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# On Cesàro and Abel-Poisson means of hexagonal Fourier series

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**Abstract:** Approximation properties of Cesàro and Abel-Poisson means of hexagonal Fourier series are studied. The degree of approximation by these means of hexagonal Fourier series of functions, which are continuous and periodic with respect to the hexagon lattice, is estimated in terms of the modulus of continuity of functions.

**Key words:** Abel Poisson-means, Cesàro means, degree of approximation, hexagonal Fourier series

## 1. Introduction

Approximation properties of Cesàro $(C,\delta)$ and Abel-Poisson means of Fourier series of $2\pi$-periodic functions of a real variable were studied by many mathematicians. The monographs [16] and [17] contain a lot of results about convergence and the degree of approximation of these means. In the same monographs, there are also results regarding approximation properties of Cesàro and Abel-Poisson means of functions of two or more variables. Approximation properties of functions of two or more variables are studied usually by assuming that the functions are periodic with respect to each of their variables. In the approximation theory of functions of several real variables, another definition of periodicity are also used. The periodicity defined by lattices is the most useful periodicity. This definition of periodicity allows us to study trigonometric approximation problems on nontensor product domains of the Euclidean space, for instance, on regular hexagons in the plane.

A lattice in the $d-$ dimensional Euclidean space $\mathbb{R}^d$ is the discrete subgroup

$$L_A := A\mathbb{Z}^d = \{Ak : k \in \mathbb{Z}^d\},$$

where $A$ is a $d \times d$ matrix with linearly independent columns, which is called the generator matrix of the lattice $L_A$. A bounded set $\Omega \subset \mathbb{R}^d$ is said to tile $\mathbb{R}^d$ with the lattice $L_A$ if

$$\sum_{k\in\mathbb{Z}^d} \chi_\Omega(x + Ak) = 1$$

for almost all $x \in \mathbb{R}^d$. The set $\Omega$ is called a spectral set for the lattice $L_A$ if it tiles $\mathbb{R}^d$ with $L_A$. For a given lattice, the spectral set is not unique. We fix $\Omega$ such that $\Omega$ contains the point $0$ in its interior and the tiling holds pointwise and without overlapping, that is

$$\sum_{k\in\mathbb{Z}^d} \chi_\Omega(x + Ak) = 1$$

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for all $x \in \mathbb{R}^d$ and

$$(\Omega + Ak) \cap (\Omega + Aj) = \emptyset$$

for $k \neq j$. For example we take $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ for the standard lattice $L_{I_d} = \mathbb{Z}^d$.

Let $\Omega$ be a spectral set for the lattice $L_A$. Consider the space $L^2(\Omega)$, which is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{L^2(\Omega)} = \frac{1}{|\Omega|}\int_\Omega f(x)\overline{g(x)}dx,$$

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. The following theorem of B. Fuglede allows us to define Fourier series on the spectral set $\Omega$:

**Theorem 1.1 ([2])** *The open bounded set $\Omega$ is a spectral set for the lattice $L_A$ if and only if the set*

$$\{e^{2\pi i\langle A^{-tr}k,x\rangle} : k \in \mathbb{Z}^d\}$$

*is an orthonormal basis of $L^2(\Omega)$.*

Hence, if $\Omega$ is a spectral set for the lattice $L_A$, the Fourier series of a function $f \in L^2(\Omega)$ becomes

$$\sum_{k\in\mathbb{Z}^d} \widehat{f}_k e^{2\pi i\langle A^{-tr}k,x\rangle},$$

where

$$\widehat{f}_k = \frac{1}{|\Omega|}\int_\Omega f(x)e^{-2\pi i\langle A^{-tr}k,x\rangle}dx, \quad k \in \mathbb{Z}^d$$

are Fourier coefficients.

A function defined on $\mathbb{R}^d$ is called periodic with respect to the lattice $L_A$ or $A-$periodic if

$$f(x + Ak) = f(x)$$

for all $k \in \mathbb{Z}^d$.

More detailed information on Fourier analysis with lattices can be found in [11].

## 2. Hexagonal Fourier series

Besides the rectangular domain, the simplest and the most useful spectral set is a regular hexagon on the plane $\mathbb{R}^2$. The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H = \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}$$

and

$$\Omega_H = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_2, \frac{\sqrt{3}}{2}x_1 \pm \frac{1}{2}x_2 < 1\}.$$

We denote the plane $t_1 + t_2 + t_3 = 0$ by $\mathbb{R}^3_H$, that is

$$\mathbb{R}^3_H = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\}.$$

Elements of the set $\mathbb{R}_H^3$ are called homogeneous coordinates. Under the transform

$$t_1 := -\frac{x_2}{2} + \frac{\sqrt{3}x_1}{2}, t_2 := x_2, t_3 := -\frac{x_2}{2} - \frac{\sqrt{3}x_1}{2},$$

the hexagon $\Omega_H$ becomes

$$\Omega = \{(t_1, t_2, t_3) \in \mathbb{R}_H^3 : -1 \leq t_1, t_2, -t_3 < 1\}.$$

We use bold letters $\mathbf{t}$, $\mathbf{s}$, etc. to denote homogeneous coordinates. Also we denote the subset of $\mathbb{R}_H^3$ consists of points with integer components by $\mathbb{Z}_H^3$.

A function $f : \mathbb{R}^2 \to \mathbb{C}$ is periodic with respect to the hexagonal lattice, or briefly $H-$periodic if

$$f(x + Hk) = f(x), \quad k \in \mathbb{Z}^2.$$

If we define

$$\mathbf{t} \equiv \mathbf{s} \pmod{3} \Longleftrightarrow t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \pmod{3}$$

for $\mathbf{t} = (t_1, t_2, t_3), \mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}_H^3$, it follows that the function $f$ is $H-$periodic if and only if

$$f(\mathbf{t} + \mathbf{s}) = f(\mathbf{t})$$

whenever $\mathbf{s} \equiv \mathbf{0}$ (mod3), in terms of homogeneous coordinates. Also,

$$\int_\Omega f(\mathbf{t} + \mathbf{s}) d\mathbf{t} = \int_\Omega f(\mathbf{t}) d\mathbf{t}, \;\; \mathbf{s} \in \mathbb{R}_H^3 \tag{2.1}$$

for the $H-$periodic function $f$ ([15]).

$L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_H := \frac{1}{|\Omega|} \int_\Omega f(\mathbf{t}) \overline{g(\mathbf{t})} d\mathbf{t},$$

where $|\Omega|$ denotes the area of $\Omega$. The functions

$$\varphi_{\mathbf{j}}(\mathbf{t}) := e^{\frac{2\pi i}{3} \langle \mathbf{j}, \mathbf{t} \rangle}, \;\; \mathbf{j} \in \mathbb{Z}_H^3, \mathbf{t} \in \mathbb{R}_H^3,$$

where $\langle \mathbf{j}, \mathbf{t} \rangle$ is the usual Euclidean inner product of $\mathbf{j}$ and $\mathbf{t}$, are $H-$periodic, and by Theorem 1.1, the set

$$\{\varphi_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}_H^3\}$$

becomes an orthonormal basis for $L^2(\Omega)$. See [11, 15] for details.

