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Research Article

Linear Diophantine Fuzzy Weighted Hamy Mean Operator with Application to Multi-Attribute Decision-Making Problems

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Abstract: Existing multi-attribute decision-making methods often fail to adequately capture complex uncertainty and attribute interrelationships in real-world problems, potentially leading to suboptimal outcomes. To overcome this, we develop a robust linear Diophantine fuzzy sets based framework and introduce four novel Hamy mean operators such as linear Diophantine fuzzy Hamy mean, linear Diophantine fuzzy weighted Hamy mean, linear Diophantine fuzzy dual Hamy mean, and linear Diophantine fuzzy weighted dual Hamy mean operators, which empower decision-makers through reference parameters and effectively model interdependent attributes. The practical impact of our approach is demonstrated through a critical emergency shelter material selection case study, where the proposed operators successfully identified "3D Printed Recycled Materials" as the optimal choice, validating its decision alignment with expert judgment. A detailed comparative analysis confirms the superiority of our model, showing a higher ranking consistency and significantly better performance in handling high hesitancy and conflicting criteria compared to existing methods. This research provides a more reliable and robust decision-making tool for practical applications in emergency management, supply chain logistics, and strategic planning, where managing complex uncertainty is crucial.

***Keywords*:** Linear Diophantine Fuzzy Set (LDFS), Hamy mean operator, multi-attribute group decision making, materials for building emergency shelters

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Abbreviation

DM	Decision Making
MADM	Multi-Attribute Decision-Making
LDFHMO	Linear Diophantine Fuzzy Hamy Mean Operator
LDFWHMO	Linear Diophantine Fuzzy Weighted Hamy Mean Operator

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LDFDHMO	Linear Diophantine Fuzzy Dual Hamy Mean Operator
LDFWDHMO	Linear Diophantine Fuzzy Weighted Dual Hamy Mean Operator

1. Introduction

Nowadays, decision making is a common process in daily life, focused on choosing the optimal choice from among the available alternatives based on various known attributes. Multi-attribute decision making is a specialized branch of Decision Making (DM) that involves cognitive processes, enabling individuals to make informed decisions based on multiple attributes or criteria. The crucial step in handling and aggregating information from multiple sources is data collection. Traditionally, this information is presented as crisp numbers. However, when dealing with human cognition, it can be challenging to accurately represent situations using rudimentary data management techniques based on crisp numbers. Decision-Makers (DMk's) may encounter unclear conclusions and uncertain decisions when employing these methods. To address this issue, Zadeh proposed Fuzzy Sets (FSs) in 1965 as a way to generalize crisp sets [1]. In FSs, the degree to which certain criteria are met is expressed through membership functions, with their complements indicating insufficient degrees. Bellman and Zadeh investigated the Multi-Attribute Decision-Making (MADM) process within FS theory in 1970 [2]. In the late 1970s, Yager and Zimmermann advanced the field of DM problem-solving in fuzzy environments [3, 4].

The Membership Degree (MD) is not always correlated with the Non-Membership Degree (NMD) in real-world situations. In such instances, the concept of Intuitionistic FSs (IFSs), as defined by Atanassov [5], has proven highly effective. It represents a significant extension of FSs with broad applicability. IFSs have been utilized across numerous domains, such as optimization problems, medical diagnostics, and DM [6]. To integrate job information and choose the best job utilizing Intuitionistic Fuzzy (IF) data, Seikh and Mandal [7] created Dombi Aggregation Operators (AOs). Senapati et al. [8] presented IF Aczel-Alsina operators and used them to find transit-sharing strategies that are sustainable. In the IF context, Gohain et al. [9] introduced a symmetric distance measure that was used to solve clustering and pattern recognition issues. Ke et al. [10] created a rating system for IFSs and used it to choose locations for solar projects aimed at reducing poverty. Wan and Yi [11] introduced power average operators for trapezoidal IF numbers, which use rigorous t-norms and t-conorms.

