\psi}\omega^2, \tag{3.11}$$

but the last one implies

$$\tau = a^{-2}b^{-2}\rho + \varphi\lambda a^{-3}b^2 + \psi\mu b^{-1}. \tag{3.12}$$

The system of four covariant equations contains four secondary forms $d\lambda$, $d\mu$, $d\varphi, d\psi$ and two linearly independent primary forms ω^1, ω^2. The rank of the corresponding bilinear system is $s_1 = 4$. After using Cartan's Lemma four new linearly independent coefficients $\rho, \sigma, \tilde{\varphi}, \tilde{\psi}$ occurre; since the number of these coefficients is equal to s_1, the Cartan test condition is satisfied, the system is compatible and determines the considered surface with arbitrariness of four real holomorphic functions of one real argument (see [3], [2]).

4 The Case of Weak Harmonic Surfaces

To continue here with weak biharmonic surfaces M^2 in $\mathbf{E}^4$, one has to exterior differentiate again, find the quantitative h^α_{iikk} and build the condition $\Delta^D H = 0$. The equation (3) gives for $\alpha = 3$ and $i = j = k = 1$

$$d\varphi = 3\lambda a(\lambda\omega^1 + \mu\omega^2) - \mu a^2 b^{-1} ab(\mu b^{-2}\omega^1 + \lambda a^2\omega^2) + h^3_{1111}\omega^1 + h^3_{1112}\omega^2,$$

so it is seen that $h^3_{1111} = \tilde{\varphi} - 3\lambda^2 a + \mu^2 a^3 b^{-2}$.

For $\alpha = 3$, $i = j = 1$ and $k = 2$ we infer

$$h^3_{1122} = a(\sigma + 2\lambda^2 - 2\mu^2) - a^{-1}b^2\lambda^2 + \mu\varphi,$$

and the same expression has h^3_{2211}. This further leads to $h^3_{2222} = a^{-1}b(b\sigma + 3\lambda\psi) + 3a\mu^2$, but $\alpha = 4$ and

$$h^4_{1111} = -ab^{-1}(a\sigma + 3\mu\varphi) - 3b\lambda^2, \;\; h^4_{1122} = h^4_{2211} = -b(\sigma - 2\lambda^2 + 2\mu^2) + a^2b^{-1}\mu^2 - \lambda\psi,$$

$$h^4_{2222} = \tilde{\psi} - 3b\mu^2 - a^{-2}b^3\lambda^2.$$

Now the conditions $\Delta^D H^3 = \Delta^D H^4 = 0$ imply

$$\tilde{\varphi} = -(2a + a^{-1}b^2)\sigma + \tilde{\Phi}, \qquad \tilde{\psi} = -(2b + a^2b^{-1})\sigma + \tilde{\Psi}, \tag{4.13}$$

where $\tilde{\Phi}$ and $\tilde{\Psi}$ are some algebraic expressions of $a, b, \lambda, \mu, \varphi, \psi$ with constant coefficients.

We replace these expressions of $\tilde{\Phi}$ and $\tilde{\Psi}$ into (13) and then make substitutions of the latter into (10), (11); the two obtained equations hold, in view of (12).

Exterior differentiation leads now to following covariant equations:

$$\begin{gathered} d\rho \wedge \omega^1 + d\sigma \wedge \omega^2 + \Pi\omega^1 \wedge \omega^2 = 0, \\ d\sigma \wedge \omega^1 + a^{-2}b^2 d\rho \wedge \omega^2 + \Gamma\omega^1 \wedge \omega^2 = 0, \\ (2a + a^{-1}b^2)d\sigma \wedge \omega^1 + \Xi\omega^1 \wedge \omega^2 = 0, \\ (2a + a^2b^{-1})d\sigma \wedge \omega^2 + \Upsilon\omega^1 \wedge \omega^2 = 0, \end{gathered}$$

where the capital Greek letters denote some algebraic expressions of $a, b, \lambda, \mu, \varphi, \psi$. Cartan's Lemma gives now from the first equation

$$d\rho = \alpha\omega^1 + \beta\omega^2, \qquad d\sigma + \Pi\omega^1 = \beta\omega^1 + \gamma\omega^2.$$

We substitute this into the other three equations, and obtain

$$-\gamma + a^{-2}b^2\alpha + \Gamma = 0, \qquad (2a + a^{-1}b^2)\gamma = \Xi, \qquad (2b + a^2b^{-1})\beta = -\Upsilon.$$

Hence the system needs new differential prolongation which leads to some algebraic relations among the old coefficients. Due to certain technical difficulties, this will be not done in the present paper.