For every natural number $n$ we define the set

$$\mathbb{H}_n := \{\mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}_H^3 : -n \leq j_1, j_2, j_3 \leq n\},$$

which consists of points inside the hexagon $n\overline{\Omega}$ with integer components.

The hexagonal Fourier series of an $H-$periodic function $f \in L^2(\Omega)$ is defined by

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}} \varphi_{\mathbf{j}}(\mathbf{t}), \tag{2.2}$$

where

$$\widehat{f}_{\mathbf{j}} = \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t})\overline{\varphi_{\mathbf{j}}(\mathbf{t})}d\mathbf{t}, \quad \mathbf{j}\in\mathbb{Z}_H^3.$$

The $n$-th hexagonal partial sum of (2.2) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j}\in\mathbb{H}_n} \widehat{f}_{\mathbf{j}}\varphi_{\mathbf{j}}(\mathbf{t}) = \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t}-\mathbf{s})D_n(\mathbf{s})d\mathbf{s}, \tag{2.3}$$

where the second equality of (2.3) follows from (2.1) with the kernel $D_n$ defined by

$$D_n(\mathbf{t}) := \sum_{\mathbf{j}\in\mathbb{H}_n} \varphi_{\mathbf{j}}(\mathbf{t}).$$

The kernel $D_n$ is an analogue of the Dirichlet kernel for the classical Fourier series. It is known by [11] (see also [15]) that

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}), \tag{2.4}$$

where

$$\Theta_n(\mathbf{t}) := \frac{\sin\frac{(n+1)(t_1-t_2)\pi}{3}\sin\frac{(n+1)(t_2-t_3)\pi}{3}\sin\frac{(n+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}} \tag{2.5}$$

for $\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}_H^3$.

The degree of approximation by various means of hexagonal Fourier series was studied in [3–10]. In [3–5], some estimates for the degree of approximation of $(C,1)$ and Abel-Poisson means of hexagonal Fourier series were obtained. But, in these works, there are no estimates in terms of the moduli of continuity of functions. In this article, the degree of approximation of $(C,1)$ and Abel-Poisson means of hexagonal series of a function belonging to $C(\overline{\Omega})$ is estimated directly in terms of the modulus of continuity of the function.

## 3. Approximation by $(C,\delta)$ means on hexagonal domain

For $\delta>0$ we denote the Cesàro $(C,\delta)$ means of the series (2.2) by $S_n^{(\delta)}(f)$, i. e.

$$S_n^{(\delta)}(f)(\mathbf{t}) = \frac{1}{A_n^{\delta}}\sum_{k=0}^{n} A_{n-k}^{\delta-1}S_k(f)(\mathbf{t}), \quad A_n^{\delta} = \binom{n+\delta}{\delta}.$$

It is easy to show that

$$S_n^{(\delta)}(f)(\mathbf{t}) = \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t}-\mathbf{s})K_n^{(\delta)}(\mathbf{s})d\mathbf{s}, \tag{3.1}$$

where

$$K_n^{(\delta)}(\mathbf{t}) = \frac{1}{A_n^{\delta}}\sum_{k=0}^{n} A_{n-k}^{\delta-1}D_k(\mathbf{t}).$$

If we set $\Theta_{-1}(\mathbf{t}) := 0$, we can express the kernel $K_n^{(\delta)}$ as

$$K_n^{(\delta)}(\mathbf{t}) = \frac{1}{A_n^{\delta}}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t})\right) \tag{3.2}$$

by aim of (2.4). Since

$$\frac{1}{|\Omega|}\int_{\Omega} D_k(\mathbf{t})d\mathbf{t} = 1$$

and

$$A_n^{\delta} = \sum_{k=0}^{n} A_k^{\delta-1}$$

we have

$$\frac{1}{|\Omega|}\int_{\Omega} K_n^{(\delta)}(\mathbf{t})d\mathbf{t} = 1. \tag{3.3}$$

Let $C(\overline{\Omega})$ be the Banach space of $H-$periodic continuous functions $f : \mathbb{R}_H^3 \to \mathbb{C}$, whose norm is the uniform norm:

$$\|f\|_{C(\overline{\Omega})} = \sup_{\mathbf{t}\in\overline{\Omega}} |f(\mathbf{t})| .$$

The modulus of continuity of a function $f \in C(\overline{\Omega})$ is defined by

$$\omega_f(u) := \sup_{0<\|\mathbf{t}\|\leq u} \|f - f(\cdot + \mathbf{t})\|_{C(\overline{\Omega})}, \quad u > 0$$

where

$$\|\mathbf{t}\| = \max\{|t_1|, |t_2|, |t_3|\}, \quad \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3.$$

$\omega_f(\cdot)$ is a nonnegative and nondecreasing function which satisfies

$$\omega_f(\lambda u) \leq (1 + \lambda)\omega_f(u) \tag{3.4}$$

for $\lambda > 0$ ([15]).

In the rest of the paper we shall write $A \lesssim B$ for the quantities $A$ and $B$ if there exists a constant $K > 0$ ($K$ is an absolute constant, or a constant depending only on parameters which are not important for the questions involve in the paper) such that $A \leq KB$ holds.

**Lemma 3.1** *The inequality*

$$\left|\frac{1}{A_n^{\delta}}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\cos(2k+1)t\right| \lesssim \frac{1}{(n+1)^{\delta}(\sin t)^{\delta}} + \frac{1}{(n+1)\sin t} \tag{3.5}$$

*holds for* $0 < \delta < 1, 0 < t < \pi$ *and for every natural number* $n$.

**Proof** It is known that ([17, page 94])

$$\left|\frac{1}{A_n^{\delta}\sin\frac{t}{2}}\sum_{k=0}^{n} A_{n-k}^{\delta-1}e^{i\left((2k+1)\frac{t}{2}\right)}\right| \lesssim \frac{1}{A_n^{\delta}\left(2\sin\frac{t}{2}\right)^{\delta+1}} + \frac{2A_{n+1}^{\delta-1}}{A_n^{\delta}\left(2\sin\frac{t}{2}\right)^{2}}$$

for $0 < t < 2\pi$. Hence, we have for $0 < t < 2\pi$

$$\left|\frac{1}{A_n^{\delta}\sin\frac{t}{2}}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\cos(2k+1)\frac{t}{2}\right| \lesssim \frac{1}{A_n^{\delta}\left(2\sin\frac{t}{2}\right)^{\delta+1}} + \frac{2A_{n+1}^{\delta-1}}{A_n^{\delta}\left(2\sin\frac{t}{2}\right)^{2}}$$

since $|\Re(z)| \leq |z|$ for every complex number $z$. This implies

$$\left|\frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\cos(2k+1)t\right| \lesssim \frac{1}{A_n^\delta\,(\sin t)^\delta} + \frac{A_{n+1}^{\delta-1}}{A_n^\delta \sin t} \quad (0 < t < \pi).$$

Since

$$\frac{A_{n+1}^{\delta-1}}{A_n^\delta} = \frac{\delta}{n+1}$$

we get

$$\left|\frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\cos(2k+1)t\right| \lesssim \frac{1}{A_n^\delta\,(\sin t)^\delta} + \frac{1}{(n+1)\sin t}.$$