There are numerous instances in which a DMk may supply MD and NMD for a certain attribute such that their sum exceeds one. To address this, Yager [12] presented the Pythagorean Fuzzy Set (PyFS), an extension of the IFS concept, which requires that the total of the squares of the MD and NMD does not exceed one. Farhadinia [13] proposed a PyF DM technique based on similarity measures. Garg [14] enhanced PyFSs by incorporating more comprehensive operating guidelines and related aggregation operators. PyF Dombi AOs were introduced and discussed in [15]. Garg [16] applied Einstein t-norm operational standards to PyF numbers. In [17], symmetric PyFS AOs were developed. Zeng [18] explored probability and ordered Weighted Averaging Operators (WAOs) within the PyFS system. Garg [19] proposed different decision-making strategies to handle Multi-Attribute Group Decision-Makin (MAGDM) issues under the PyFS system. For urban strategy, linguistic Pythagorean fuzzy Aczél-Alsina operators supported smart-city multi-attribute group decision-making, showcasing scalability to complex policy contexts [20]. Furthermore, Einstein ordered-weighted operators for Pythagorean fuzzy hypersoft sets broadened the hypersoft paradigm for Multi-Criteria Decision-Making (MCDM), integrating preference-driven weighting with expressive uncertainty modeling [21]. Deqing [22] suggested several distance measures that consider all four parameters of Pythagorean fuzzy sets and numbers. Firozja [23] introduced a novel similarity metric for PyFs utilizing triangle conorms.

Although IFS and PyFS have been widely explored, they have limitations in representing fuzzy information, particularly in scenarios characterized by extreme complexity and contradiction. IFS and PyFS fail to effectively represent decision-makers' assessments when the squared sum of MD and NMD exceeds one, yet the sum of their cubed or higher powers remains below one. To address these shortcomings and enhance the ability to represent fuzzy information, Yager [24] introduced the q-Rung Orthopair Fuzzy Set (q-ROFS). The defining feature of q-ROFS is that the sum of the q-th powers of MD and NMD is fixed at 1 for $q > 1$. When $q = 1$ and $q = 2$, q-ROFS respectively corresponds to IFS and PyFS. Building on this base, an Einstein-average aggregation framework for q-rung orthopair fuzzy soft sets showed how medical diagnostic judgments can be modeled while preserving the operator's desirable monotonicity and boundedness [25]. Interval-valued q-ROF information was then paired with Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) to rank unmanned aerial vehicles for precision agriculture, demonstrating the approach on a realistic multi-attribute alternative set [26]. Interactive aggregation operators extended the q-Rung Orthopair Fuzzy Set (ROFSS) toolkit to capture inter-criterion effects within a unified multicriteria decision process [27]. In the realm of information AOs, researchers have created various AOs within the q-ROF framework, including the family of q-ROF power AOs [28] and a series of q-ROF linguistic AOs [29]. Additionally, q-ROF AOs based on the Heronian mean operator [30] have been developed. Concerning decision-making methods, studies have integrated q-ROFS with distance measures [31] and the TOPSIS method [32] to formulate multi-attribute decision-making approaches. For public-health logistics, an Evaluation based on Distance from Average Solution (EDAS) pipeline under interval-valued q-ROF soft sets prioritized biomedical-waste disposal techniques, illustrating robustness in sensitive, high-stakes choices [33]. Also, Einstein geometric operators furnished alternative aggregation behavior for q-ROFSS-based MCDM, enabling sensitivity control via the operator family's parameterization [34]. A complementary group decision-making scheme used interval-valued probabilistic linguistic T-spherical fuzzy information to select a cloud storage provider, highlighting richer linguistic expressiveness for panel decisions [35]. A CRiteria Importance Through Intercriteria Correlation (CRITIC)-weighted EDAS model with linguistic T-spherical fuzzy Hamacher operators further refined objective weighting and aggregation interplay for group decisions [36].