Further we restrict ourselves to a particular case.

5 Subcase of surfaces whose one family of curvature lines consists of geodesics

Let further $\lambda = 0$. Then $\omega^2_1 = \mu\omega^2$, thus for the curvature lines determined by $\omega^2 = 0$ there hold $dx = e_1\omega^1$, $de_1 = ae_3\omega^1$, so that these lines are geodesics.

From (9)-(12) it follows that now $\rho = \sigma = 0$, $\tau = \psi\mu b^{-1}$, $d\varphi = \tilde{\varphi}\omega^1$, $d\psi = -3\mu\psi\omega^1 + \tilde{\psi}\omega^2$. The covariant equations are

$$\begin{array}{ll} \left(d\mu + \mu^2\omega^1\right) \wedge \omega^2 = 0, & d\varphi \wedge \omega^1 = 0 \\ -bd\mu \wedge \omega^1 + (d\psi + 2\mu\psi) \wedge \omega^2 = 0, & (bd\mu - \psi\mu\omega^2) \wedge \omega^1 = 0. \end{array} \tag{5.14}$$

They contain three secondary forms $d\mu, d\varphi, d\psi$, and the rank of the corresponding bilinear system is $s_1 = 3$. The Cartan's Lemma gives $d\mu = -\mu^2\omega^1 + b^{-1}\psi\mu\omega^2$ and the expressions above for $d\varphi$ and $d\psi$. Hence the number of new coefficients $\tilde{\varphi}$, $\tilde{\psi}$ is less than s_1 and a new differential prolongation is needed. The equation containing $d\mu$ gives by exterior differentiation trivial identity, but the other for $d\varphi$ and $d\psi$ give

$$d\tilde{\varphi} \wedge \omega^1 = 0, \qquad \Big[d\tilde{\psi} + (4\tilde{\psi}\mu + 3b^{-1}\psi\mu)\omega^1\Big] \wedge \omega^2 = 0.$$

Here $s_1 = 2$ and after using Cartan's Lemma, two new coefficients occur. Hence the system is compatible and determines the considered surface with arbitrariness of two real holomorphic functions of one real argument.

We further look for the weak harmonic surfaces among the considered ones. We have

$$h^3_{1111} = \tilde{\varphi} + a^3b^{-2}\mu^2,\ h^3_{1122} = h^3_{2211} = -2a\mu^2 + \mu\varphi,\ h^3_{2222} = 3a\mu^2$$

$$h^4_{1111} = -3ab^{-1}\mu\varphi,\ h^4_{1122} = h^4_{2211} = -2b\mu^2 + a^2b^{-1}\mu^2,\ h^4_{2222} = \tilde{\psi} - 3b\mu^2.$$

Therefore $\Delta^D H^3 = \Delta^D H^4 = 0$ lead to

$$\tilde{\varphi} = \mu^2 a(1 - a^2b^2) - 2\mu\varphi \tag{5.15}$$

$$\tilde{\psi} = b(2a^2b^{-2} - 1)\mu^2 + 3ab^{-1}\mu\varphi. \tag{5.16}$$

We substitute these expressions into the previous equations and then exteriorly differentiate the latter. The equation $d\varphi = \tilde{\varphi}\omega^1$ gives then $b^{-1}\mu\psi(\varphi - a\mu) = 0$. Then either (*i*) $\mu = 0$ or (*ii*) $\psi = 0$ or (*iii*) $\varphi = a\mu = 0$.