This inequality and the fact $A_n^\delta \approx (n+1)^\delta$ yields (3.5). □

**Theorem 3.2** *For* $0 < \delta < 1$ *the kernel* $K_n^{(\delta)}$ *satisfies the inequality*

$$\int_\Omega \left|K_n^{(\delta)}(\boldsymbol{t})\right| d\boldsymbol{t} \lesssim \log(n+1). \tag{3.6}$$

**Proof**

$$\int_\Omega \left|K_n^{(\delta)}(\mathbf{t})\right| d\mathbf{t} = \frac{1}{A_n^\delta}\int_\Omega \left|\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t})\right)\right| d\mathbf{t}.$$

Since the function

$$\mathbf{t} \to \left|\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t})\right)\right|$$

is symmetric with respect to variables $t_1, t_2$ and $t_3$, where $\mathbf{t} = (t_1, t_2, t_3) \in \Omega$, it is sufficient to estimate the integral over the triangle

$$\begin{aligned}\Delta := &\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3 : 0 \leq t_1, t_2, -t_3 \leq 1\}\\ = &\{(t_1, t_2) : t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1\},\end{aligned}$$

which is one of the six equilateral triangles in $\overline{\Omega}$. By considering (2.5),

$$\begin{aligned}&\frac{1}{A_n^\delta}\int_\Delta \left|\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t})\right)\right| d\mathbf{t}\\ &= \frac{1}{A_n^\delta}\int_\Delta \left|\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\begin{array}{c}\frac{\sin\frac{(k+1)(t_1-t_2)\pi}{3}\sin\frac{(k+1)(t_2-t_3)\pi}{3}\sin\frac{(k+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\\ -\frac{\sin\frac{k(t_1-t_2)\pi}{3}\sin\frac{k(t_2-t_3)\pi}{3}\sin\frac{(k(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\end{array}\right)\right| d\mathbf{t}.\end{aligned}$$

If we use the change of variables

$$s_1 := \frac{t_1 - t_3}{3} = \frac{2t_1 + t_2}{3},\, s_2 := \frac{t_2 - t_3}{3} = \frac{t_1 + 2t_2}{3} \tag{3.7}$$

the integral becomes

$$3\frac{1}{A_n^{\delta}}\int_{\tilde{\Delta}}\left|\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\begin{array}{c}\frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1)\pi)}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin((-s_1)\pi)}\\ -\frac{\sin(k(s_1-s_2)\pi)\sin(k(s_2\pi)\sin(k(-s_1)\pi)}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin((-s_1)\pi)}\end{array}\right)\right| ds_1 ds_2,$$

where $\tilde{\Delta}$ is the image of $\Delta$ in the plane, that is

$$\tilde{\Delta} := \{(s_1, s_2) : 0 \leq s_1 \leq 2s_2, 0 \leq s_2 \leq 2s_1, s_1 + s_2 \leq 1\}.$$

Since the integrated function is symmetric with respect to $s_1$ and $s_2$, estimating the integral over the triangle

$$\Delta^* := \{(s_1, s_2) \in \tilde{\Delta} : s_1 \leq s_2\} = \{(s_1, s_2) : s_1 \leq s_2 \leq 2s_1, s_1 + s_2 \leq 1\},$$

which is the half of $\tilde{\Delta}$, will be sufficient. The change of variables

$$s_1 := \frac{u_1 - u_2}{2}, s_2 := \frac{u_1 + u_2}{2} \tag{3.8}$$

transforms the triangle $\Delta^*$ to the triangle

$$\Gamma := \{(u_1, u_2) : 0 \leq u_2 \leq \frac{u_1}{3}, 0 \leq u_1 \leq 1\}.$$

Thus we must estimate

$$I_n^{(\delta)} := \frac{1}{A_n^{\delta}}\int_{\Gamma}\left|\sum_{k=0}^{n} A_{n-k}^{\delta-1} D_k^*(u_1, u_2)\right| du_1 du_2,$$

where

$$\begin{aligned} D_k^*(u_1, u_2) :=& \frac{\sin((k+1)u_2\pi)\sin((k+1)\frac{u_1+u_2}{2}\pi)\sin((k+1)\frac{u_1-u_2}{2}\pi)}{\sin(u_2\pi)\sin(\frac{u_1+u_2}{2}\pi)\sin(\frac{u_1-u_2}{2}\pi)} \\ &- \frac{\sin(ku_2\pi)\sin(k\frac{u_1+u_2}{2}\pi)\sin(k\frac{u_1-u_2}{2}\pi)}{\sin(u_2\pi)\sin(\frac{u_1+u_2}{2}\pi)\sin(\frac{u_1-u_2}{2}\pi)}. \end{aligned}$$

Using elementary trigonometric identities yields

$$D_k^*(u_1, u_2) = D_{k,1}^*(u_1, u_2) + D_{k,2}^*(u_1, u_2) + D_{k,3}^*(u_1, u_2),$$

where

$$\begin{aligned} D_{k,1}^*(u_1, u_2) &= 2\cos\left(\left(k + \frac{1}{2}\right)u_2\pi\right)\frac{\sin(\frac{1}{2}u_2\pi)\sin((k+1)\frac{u_1+u_2}{2}\pi)\sin((k+1)\frac{u_1-u_2}{2}\pi)}{\sin(u_2\pi)\sin(\frac{u_1+u_2}{2}\pi)\sin(\frac{u_1-u_2}{2}\pi)}, \\ D_{k,2}^*(u_1, u_2) &= 2\cos\left(\left(k + \frac{1}{2}\right)\frac{u_1+u_2}{2}\pi\right)\frac{\sin(ku_2\pi)\sin(\frac{1}{2}\frac{u_1+u_2}{2}\pi)\sin((k+1)\frac{u_1-u_2}{2}\pi)}{\sin(u_2\pi)\sin(\frac{u_1+u_2}{2}\pi)\sin(\frac{u_1-u_2}{2}\pi)}, \\ D_{k,3}^*(u_1, u_2) &= 2\cos\left(\left(k + \frac{1}{2}\right)\frac{u_1-u_2}{2}\pi\right)\frac{\sin(ku_2\pi)\sin(k\frac{u_1+u_2}{2}\pi)\sin(\frac{1}{2}\frac{u_1-u_2}{2}\pi)}{\sin(u_2\pi)\sin(\frac{u_1+u_2}{2}\pi)\sin(\frac{u_1-u_2}{2}\pi)}. \end{aligned}$$

We can write $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 := \{(u_1,u_2) \in \Gamma : u_1 \leq \frac{1}{n+1}\},$$

$$\Gamma_2 := \{(u_1,u_2) \in \Gamma : u_1 \geq \frac{1}{n+1}, u_2 \leq \frac{1}{3(n+1)}\},$$

$$\Gamma_3 := \{(u_1,u_2) \in \Gamma : u_1 \geq \frac{1}{n+1}, u_2 \geq \frac{1}{3(n+1)}\}.$$