1.1 *Literature review*

Several techniques are available for addressing difficult DM issues in fields such as medical, business, engineering, agriculture, artificial intelligence, and computational intelligence. It is essential to evaluate items using a variety of characteristics or standards in real-world DM situations. These methods fall under the categories of MADM. Researchers have concentrated on different frameworks, including FSs, IFSs, PyFSs, and q-ROFSs, to create novel DM techniques using ordered pairs of MD and NMD. These assessments are typically based on evaluations from DMk or experts. However, these assessments often fall short in several difficult real-life situations. To improve the evaluation of these grades in decision analysis, Riaz and Hashmi [37] presented the idea of Linear Diophantine Fuzzy Sets (LDFSs) in 2019, showing that LDFSs provide a more comprehensive framework than IFS, PyFS, and q-ROFSs. LDFSs provide a more informative and effective representation than these previous models. A key advantage of LDFSs is their Reference Parameters (RPs), which allow for more range in MD and NMD compared to IFS, PyFS, and q-ROFSs. In LDFSs, RPs are restricted and must sum to no more than 1. Many researchers have since explored LDFSs theory to enhance its applicability across various fields. For instance, Ayub et al. [38] developed algebraic characteristics and relationships for LDFSs. Iampan et al. [39] created Einstein's AOs for LDFSs and applied them to DM problems. Additionally, Riaz et al. [40] focused on LDFS prioritized AOs for solving DM challenges. Kamac [41] expanded on LDFSs by developing complex LDFSs, defining cosine similarity measures, and exploring their uses. Almaghrabi et al. [42] introduced a novel method to q-rung LDFSs, while Qiyas et al. [43] created various distance measures for q-rung LDF numbers.

To explore the relationships among different attributes, the Hamy Mean (HM) operator is a widely applicable, versatile, and powerful tool for managing complex and inconsistent information in real-life situations. This operator has been utilized across various fields by several researchers. Initially, Hara et al. [44] presented the HM operator ideas to analyze the relationships among different attributes using various parameters. Subsequently, Wu et al. [45] applied the Hamy Mean Operator (HMO) within the framework of Interval-Valued IFS (IVIFS), developing both HMO and weighted HMO for MAGDM applications. Wu et al. [46] explored service quality in tourism by applying Dombi t-norm-based HMO using a MADM technique within the framework of IVIFSs. Wang et al. [47] used the HMO over q-ROFS for selecting corporate resource management systems. Liu and Wang [48] worked on IFSs and developed several interactive HMO for IFSs. Liu and Liu [49] investigated linguistic IFSs and proposed HMO for MAGDM problems. Liu and You [50] studied linguistic HMO in neutrosophic contexts for MADM. The ideas of hesitant fuzzy linguistic power HM aggregation operators were introduced by Liu et al. [51], while Wei et al. [52] introduced dual hesitant Pythagorean fuzzy hamy mean operator. Li et al. [53] presented the IF Dombi HMO and applied it to a MAGDM problem to select the best vehicle for a transportation firm. Wu et al. [45] introduced HMO for IVIFSs, and Li et al. [53] introduced Dombi HMO for IFSs. Wu et al. [46]

further proposed Dombi HMO for IVIFSs, while Liang [54] also presented HMO for IFSs, and Li et al. [55] investigated HMO for Picture Fuzzy Sets (PFSs).

1.2 *Problem statement*

The integration of the Hamy mean operator with linear Diophantine fuzzy sets remains an unresolved challenge in multi-attribute decision-making. While the HM operator is powerful for capturing interrelationships between multiple input arguments, and LDFS provides a superior framework for modeling uncertainty with reference parameters, no existing methodology successfully combines their strengths. This creates a significant gap: there is no aggregation tool capable of simultaneously:

• Capturing the complex interdependencies among multiple criteria.

• Leveraging the expressive power of LDFS to handle high-dimensional uncertainty with controlled degrees of membership and non-membership.

1.3 *Motivational factors*

The absence of a dedicated HM operator for LDFS severely limits the effectiveness of decision-making in complex, real-world environments. The motivation for this research is therefore threefold:

1. To Overcome the Limitations of Isolated Tools: Using standard aggregation operators with LDFS data fails to capture the crucial interdependencies among criteria. For instance, in supplier selection, the criteria 'cost', 'quality', and 'delivery time' are often interrelated. A simple weighted average cannot model how an improvement in 'quality' might justifiably affect the evaluation of 'cost'. Our work is motivated by the need for an operator that respects these complex relationships within the rich LDFS environment.