(*i*) If $\mu = 0$ then $\tilde{\psi} = 0$; thus $d\psi = 0$ and $\psi = \psi_0 =$ const.; also $\tilde{\varphi} = 0$ from (15), and hence $\varphi = \varphi_0 =$ const. Moreover, $\omega_1^2 = \omega_3^4 = 0$. From (5) and (6) it follows that the components of the third fundamental form, except two, are equal to zero. But two last ones are $h^3_{111} = \varphi_0$, $h^4_{222} = \psi_0$, hence constants. Thus the surface is trivially 2-parallel and turns to be a product of two plane Cornu spirals.

(*ii*) If $\mu \neq 0, \psi = 0$, then also $\tilde{\psi} = 0$, due to (14), and (16) gives $\varphi = \frac{1}{3}a(a^{-2}b^{-2}\text{-}2)\mu$. From this, we infer by differentiation

$$d\varphi = -\frac{1}{9}a(8 - 8a^{-2}b^2 - a^4b^4)\mu^2\omega^1.$$

Now (15) implies

$$\tilde{\varphi} = -\frac{1}{3}a(a^2b^{-2} + 2a^{-2}b^2 + 3)\mu^2.$$

Substitution into $d\varphi = \tilde{\varphi}\omega^1$ leads to

$$b^6 + 2a^2b^4 - 17a^4b^2 - 3a^6 = 0,$$

which shows that between a and b there is a relation $b = \kappa_0 a$, where κ_0^2 is a positive solution of the equation

$$x^3 + 2x^2 - 17\lambda - 3 = 0.$$

Elementary calculations show that this equation really has a positive solution between 3 and 4. Let us substitute $b = \kappa_0 a$ into the equation $db = -\mu b\omega^1$; the result is $da = -a\mu\omega^1$. This shows that there must be the equality $\varphi = -a\mu$, but using here the obtained above expression for φ the result is a contradiction $\kappa_0^2 a^{-4} = -1$. Hence the surface in the case (*ii*) does not exist.

(*iii*) If $\mu\psi \neq 0$, then necessarily $\varphi = a\mu$; but this gives by differentiation $\tilde{\varphi}\omega^1 = \varphi\omega^1\mu + a(-\mu^2\omega^1 + b^{-1}\psi\mu\omega^2)$, and thus a result which contains a contradiction: here the coefficient of ω^2 cannot be zero!

Hence the surface in the case (*iii*) also does not exist.

As a result the following statement can be formulated.

Theorem 2. *In $\boldsymbol{E}^4$ in the class of surfaces with flat connection $\bar{\nabla}$, whose one family of curvature lines consists of geodesics, a surface is weak biharmonic if and only if it is 2-parallel.*

The Proposition 2 and Theorem 2 let us assume the following

Conjecture. A locally Euclidean submanifold M^m in $\mathbf{E}^n$ is weak harmonic if and only if it is 2-parallel.

To verify this conjecture a lot of work has to be done. First the case of general surfaces with flat $\overline{\nabla}$ in $\mathbf{E}^4$ must be finished; the case $m > 2$, $n > 4$ is not completely studied as well.

Note that in the classical case of surfaces M^2 in $\mathbf{E}^3$ the conjecture is true, and can be verified by taking $a = 0$ in the formulas above. Then $\varphi = 0$ and from (10) $\tilde{\varphi} = 0$, but the first relations of section 3, right-hand column, give $\omega_1^2 = -bh_{122}^4$, thus in (7) $\lambda = 0$ and due to (9) $\rho = \sigma = 0$. Further, (16) reduces to $\tilde{\psi} = -b\mu^2$, from (12) $\tau = b^{-1}\psi\mu$, and as a result $d\psi = -3\mu\psi\omega^1 - b\mu^2\omega^2$. This by exterior differentiation implies $\mu(3\psi^2 - b^2\mu^2) = 0$. Here $\mu = 0$ gives (*i*) as before, but $\mu \neq 0$, $\psi = \pm\frac{1}{\sqrt{3}}b\mu$ leads to a contradiction, like (*iii*).

We conclude with an announcement, that in $\mathbf{E}^3$ there exist surfaces with $K \neq 0$, which are weak biharmonic, but not 2-parallel. By means of the Cartan's theory it can be shown that these exist with arbitrariness of two real holonomic functions of the real argument.

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