Hence we have $I_n^{(\delta)} = I_{n,1}^{(\delta)} + I_{n,2}^{(\delta)} + I_{n,3}^{(\delta)}$, where

$$I_{n,j}^{(\delta)} := \frac{1}{A_n^\delta} \int_{\Gamma_j} \left| \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_k^*(u_1,u_2) \right| du_1 du_2 \quad (j = 1,2,3).$$

We shall use well known inequalities

$$\left| \frac{\sin(nt)}{\sin t} \right| \leq n \quad (n \in \mathbb{N}) \tag{3.9}$$

and

$$\sin t \geq \frac{2}{\pi} t \quad \left(0 \leq t \leq \frac{\pi}{2}\right) \tag{3.10}$$

to estimate the integrals $I_{n,j}^{(\delta)}$ $(j = 1,2,3)$. Simple calculations give

$$\frac{\sin\left(\frac{u_2\pi}{2}\right)}{\sin(u_2\pi)} \leq 1, \frac{\sin\left(\frac{1}{2}\frac{u_1+u_2}{2}\pi\right)}{\sin\left(\frac{u_1+u_2}{2}\pi\right)} \leq 1, \frac{\sin\left(\frac{1}{2}\frac{u_1-u_2}{2}\pi\right)}{\sin\left(\frac{u_1-u_2}{2}\pi\right)} \leq 1$$

for $(u_1,u_2) \in \Gamma$. By considering these inequalities and (3.9) we get

$$\left|D_{k,1}^*(u_1,u_2)\right| \leq (k+1)^2, \left|D_{k,2}^*(u_1,u_2)\right| \leq k(k+1), \left|D_{k,3}^*(u_1,u_2)\right| \leq k^2 \tag{3.11}$$

for $(u_1,u_2) \in \Gamma_1$. Hence,

$$\begin{aligned} I_{n,1}^{(\delta)} &\leq \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^{\delta-1} (k+1)^2 \int_{\Gamma_1} du_1 du_2 \\ &= \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^{\delta-1} (k+1)^2 \frac{1}{6(n+1)^2} \leq \frac{1}{6} \leq 1, \end{aligned} \tag{3.12}$$

since ([17, page 77])

$$\sum_{k=0}^{n} A_{n-k}^{\delta-1} = A_n^\delta. \tag{3.13}$$

We can write the rectangle $\Gamma_2$ as the union of rectangles

$$\Gamma_2^{'} = \{(u_1,u_2) \in \Gamma_2 : u_2 \leq \frac{1}{(n+1)^2}\}$$

$$\Gamma_2^{''} = \{(u_1,u_2) \in \Gamma_2 : u_2 \geq \frac{1}{(n+1)^2}\}.$$

It is easy to see that

$$\sin\left(\frac{u_1+u_2}{2}\pi\right) \geq \frac{\sqrt{3}}{2}\sin\left(\frac{u_1\pi}{2}\right) \quad \text{and} \quad \sin\left(\frac{u_1-u_2}{2}\pi\right) \geq \sin\left(\frac{u_1\pi}{3}\right) \tag{3.14}$$

for $(u_1, u_2) \in \Gamma$. Considering these inequalities and (3.10) yields

$$\left|D^*_{k,1}(u_1,u_2)\right| \lesssim \frac{1}{u_1^2} \quad \text{and} \quad \left|D^*_{k,j}(u_1,u_2)\right| \lesssim \frac{k}{u_1} \quad (j=2,3) \tag{3.15}$$

for $(u_1, u_2) \in \Gamma_2^{'}$. Hence

$$\begin{aligned}
&\frac{1}{A_n^\delta}\int_{\Gamma_2^{'}}\left|\sum_{k=0}^{n} A_{n-k}^{\delta-1} D^*_{k,1}(u_1,u_2)\right| du_1du_2 \\
&\leq \frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\int_{\Gamma_2^{'}}\left|D^*_{k,1}(u_1,u_2)\right| du_1du_2\right) \\
&\lesssim \frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\int_{\Gamma_2^{'}}\frac{1}{u_1^2} du_1du_2\right) \\
&= \frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\int_0^{\frac{1}{3(n+1)^2}}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1^2} du_1du_2\right) \\
&\leq \frac{1}{n+1},
\end{aligned}$$

and for $j = 2, 3$

$$\begin{aligned}
&\frac{1}{A_n^\delta}\int_{\Gamma_2^{'}}\left|\sum_{k=0}^{n} A_{n-k}^{\delta-1} D^*_{k,j}(u_1,u_2)\right| du_1du_2 \\
&\lesssim \frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}k\left(\int_{\Gamma_2^{'}}\frac{1}{u_1} du_1du_2\right) \\
&\leq \frac{n}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}\left(\int_0^{\frac{1}{3(n+1)^2}}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1} du_1du_2\right) \\
&= \frac{n}{3(n+1)^2}\log(n+1) \\
&\leq \frac{\log(n+1)}{n+1}.
\end{aligned}$$

For $(u_1, u_2) \in \Gamma_2^{''} \cup \Gamma_3$ we need another expression of the function $D_k^*$. Since

$$\sin(2x) + \sin(2y) + \sin(2z) = -4\sin x \sin y \sin z$$

for $x + y + z = 0$, we have

$$D_k^*(u_1,u_2) = H^*_{k,1}(u_1,u_2) + H^*_{k,2}(u_1,u_2) + H^*_{k,3}(u_1,u_2),$$

where

$$H^*_{k,1}(u_1,u_2) := \frac{1}{2}\frac{\cos((2k+1)u_2\pi)}{\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$

$$H^*_{k,2}(u_1,u_2) := -\frac{1}{2}\frac{\cos\left((2k+1)\frac{u_1+u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$

$$H^*_{k,3}(u_1,u_2) := \frac{1}{2}\frac{\cos\left((2k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)}.$$

By (3.10) and (3.5) we have

$$\begin{aligned}\left|\frac{1}{A^\delta_n}\sum_{k=0}^{n} A^{\delta-1}_{n-k} H^*_{k,1}(u_1,u_2)\right| &\lesssim \frac{1}{u_1^2}\left|\frac{1}{A^\delta_n}\sum_{k=0}^{n} A^{\delta-1}_{n-k}\cos((2k+1)u_2\pi)\right| \\ &\lesssim \frac{1}{u_1^2}\left(\frac{1}{(n+1)^\delta(\sin(u_2\pi))^\delta}+\frac{1}{(n+1)\sin(u_2\pi)}\right) \\ &\le \frac{1}{u_1^2}\left(\frac{1}{(n+1)^\delta u_2^\delta}+\frac{1}{(n+1)u_2}\right).\end{aligned}$$

Since $(n+1)u_2 < 1$ for $(u_1,u_2)\in\Gamma_2''$,

$$\left|\frac{1}{A^\delta_n}\sum_{k=0}^{n} A^{\delta-1}_{n-k} H^*_{k,1}(u_1,u_2)\right| \lesssim \frac{1}{u_1^2}\frac{1}{(n+1)u_2}, \quad (u_1,u_2)\in\Gamma_2'', \tag{3.16}$$

and since $3(n+1)u_2 \ge 1$ for $(u_1,u_2)\in\Gamma_3$, we get

$$\left|\frac{1}{A^\delta_n}\sum_{k=0}^{n} A^{\delta-1}_{n-k} H^*_{k,1}(u_1,u_2)\right| \lesssim \frac{1}{u_1^2}\frac{1}{(n+1)^\delta u_2^\delta}, \quad (u_1,u_2)\in\Gamma_3. \tag{3.17}$$