2. To Fully Utilize the Expressive Power of LDFS: LDFSs key innovation is the use of reference parameters, which allow experts to precisely calibrate their membership and non-membership degrees. However, without specialized operators like the HM, this nuanced information is aggregated using generic methods, potentially diluting the accuracy and flexibility that LDFS offers. This research is motivated by the need to develop aggregation techniques that preserve and process this sophisticated information structure without loss of fidelity.

3. To Enhance Decision-Making Robustness in Uncertainty: Many existing fuzzy extensions (like intuitionistic or Pythagorean fuzzy sets) become less effective under high uncertainty or when expert judgments exhibit significant hesitancy. The configurable nature of LDFS, combined with the relationship-modeling capability of the HM operator, provides a more robust foundation for decisions where information is imperfect and criteria are not independent. Our motivation is to provide decision-makers with a more reliable and adaptable mathematical tool for such challenging scenarios.

1.4 *Aims and objectives*

The objectives of our work are:

1. For the aggregation of Linear Diophantine Fuzzy Sets (LDFNs) first to develop some novel operators, i.e. Linear Diophantine Fuzzy Hamy Mean (LDFHM) operator, Linear Diophantine Fuzzy Weighted Hamy Mean (LDFWHM) operator, Linear Diophantine Fuzzy Dual Hamy Mean (LDFDHM) operator, and Linear Diophantine Fuzzy Weighted Dual Hamy Mean (LDFWDHM) operator.

2. Analyze the new operator's characteristics and specific cases.

3. Create a new MAGDM strategy that includes the LDFHM, LDFWHM, LDFDHM, and LDFWDHM operators.

4. The usefulness of the created algorithm is shown with a numerical example.

5. Conduct a comparison analysis of the suggested method against many existing methodologies.

1.5 *Structure of the paper*

The article is organized as follows. Section 2 introduces foundational definitions and basic properties of LDFNs, as well as preliminaries on the HM and Dual Hamy Mean (DHM) operators. In section 3, we develop the HM and DHM operators for LDFNs, establish their properties, and provide illustrative examples. Section 4 describes the objective weighting scheme based on the deviation-maximization method. Section 5 presents the LDF-MAGDM framework. In section 6, we apply the proposed method to a practical problem and analyze its sensitivity to the parameter k. Section 7 compares our approach with prior work and explain the limitations and shortcomings of our work. In section 8 concludes the paper and also discuss the future work.

2. Preliminaries

In this section, we review fundamental concepts related to FSs, IFSs, LDFSs, Hamy mean operator and Dual Hamy mean operator. These foundational elements are employed in the development of a hybrid structure referred to as LDFSs. Throughout the paper, we consistently use $\mathfrak{A}$ as a fixed sample space or as the universe. Also μ and ν represent MD and NMD respectively, and α and β serving as RPs corresponding to μ and ν, respectively.

Definition 1 [1] Assume $\mathfrak{A}$ is not empty set. A FS $\mathfrak{F}$ with in $\mathfrak{A}$ is defined as:

$$\mathfrak{F} = \{\langle a,\ \mu_{\mathfrak{F}}(a)\rangle | a \in \mathfrak{A}\}, \tag{1}$$

where $\mu_{\mathfrak{F}}(a) \in [0,\ 1]$ denotes the MD of $\mathfrak{F}$.

Attanassov [5] suggested the idea of membership grades and non-membership grades with the condition that the sum of membership grade and non-membership grade cannot be larger than 1.

Definition 2 [5] The IFSs L is defined for every element in a universal set $\mathfrak{A} \neq \emptyset$ as follows:

$$L = \{(a,\ \mu_L(a),\ \nu_L(a)) | a \in \mathfrak{A}\}, \tag{2}$$

where $\mu_L(a) : \mathfrak{A} \to [0,\ 1]$ and $\nu_L(a) : \mathfrak{A} \to [0,\ 1]$ are functions representing the MD and NMD respectively, for every element $a \in \mathfrak{A}$. It holds that $0 \leq \mu_L(a) + \nu_L(a) \leq 1$ for all $a \in \mathfrak{A}$. The hesitation or indeterminacy component is as follows: $\phi(a) = 1 - (\mu_L(a) + \nu_L(a))$.

2.1 *Linear diophantine fuzzy sets*

In this part, we present a new idea known as LDFSs [37]. In number theory, the suggested model is shares similarities with the well-known Linear Diophantine equation $cx + dy = e$. Given the limitations of existing models like IFS, PFSs, and q-ROFSs regarding membership/non-membership grades, we address this issue by introducing LDFSs with the incorporation of RPs. This addition removes constraints on MD and NMD grades, allowing the DMk the freedom to select grades without limitations. The framework of LDFS further organizes problems by employing various types of RPs.

Definition 3 [37] Let $\mathfrak{A} \neq \emptyset$ universal set. A LDFS $\mathfrak{L}$ In $\mathfrak{A}$ can be characterized by

$$\mathfrak{L} = \{(a,\ (\mu_{\mathfrak{L}}(a),\ \nu_{\mathfrak{L}}(a)),\ (\alpha_{\mathfrak{L}}(a),\ \beta_{\mathfrak{L}}(a)) : a \in \mathfrak{A})\}, \tag{3}$$

where $\mu_{\mathfrak{L}}(a),\ \nu_{\mathfrak{L}}(a),\ \alpha_{\mathfrak{L}}(a),\ \beta_{\mathfrak{L}}(a) \in [0,\ 1]$ are MD, NMD and RPs respectively. These degrees satisfy the condition $0 \leq (\alpha_{\mathfrak{L}}(a))\mu_{\mathfrak{L}}(a) + \beta_{\mathfrak{L}}(a)\nu_{\mathfrak{L}}(a) \leq 1\ \forall a \in \mathfrak{A}$ with $0 \leq \alpha_{\mathfrak{L}}(a) + \beta_{\mathfrak{L}}(a) \leq 1$.

Definition 4 [37] Let $\vartheta_\mathfrak{c} = (\langle \mu_\mathfrak{c},\ \nu_\mathfrak{c} \rangle,\ \langle \alpha_\mathfrak{c},\ \beta_\mathfrak{c} \rangle)$ be two LDFNs with $\mathfrak{c}$ = 1, 2, and $\kappa > 0$. Then, the following conditions hold:

1. $\vartheta_1^c = (\langle \nu_1,\ \mu_1 \rangle,\ \langle \beta_1,\ \alpha_1 \rangle)$
2. $\vartheta_1 = \vartheta_2 \iff \mu_1 = \mu_2,\ \nu_1 = \nu_2,\ \alpha_1 = \alpha_2,\ \beta_1 = \beta_2$
3. $\vartheta_1 \subseteq \vartheta_2 \iff \mu_1 \leq \mu_2,\ \nu_1 \geq \nu_2,\ \alpha_1 \leq \alpha_2,\ \beta_1 \geq \beta_2$
4. $\vartheta_1 \cup \vartheta_2 = (\langle \sup\{\mu_1,\ \mu_2\},\ \inf\{\nu_1,\ \nu_2\} \rangle,\ \langle \sup\{\alpha_1,\ \alpha_2\},\ \inf\{\beta_1,\ \beta_2\} \rangle)$
5. $\vartheta_1 \cap \vartheta_2 = (\langle \inf\{\mu_1,\ \mu_2\},\ \sup\{\nu_1,\ \nu_2\} \rangle,\ \langle \inf\{\alpha_1,\ \alpha_2\},\ \sup\{\beta_1,\ \beta_2\} \rangle)$
6. $\vartheta_1 \oplus \vartheta_2 = (\langle \mu_1 + \mu_2 - \mu_1\mu_2,\ \nu_1\nu_2 \rangle,\ \langle \alpha_1 + \alpha_2 - \alpha_1\alpha_2,\ \beta_1\beta_2 \rangle)$
7. $\vartheta_1 \otimes \vartheta_2 = (\langle \mu_1\mu_2,\ \nu_1 + \nu_2 - \nu_1\nu_2 \rangle,\ \langle \alpha_1\alpha_2,\ \beta_1 + \beta_2 - \beta_1\beta_2 \rangle)$
8. $\kappa\vartheta_1 = (\langle 1 - (1 - \mu_1)^\kappa,\ \nu_1^\kappa \rangle,\ \langle 1 - (1 - \alpha_1)^\kappa,\ \beta_1^\kappa \rangle)$
9. $\vartheta^\kappa = (\langle \mu_1)^\kappa,\ 1 - (1 - \nu_1)^\kappa \rangle,\ \langle \alpha_1^\kappa,\ 1 - (1 - \beta_1)^\kappa \rangle)$.