By similar way we obtain

$$\left|\frac{1}{A^\delta_n}\sum_{k=0}^{n} A^{\delta-1}_{n-k} H^*_{k,j}(u_1,u_2)\right| \lesssim \frac{1}{u_1^{\delta+1}}\frac{1}{(n+1)^\delta u_2}, \quad (u_1,u_2)\in\Gamma_2''\cup\Gamma_3, \tag{3.18}$$

for $j=2,3$. By (3.17),

$$\begin{aligned}\frac{1}{A^\delta_n}\int_{\Gamma_2''}\left|\sum_{k=0}^{n} A^{\delta-1}_{n-k} H^*_{k,1}(u_1,u_2)\right| du_1du_2 &= \int_{\Gamma_2''}\left|\frac{1}{A^\delta_n}\sum_{k=0}^{n} A^{\delta-1}_{n-k} H^*_{k,1}(u_1,u_2)\right| du_1du_2 \\ &\lesssim \frac{1}{n+1}\int_{\frac{1}{3(n+1)^2}}^{\frac{1}{3(n+1)}}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1^2}\frac{1}{u_2}du_1du_2 \\ &= \frac{n}{n+1}\log(n+1) \le \log(n+1).\end{aligned}$$

By (3.18) we get

$$\begin{aligned}\frac{1}{A_n^\delta}\int_{\Gamma_2''}\left|\sum_{k=0}^{n} A_{n-k}^{\delta-1}H_{k,j}^*(u_1,u_2)\right|du_1du_2 &= \int_{\Gamma_2''}\left|\frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}H_{k,j}^*(u_1,u_2)\right|du_1du_2\\ &\lesssim \frac{1}{(n+1)^\delta}\int_{\frac{1}{3(n+1)^2}}^{\frac{1}{3(n+1)}}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1^{\delta+1}}\frac{1}{u_2}du_1du_2\\ &= \frac{\log(n+1)}{(n+1)^\delta}\left((n+1)^\delta-1\right)\\ &\le \log(n+1)\end{aligned}$$

for $j=2,3$. Combining these estimates yields

$$I_{n,2}^{(\delta)} \lesssim \log(n+1).$$

By (3.17)

$$\begin{aligned}\frac{1}{A_n^\delta}\int_{\Gamma_3}\left|\sum_{k=0}^{n} A_{n-k}^{\delta-1}H_{k,1}^*(u_1,u_2)\right|du_1du_2 &= \int_{\Gamma_3}\left|\frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}H_{k,1}^*(u_1,u_2)\right|du_1du_2\\ &\lesssim \frac{1}{(n+1)^\delta}\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\int_{3u_2}^{1}\frac{1}{u_1^2}\frac{1}{u_2^\delta}du_1du_2\\ &= \frac{1}{(n+1)^\delta}\frac{3^\delta}{\delta}\left((n+1)^\delta-1\right)\lesssim 1,\end{aligned}$$

and for $j=2,3$, (3.18) gives

$$\begin{aligned}\frac{1}{A_n^\delta}\int_{\Gamma_3}\left|\sum_{k=0}^{n} A_{n-k}^{\delta-1}H_{k,j}^*(u_1,u_2)\right|du_1du_2 &= \int_{\Gamma_3}\left|\frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}H_{k,j}^*(u_1,u_2)\right|du_1du_2\\ &\lesssim \frac{1}{(n+1)^\delta}\int_{\frac{1}{n+1}}^{1}\int_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}}\frac{1}{u_1^{\delta+1}}\frac{1}{u_2}du_2du_1\\ &= \frac{1}{A_n^\delta}\int_{\frac{1}{n+1}}^{1}\log\left((n+1)u_1\right)\frac{1}{u_1^{\delta+1}}du_1\\ &\le \frac{\log(n+1)}{(n+1)^\delta}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1^{\delta+1}}du_1\\ &= \frac{\log(n+1)}{(n+1)^\delta}\frac{1}{\delta}\left((n+1)^\delta-1\right)\lesssim \log(n+1).\end{aligned}$$

Thus we have

$$I_{n,3}^{(\delta)} \lesssim \log(n+1),$$

and hence (3.6) follows. □

**Theorem 3.3** *The estimate*

$$\int_{\Omega} \|\boldsymbol{t}\| \left| K_n^{(\delta)}(\boldsymbol{t}) \right| d\boldsymbol{t} \lesssim \frac{\log(n+1)}{(n+1)^{\delta}} \tag{3.19}$$

*holds for* $0 < \delta < 1$.

**Proof** As in proof of Theorem 3.2, it is sufficient to estimate the integral on the triangle $\Delta$. By transforms (3.7) and (3.8) estimating the integral

$$\int_{\Delta} \|\mathbf{t}\| \left| K_n^{(\delta)}(\mathbf{t}) \right| d\mathbf{t}$$

is equivalent to estimating the integral

$$\int_{\Gamma} u_1 \left| \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_k^{*}(u_1, u_2) \right| du_1 du_2.$$

We can write

$$\int_{\Gamma} u_1 \left| \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_k^{*}(u_1, u_2) \right| du_1 du_2 = J_{n,1}^{(\delta)} + J_{n,2}^{(\delta)} + J_{n,3}^{8\delta)},$$

where

$$J_{n,j}^{(\delta)} := \int_{\Gamma_j} u_1 \left| \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_k^{*}(u_1, u_2) \right| du_1 du_2, \quad j = 1, 2, 3.$$

Considering (3.11) yields

$$\begin{aligned} J_{n,1}^{(\delta)} &= \frac{1}{A_n^{\delta}} \int_{\Gamma_1} u_1 \left| \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_k^{*}(u_1, u_2) \right| du_1 du_2 \\ &\lesssim \int_{\Gamma_1} u_1 (n+1)^2 du_1 du_2 = (n+1)^2 \int_0^{\frac{1}{3(n+1)}} \int_{3u_2}^{\frac{1}{n+1}} u_1 du_1 du_2 \\ &= (n+1)^2 \int_0^{\frac{1}{3(n+1)}} \left( \frac{1}{(n+1)^2} - (3u_2)^2 \right) du_2 \le \frac{1}{n+1}. \end{aligned}$$