Definition 5 [37] Let $\vartheta_\mathfrak{c} = (\langle \mu_\mathfrak{c},\ \nu_\mathfrak{c} \rangle,\ \langle \alpha_\mathfrak{c}.\beta_\mathfrak{c} \rangle)$ represents an LDFN. The Score Function (SF) on $\vartheta_\mathfrak{c}$ can be described by the mapping $P : \text{LDFN}(\mathfrak{A}) \to [-1,\ 1]$, defined as follows:

$$P_{\vartheta_\mathfrak{c}} = P(\vartheta_\mathfrak{c}) = \frac{1}{2}\left[(\mu_\mathfrak{c} - \nu_\mathfrak{c}) + (\alpha_\mathfrak{c} - \beta_\mathfrak{c})\right] \tag{4}$$

where LDFN($\mathfrak{A}$) denotes a collection of LDFNs on $\mathfrak{A}$.

Definition 6 [37] Let $\vartheta_\mathfrak{c} = (\langle \mu_\mathfrak{c},\ \nu_\mathfrak{c} \rangle,\ \langle \alpha_\mathfrak{c}.\beta_\mathfrak{c} \rangle)$ represents an LDFN. The mapping $\psi : \text{LDFN}(\mathfrak{A}) \to [0,\ 1]$ represents the Accuracy Function (AF) and is defined as:

$$\psi_{\vartheta_\mathfrak{c}} = \psi(\vartheta_\mathfrak{c}) = \frac{1}{2}\left[\frac{\mu_\mathfrak{c} + \nu_\mathfrak{c}}{2} + (\alpha_\mathfrak{c} + \beta_\mathfrak{c})\right] \tag{5}$$

where LDFN($\mathfrak{A}$) denotes a collection of LDFNs on $\mathfrak{A}$.

Definition 7 [37] Let ϑ_1 and ϑ_2 be two LDFNs. By utilizing the SF and AF, a straightforward comparison between these two LDFNs can be made as follows:

1. If $P_{\vartheta_1} < P_{\vartheta_2}$, then $\vartheta_1 < \vartheta_2$,
2. If $P_{\vartheta_1} > P_{\vartheta_2}$, then $\vartheta_1 > \vartheta_2$,
3. If $P_{\vartheta_1} = P_{\vartheta_2}$, then,

- If $\psi_{\vartheta_1} < \psi_{\vartheta_2}$, then $\vartheta_1 < \vartheta_2$,
- If $\psi_{\vartheta_1} > \psi_{\vartheta_2}$, then $\vartheta_1 > \vartheta_2$,
- If $\psi_{\vartheta_1} = \psi_{\vartheta_2}$, then $\vartheta_1 \approx \vartheta_2$.

Definition 8 [37] The quadratic SF for an LDFN is a mathematical function $J : \text{LDFN}(\mathfrak{A}) \to [-1,\ 1]$, and it is described as

$$J_{\vartheta_\mathfrak{c}} = J(\vartheta_\mathfrak{c}) = \frac{1}{2}\left[(\mu_\mathfrak{c}^2 - \nu_\mathfrak{c}^2) + (\alpha_\mathfrak{c}^2 - \beta_\mathfrak{c}^2)\right]. \tag{6}$$

Definition 9 [37] The quadratic AF for an LDFN $\vartheta_\mathfrak{c}$ is a mathematical function $\phi : \text{LDFN}(\mathfrak{A}) \to [0,\ 1]$, and it is defined as

$$\phi_{\vartheta_\mathfrak{c}} = \phi(\vartheta_\mathfrak{c}) = \frac{1}{2}\left(\frac{\mu_\mathfrak{c}^2 + \nu_\mathfrak{c}^2}{2} + (\alpha_\mathfrak{c}^2 + \beta_\mathfrak{c}^2)\right). \tag{7}$$

Definition 10 [37] Let ϑ_1 and ϑ_2 be two LDFNs. By utilizing the Quadratic Score Function (QSF) and Quadratic Accuracy Function (QAF), a straightforward comparison between these two LDFNs can be made as follows:

1. If $J_{\vartheta_1} < J_{\vartheta_2}$, then $\vartheta_1 < \vartheta_2$,
2. If $J_{\vartheta_1} > J_{\vartheta_2}$, then $\vartheta_1 > \vartheta_2$,
3. If $J_{\vartheta_1} = J_{\vartheta_2}$, then:

• If $\phi_{\vartheta_1} < \phi_{\vartheta_2}$, then $\vartheta_1 < \vartheta_2$,
• If $\phi_{\vartheta_1} > \phi_{\vartheta_2}$, then $\vartheta_1 > \vartheta_2$,
• If $\phi_{\vartheta_1} = \phi_{\vartheta_2}$, then $\vartheta_1 \approx \vartheta_2$.

Definition 11 [37] Another formulation of the score function, known as the Expectation Score Function (ESF) on LDFN($\mathfrak{A}$), is defined by the function M : LDFN($\mathfrak{A}$) $\rightarrow [0,\ 1]$, expressed as

$$M_{\vartheta_\mathfrak{c}} = M(\vartheta_\mathfrak{c}) = \frac{1}{2}\left(\frac{(\mu_\mathfrak{c} - \nu_\mathfrak{c} + 1)}{2} + \frac{(\alpha_\mathfrak{c} - \beta_\mathfrak{c} + 1)}{2}\right), \tag{8}$$

the ESF is a modified version of the score function, with values bounded in the interval $[0,\ 1]$ instead of $[-1,\ 1]$.

Definition 12 [56] Let $\vartheta_1 = (\langle\mu_1,\ \nu_1\rangle,\ \langle\alpha_1,\ \beta_1\rangle)$ and $\vartheta_2 = (\langle\mu_2,\ \nu_2\rangle,\ \langle\alpha_2,\ \beta_2\rangle)$ be two LDFNs then the Hamming distance denoted by $D(\vartheta_1,\ \vartheta_2)$ is defined as

$$D(\vartheta_1,\ \vartheta_2) = \frac{1}{4}(|\mu_1 - \mu_2| + |\nu_1 - \nu_2| + |\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|). \tag{9}$$

2.2 *Hamy mean operator*

Following the foundational aggregation operators, this section delves into the Hamy Mean operator [44] and its adaptation to the linear Diophantine fuzzy environment. The HM operator is a powerful aggregation tool known for its ability to capture the interrelationships between multiple input arguments, a feature that simpler operators like the weighted average or geometric mean lack.

Definition 13 [44] Consider the collection F_ι $(\iota = 1,\ 2,\ 3,\ \ldots,\ \mathfrak{J})$ of non-negative real numbers and $k = 1,\ 2,\ \ldots,\ \mathfrak{J}$. $\text{HM}^{(k)}(F_1,\ F_2,\ \ldots,\ F_\mathfrak{J})$ is the representation of the Hamy mean operator which has the following definition:

$$\text{HM}^{(k)}(F_1,\ F_2,\ \ldots,\ F_\mathfrak{J}) = \frac{\sum\limits_{1\le\iota_1<\ldots<\iota_k\le\mathfrak{J}}\left(\prod\limits_{\mathfrak{c}=1}^{k} F_{\iota_\mathfrak{c}}\right)^{\frac{1}{k}}}{C_\mathfrak{J}^k}, \tag{10}$$

the Hamy mean operator denoted as $\text{HM}^{(k)}$ is defined by considering all possible combinations of k-tuples from the set $(1,\ 2,\ 3,\ \ldots,\ \mathfrak{J})$. The number of such combinations is given by the binomial coefficient $C_\mathfrak{J}^k$, calculated as $\dfrac{\mathfrak{J}!}{k!(\mathfrak{J}-k)!}$.