By (3.15),

$$\begin{aligned} \int_{\Gamma_2'} u_1 \left| \frac{1}{A_n^{\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_{k,1}^{*}(u_1, u_2) \right| du_1 du_2 &= \frac{1}{A_n^{\delta}} \int_{\Gamma_2'} u_1 \left| \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_{k,1}^{*}(u_1, u_2) \right| du_1 du_2 \\ &\lesssim \int_{\Gamma_2'} u_1 \frac{1}{u_1^2} du_1 du_2 \\ &= \int_0^{\frac{1}{3(n+1)^2}} \int_{\frac{1}{n+1}}^{1} \frac{1}{u_1} du_1 du_2 \\ &= \log(n+1) \int_0^{\frac{1}{3(n+1)^2}} du_2 \\ &\le \frac{\log(n+1)}{(n+1)^2}, \end{aligned}$$

and for $j = 2, 3$,

$$\begin{aligned}\int_{\Gamma_2'} u_1 \left| \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_{k,j}^*(u_1, u_2) \right| du_1 du_2 &= \frac{1}{A_n^\delta} \int_{\Gamma_2'} u_1 \left| \sum_{k=0}^{n} A_{n-k}^{\delta-1} D_{k,j}^*(u_1, u_2) \right| du_1 du_2 \\ &\lesssim n \int_{\Gamma_2'} u_1 \frac{1}{u_1} du_1 du_2 \\ &= n \int_0^{\frac{1}{3(n+1)^2}} \int_{\frac{1}{n+1}}^{1} du_1 du_2 \leq \frac{1}{n+1}.\end{aligned}$$

By (3.16),

$$\begin{aligned}\int_{\Gamma_2''} u_1 \left| \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^{\delta-1} H_{k,1}^*(u_1, u_2) \right| du_1 du_2 &\lesssim \frac{1}{n+1} \int_{\Gamma_2''} u_1 \frac{1}{u_1^2 u_2} du_1 du_2 \\ &= \frac{1}{n+1} \int_{\frac{1}{3(n+1)^2}}^{\frac{1}{3(n+1)}} \int_{\frac{1}{n+1}}^{1} \frac{1}{u_1 u_2} du_1 du_2 \\ &= \frac{(\log(n+1))^2}{n+1},\end{aligned}$$

and by (3.18),

$$\begin{aligned}\int_{\Gamma_2''} u_1 \left| \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^{\delta-1} H_{k,j}^*(u_1, u_2) \right| du_1 du_2 &\lesssim \frac{1}{(n+1)^\delta} \int_{\Gamma_2''} u_1 \frac{1}{u_1^{\delta+1} u_2} du_1 du_2 \\ &= \frac{1}{(n+1)^\delta} \int_{\frac{1}{3(n+1)^2}}^{\frac{1}{3(n+1)}} \int_{\frac{1}{n+1}}^{1} \frac{1}{u_1^\delta u_2} du_1 du_2 \\ &= \frac{\log(n+1)}{(n+1)^\delta} \int_{\frac{1}{n+1}}^{1} \frac{1}{u_1^\delta} du_1 \\ &= \frac{\log(n+1)}{(n+1)^\delta} \frac{1}{1-\delta} \left(1 - \frac{1}{(n+1)^{1-\delta}}\right) \\ &\lesssim \frac{\log(n+1)}{(n+1)^\delta}.\end{aligned}$$

Since

$$\frac{(\log(n+1))^2}{n+1} \lesssim \frac{\log(n+1)}{(n+1)^\delta}$$

we get

$$J_{n,2}^{(\delta)} \lesssim \frac{\log(n+1)}{(n+1)^\delta}.$$

By (3.17),

$$\begin{aligned}\int_{\Gamma_3} u_1 \left|\frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1} H_{k,1}^{*}(u_1,u_2)\right| du_1du_2 &\lesssim \frac{1}{(n+1)^\delta}\int_{\Gamma_3} u_1\frac{1}{u_1^2u_2^\delta}du_1du_2\\ &= \frac{1}{(n+1)^\delta}\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\int_{3u_2}^{1}\frac{1}{u_1u_2^\delta}du_1du_2\\ &= \frac{1}{(n+1)^\delta}\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\log\left(\frac{1}{3u_2}\right)\frac{1}{u_2^\delta}du_2\\ &\le \frac{\log(n+1)}{(n+1)^\delta}\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\frac{1}{u_2^\delta}du_2\\ &\lesssim \frac{\log(n+1)}{(n+1)^\delta},\end{aligned}$$

and by (3.18)

$$\begin{aligned}\int_{\Gamma_3} u_1 \left|\frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1} H_{k,j}^{*}(u_1,u_2)\right| du_1du_2 &\lesssim \frac{1}{(n+1)^\delta}\int_{\Gamma_3} u_1\frac{1}{u_1^{\delta+1}u_2}du_1du_2\\ &= \frac{1}{(n+1)^\delta}\int_{\frac{1}{n+1}}^{1}\int_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}}\frac{1}{u_1^\delta u_2}du_2du_1\\ &= \frac{1}{(n+1)^\delta}\int_{\frac{1}{n+1}}^{1}\log\left((n+1)u_1\right)\frac{1}{u_1^\delta}du_1\\ &\le \frac{\log(n+1)}{(n+1)^\delta}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1^\delta}du_1\\ &\lesssim \frac{\log(n+1)}{(n+1)^\delta}.\end{aligned}$$

Hence we get

$$J_{n,3}^{(\delta)} \lesssim \frac{\log(n+1)}{(n+1)^\delta},$$

and so (3.19) holds. □

In [15], Y. Xu proved that if $f \in C(\overline{\Omega})$ the $(C,1)$ means $S_n^{(1)}(f)$ converges uniformly to $f$ on $\overline{\Omega}$. The degree of approximation of $(C,1)$ means of hexagonal Fourier series of functions belong to $C(\overline{\Omega})$ was studied in [3–5].

The following theorem gives the main estimate for the degree of approximation of $(C,\delta)$ means of hexagonal Fourier series.

**Theorem 3.4** *For $f \in C(\overline{\Omega})$ the estimate*

$$\left\|f - S_n^{(\delta)}(f)\right\|_{C(\overline{\Omega})} \lesssim \begin{cases}\log(n+1)\omega_f\left(\frac{\log(n+1)}{(n+1)^\delta}\right), & 0<\delta<1\\ \omega_f\left(\frac{(\log(n+1))^2}{n+1}\right), & \delta\ge 1\end{cases} \tag{3.20}$$

*holds.*

**Proof** Let $f \in C(\overline{\Omega})$. By (3.1) and (3.3) we have

$$f(\mathbf{t}) - S_n^{(\delta)}(f)(\mathbf{t}) = \frac{1}{|\Omega|}\int_\Omega \left(f(\mathbf{t}) - f(\mathbf{t}-\mathbf{s})\right) K_n^{(\delta)}(\mathbf{s})d\mathbf{s}.$$

If we set

$$d_n^{(\delta)} := \frac{1}{|\Omega|}\int_\Omega \|\mathbf{t}\| \left|K_n^{(\delta)}(\mathbf{t})\right| d\mathbf{t},$$

we obtain

$$\begin{aligned}
\left|f(\mathbf{t}) - S_n^{(\delta)}(f)(\mathbf{t})\right| &\le \frac{1}{|\Omega|}\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{s})| \left|K_n^{(\delta)}(\mathbf{s})\right| d\mathbf{s} \\
&\le \frac{1}{|\Omega|}\int_\Omega \omega_f\left(\|\mathbf{s}\|\right) \left|K_n^{(\delta)}(\mathbf{s})\right| d\mathbf{s} \\
&= \frac{1}{|\Omega|}\int_\Omega \omega_f\left(\frac{\|\mathbf{s}\|}{d_n^{(\delta)}} d_n^{(\delta)}\right) \left|K_n^{(\delta)}(\mathbf{s})\right| d\mathbf{s} \\
&\le \omega_f\left(d_n^{(\delta)}\right) \frac{1}{|\Omega|}\int_\Omega \left(1 + \frac{\|\mathbf{s}\|}{d_n^{(\delta)}}\right) \left|K_n^{(\delta)}(\mathbf{s})\right| d\mathbf{s} \\
&= \omega_f\left(d_n^{(\delta)}\right) \left(\frac{1}{|\Omega|}\int_\Omega \left|K_n^{(\delta)}(\mathbf{s})\right| d\mathbf{s} + \frac{1}{d_n^\delta}\frac{1}{|\Omega|}\int_\Omega \|\mathbf{s}\| \left|K_n^{(\delta)}(\mathbf{s})\right| d\mathbf{s}\right) \\
&= \omega_f\left(d_n^{(\delta)}\right) \left(1 + \frac{1}{|\Omega|}\int_\Omega \left|K_n^{(\delta)}(\mathbf{s})\right| d\mathbf{s}\right).
\end{aligned}$$

Hence, for $0 < \delta < 1$, (3.20) follows from (3.6) and (3.19). It follows from [15] that

$$\int_\Omega \left|K_n^{(1)}(\mathbf{t})\right| d\mathbf{t} \lesssim 1, \tag{3.21}$$

and from [4]

$$\int_\Omega \|\mathbf{t}\| \left|K_n^{(1)}(\mathbf{t})\right| d\mathbf{t} \lesssim \frac{(\log(n+1))^2}{n+1}.$$

Thus we get (3.20) for $\delta = 1$. For $\delta > 1$, we use the equality (see [14])

$$S_n^{(\delta)}(f)(\mathbf{t}) = \frac{1}{A_n^\delta}\sum_{k=0}^n A_{n-k}^{\delta-1} S_k(f)(\mathbf{t}) = \frac{1}{A_n^\delta}\sum_{k=0}^n A_{n-k}^{\delta-2} A_k^1 S_k^{(1)}(f)(\mathbf{t}).$$

Thus,

$$f(\mathbf{t}) - S_n^{(\delta)}(f)(\mathbf{t}) = \frac{1}{A_n^\delta}\sum_{k=0}^n A_{n-k}^{\delta-2} A_k^1 \left(f(\mathbf{t}) - S_k^{(1)}(f)(\mathbf{t})\right).$$

Hence, considering (3.20),

$$\begin{aligned}\left|f(\mathbf{t})-S_n^{(\delta)}(f)(\mathbf{t})\right| &\leq \frac{1}{A_n^{\delta}}\sum_{k=0}^{n}A_{n-k}^{\delta-2}A_k^{1}\left|f(\mathbf{t})-S_k^{(1)}(f)(\mathbf{t})\right|\\
&\lesssim \frac{1}{A_n^{\delta}}\sum_{k=0}^{n}A_{n-k}^{\delta-2}A_k^{1}\omega_f\left(\frac{(\log(k+1))^2}{k+1}\right)\\
&\lesssim \frac{1}{(n+1)^{\delta}}\sum_{k=0}^{n}(n-k+1)^{\delta-2}(k+1)\omega_f\left(\frac{(\log(k+1))^2}{k+1}\right)\\
&= \frac{1}{(n+1)^{\delta}}\sum_{k=0}^{n}(n-k+1)^{\delta-2}(k+1)\\
&\times \omega_f\left(\frac{(\log(n+1))^2}{n+1}\frac{n+1}{(\log(n+1))^2}\frac{(\log(k+1))^2}{k+1}\right)\\
&\leq \frac{1}{(n+1)^{\delta}}\omega_f\left(\frac{(\log(n+1))^2}{n+1}\right)\sum_{k=0}^{n}(n-k+1)^{\delta-2}(k+1)\\
&\times\left(1+\frac{n+1}{(\log(n+1))^2}\frac{(\log(k+1))^2}{k+1}\right)\\
&\leq \frac{1}{(n+1)^{\delta}}\omega_f\left(\frac{(\log(n+1))^2}{n+1}\right)\sum_{k=0}^{n}(n-k+1)^{\delta-2}(k+1)\left(1+\frac{n+1}{k+1}\right)\\
&\lesssim \frac{1}{(n+1)^{\delta-1}}\omega_f\left(\frac{(\log(n+1))^2}{n+1}\right)\sum_{k=0}^{n}(n-k+1)^{\delta-2}\\
&\leq \omega_f\left(\frac{(\log(n+1))^2}{n+1}\right).\end{aligned}$$

□

The analogue of (3.20) for classical Fourier series was obtained in [13].

## 4. Approximation by Abel-Poisson means on hexagonal domain

Abel-Poisson means of series (2.2) are defined by

$$U_r(f)(\mathbf{t}):=\sum_{k=0}^{\infty}\sum_{\mathbf{j}\in\mathbb{J}_k}r^k\widehat{f}_{\mathbf{j}}\varphi_{\mathbf{j}}(\mathbf{t})=\frac{1}{|\Omega|}\int_{\Omega}f(\mathbf{t}-\mathbf{s})P_r(\mathbf{s})d\mathbf{s},\quad 0\leq r<1,$$

where

$$\mathbb{J}_k:=\mathbb{H}_k\setminus\mathbb{H}_{k-1}$$

and

$$P_r(\mathbf{t}):=\sum_{k=0}^{\infty}\sum_{\mathbf{j}\in\mathbb{J}_k}r^k\varphi_{\mathbf{j}}(\mathbf{t})$$

is the Poisson kernel. The Poisson kernel is nonnegative, satisfies

$$\frac{1}{|\Omega|}\int_{\Omega} P_r(\mathbf{t})d\mathbf{t} = 1, \tag{4.1}$$

and has the compact formula

$$\begin{aligned} P_r(\mathbf{t}) &= \frac{(1-r)^3(1-r^3)}{q_r\left(\frac{2\pi(t_1-t_2)}{3}\right) q_r\left(\frac{2\pi(t_2-t_3)}{3}\right) q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)} \\ &+ \frac{r(1-r)^2}{q_r\left(\frac{2\pi(t_1-t_2)}{3}\right) q_r\left(\frac{2\pi(t_2-t_3)}{3}\right)} \\ &+ \frac{r(1-r)^2}{q_r\left(\frac{2\pi(t_2-t_3)}{3}\right) q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)} \\ &+ \frac{r(1-r)^2}{q_r\left(\frac{2\pi(t_3-t_1)}{3}\right) q_r\left(\frac{2\pi(t_1-t_2)}{3}\right)} \end{aligned}$$

for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_H$, where $q_r(x) = 1 - 2r\cos x + r^2$ ([15]). Also, it is easy to see that

$$\begin{aligned} P_r(\mathbf{t}) \le{}& \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_1-t_2)}{3}\right) q_r\left(\frac{2\pi(t_2-t_3)}{3}\right)} + \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_2-t_3)}{3}\right) q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)} \\ &+ \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_3-t_1)}{3}\right) q_r\left(\frac{2\pi(t_1-t_2)}{3}\right)} \end{aligned} \tag{4.2}$$

and

$$\frac{(1-r)^2}{q_r(x)q_r(y)} = \frac{1}{(1+r)^2} p_r(x)p_r(y), \tag{4.3}$$

where

$$p_r(x) = \frac{1-r^2}{q_r(x)}$$

is the classical Poisson kernel. The classical Poisson kernel satisfies the inequalities ([17, pp. 96-97])

$$p_r(x) \le \frac{2}{1-r}, \quad 0 \le x \le \pi \tag{4.4}$$

and

$$p_r(x) \lesssim \frac{1-r}{x^2}, \quad 0 < x \le \pi. \tag{4.5}$$

We have the following estimate for Abel-Poisson means of hexagonal Fourier series of $H-$periodic continuous functions.

**Theorem 4.1** *For* $f \in C(\overline{\Omega})$ *the estimate*

$$\|f - U_r(f)\|_{C(\overline{\Omega})} \lesssim \omega_f\left((1-r)\left|\log(1-r)\right|\right) \tag{4.6}$$

*holds as* $r \to 1^-$.

**Proof** If we set

$$\lambda_n^{(r)} := \frac{1}{|\Omega|}\int_\Omega \|\mathbf{t}\|\, P_r(\mathbf{t})d\mathbf{t},$$

by (4.1)

$$\begin{aligned}
|f(\mathbf{t}) - U_r(f)(\mathbf{t})| &\le \frac{1}{|\Omega|}\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{s})|\, P_r(\mathbf{s})d\mathbf{s} \\
&\le \frac{1}{|\Omega|}\int_\Omega \omega_f\left(\|\mathbf{s}\|\right) P_r(\mathbf{s})d\mathbf{s} \\
&= \frac{1}{|\Omega|}\int_\Omega \omega_f\left(\frac{\|\mathbf{s}\|}{\lambda_n^{(r)}}\lambda_n^{(r)}\right) P_r(\mathbf{s})d\mathbf{s} \\
&\le \omega_f\left(\lambda_n^{(r)}\right)\frac{1}{|\Omega|}\int_\Omega \left(1+\frac{\|\mathbf{s}\|}{\lambda_n^{(r)}}\right) P_r(\mathbf{s})d\mathbf{s} \\
&= \omega_f\left(\lambda_n^{(r)}\right)\left(\frac{1}{|\Omega|}\int_\Omega P_r(\mathbf{s})d\mathbf{s} + \frac{1}{\lambda_n^{(r)}}\frac{1}{|\Omega|}\int_\Omega \|\mathbf{s}\|\, P_r(\mathbf{s})d\mathbf{s}\right) \\
&= 2\omega_f\left(\lambda_n^{(r)}\right).
\end{aligned}$$

Since the function $\omega_f(\cdot)$ is nondecreasing we have to estimate the quantity $\lambda_n^{(r)}$. If we set

$$\begin{aligned}
Q_r(\mathbf{t}) := {}& \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_1-t_2)}{3}\right) q_r\left(\frac{2\pi(t_2-t_3)}{3}\right)} + \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_2-t_3)}{3}\right) q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)} \\
&+ \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_3-t_1)}{3}\right) q_r\left(\frac{2\pi(t_1-t_2)}{3}\right)},
\end{aligned}$$

(4.2) implies that it will be sufficient to estimate the integral

$$\int_\Delta \|\mathbf{t}\|\, Q_r(\mathbf{t})d\mathbf{t}.$$

If we use the change of variables (3.7) and (3.8) estimating the integral

$$\int_\Gamma u_1 Q_r^*(u_1,u_2)du_1du_2,$$

where

$$\begin{aligned}
Q_r^*(u_1,u_2) := {}& p_r\left(\pi(u_1+u_2)\right) p_r\left(2\pi u_2\right) \\
&+ p_r\left(\pi(u_1-u_2)\right) p_r\left(\pi(u_1+u_2)\right) \\
&+ p_r\left(\pi(u_1-u_2)\right) p_r\left(2\pi u_2\right)
\end{aligned}$$

will be sufficient. We write the triangle $\Gamma$ as $\Gamma = \Gamma_1^* \cup \Gamma_2^* \cup \Gamma_3^*$, where

$$\Gamma_1^* := \{(u_1, u_2) \in \Gamma : u_1 \le 1 - r\}$$

$$\Gamma_2^* := \{(u_1, u_2) \in \Gamma : u_1 \ge 1 - r, u_2 \le \frac{1-r}{3}\}$$

$$\Gamma_3^* := \{(u_1, u_2) \in \Gamma : u_1 \ge 1 - r, u_2 \ge \frac{1-r}{3}\},$$

and hence

$$\int_\Gamma u_1 Q_r^*(u_1, u_2) du_1 du_2 = \sum_{k=1}^{3} \left( \int_{\Gamma_k^*} u_1 Q_r^*(u_1, u_2) du_1 du_2 \right).$$

By (4.4),

$$\begin{aligned}\int_{\Gamma_1^*} u_1 Q_r^*(u_1, u_2) du_1 du_2 &= \int_0^{\frac{1-r}{3}} \left( \int_{3u_2}^{1-r} u_1 Q_r^*(u_1, u_2) du_1 \right) du_2 \\ &\lesssim \frac{1}{(1-r)^2} \int_0^{\frac{1-r}{3}} \left( \int_{3u_2}^{1-r} u_1 du_1 \right) du_2 \\ &\lesssim 1 - r.\end{aligned}$$

If we consider (4.4) and (4.5) and taking into account the inequality $u_1 - u_2 \ge \frac{2}{3} u_1$,

$$\begin{aligned}\int_{\Gamma_2^*} u_1 Q_r^*(u_1, u_2) du_1 du_2 &= \int_0^{\frac{1-r}{3}} \left( \int_{1-r}^{1} u_1 Q_r^*(u_1, u_2) du_1 \right) du_2 \\ &\lesssim \int_0^{\frac{1-r}{3}} \left( \int_{1-r}^{1} u_1 \left( \frac{1}{\pi^2 (u_1 + u_2)^2} + \frac{1}{\pi^2 (u_1 - u_2)^2} \right) du_1 \right) du_2 \\ &\le \int_0^{\frac{1-r}{3}} \left( \int_{1-r}^{1} \frac{1}{u_1} du_1 \right) du_2 = \frac{1-r}{3} \left| \log(1-r) \right| \\ &\le (1-r) \left| \log(1-r) \right|.\end{aligned}$$

Now considering (4.5) again and using $u_1 - u_2 \ge \frac{2}{3} u_1$, we get

$$\begin{aligned}\int_{\Gamma_3^*} u_1 Q_r^*(u_1, u_2) du_1 du_2 &= \int_{\frac{1-r}{3}}^{\frac{1}{3}} \left( \int_{3u_2}^{1} u_1 Q_r^*(u_1, u_2) du_1 \right) du_2 \\ &\lesssim (1-r)^2 \int_{\frac{1-r}{3}}^{\frac{1}{3}} \left( \int_{3u_2}^{1} \frac{1}{u_1 u_2^2} du_1 \right) du_2 \\ &\lesssim (1-r) \left| \log(1-r) \right|.\end{aligned}$$

□

The analogue of Theorem 4.1 was proved in [12] for Abel-Poisson means of classical Fourier series.

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