Definition 14 The HM operator possesses the following evident properties [44]:

(i) $\text{HM}^{(k)}(0,\ 0,\ \ldots,\ 0) = 0$,
(ii) $\text{HM}^{(k)}(F,\ F,\ \ldots,\ F) = F$,
(iii) $\text{HM}^{(k)}(F_1,\ F_2,\ \ldots,\ F_\mathfrak{J}) \le \text{HM}^{(k)}(g_1,\ g_2,\ \ldots,\ g_\mathfrak{J})$, if $F_\iota \le g_\iota\ \forall\ \iota$,
(iv) $\min\limits_\iota\{F_\iota\} \le \text{HM}^{(k)}(F_1,\ F_2,\ \ldots,\ F_\mathfrak{J}) \le \max\limits_\iota\{F_\iota\}$.

The HM operator is characterized by two important special cases summarized as [44]:

Case 1: When $k = 1$, $\text{HM}^{(1)}(F_1,\ F_2,\ \ldots,\ F_\mathfrak{J}) = \frac{1}{\mathfrak{J}}\sum\limits_{\mathfrak{c}=1}^{\mathfrak{J}} F_\mathfrak{c}$ the HM operator becomes the arithmetic mean operator.

Case 2: When $k = \mathfrak{J}$, $\text{HM}^{(\mathfrak{J})}(\digamma_1, \digamma_2, \ldots, \digamma_{\mathfrak{J}}) = \left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}} \digamma_{\mathfrak{c}}\right)^{\frac{1}{\mathfrak{J}}}$ it becomes the geometric mean operator.

2.2.1 *Weighted Hamy mean operator*

Definition 15 [57] Assume a collection of non-negative real numbers denoted by $\digamma_\iota$ where ι takes values from 1 to $\mathfrak{J}$ and corresponding weight vectors $p^\gamma = (p_1^\gamma, p_2^\gamma, \ldots, p_{\mathfrak{J}}^\gamma)^T$, $p_\iota^\gamma \in [0, 1]$ and $\sum_{\iota=1}^{\mathfrak{J}} p_\iota^\gamma = 1$. Then,

$$\text{WHM}^{(k)}(\digamma_1, \digamma_2, \ldots, \digamma_{\mathfrak{J}}) = \frac{\sum_{1 \le \iota_1 < \ldots < \iota_k \le \mathfrak{J}} \left(\prod_{\mathfrak{c}=1}^{k} \digamma_{\iota_{\mathfrak{c}}}^{p_{\mathfrak{c}}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^\gamma}}}{C_{\mathfrak{J}}^k}, \tag{11}$$

the weighted hamy mean operator denoted as $\text{WHM}^{(k)}$ is defined by considering each k-tuple combination $(\iota_1, \iota_2, \ldots, \iota_k)$ from the set $(1, 2, 3, \ldots, \mathfrak{J})$. The binomial coefficient is expressed as $C_{\mathfrak{J}}^k = \frac{\mathfrak{J}!}{k!(\mathfrak{J}-k)!}$.

2.3 *Dual Hamy mean operator*

Extending the HM concept, Wu et al. [58] introduced the DHM operator defined as:

Definition 16 [58] Consider a collection $\digamma_\iota (\iota = 1, 2, 3, \ldots, \mathfrak{J})$ of non-negative real numbers and $k = 1, 2, \ldots, \mathfrak{J}$. The DHM operator represented as $\text{DHM}^{(k)}(\digamma_1, \digamma_2, \ldots, \digamma_{\mathfrak{J}})$ is defined by:

$$\text{DHM}^{(k)}(\digamma_1, \digamma_2, \ldots, \digamma_{\mathfrak{J}}) = \left(\prod_{1 \le \iota_1 < \ldots < \iota_k \le \mathfrak{J}} \left(\frac{\sum_{\mathfrak{c}=1}^{k} \digamma_{\iota_{\mathfrak{c}}}}{k}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^k}}, \tag{12}$$

the dual hamy mean operator denoted as $\text{DHM}^{(k)}$ is defined by considering each k-tuple combination $(\iota_1, \iota_2, \ldots, \iota_k)$ from the set $(1, 2, 3, \ldots, \mathfrak{J})$. The binomial coefficient is expressed as $C_{\mathfrak{J}}^k = \frac{\mathfrak{J}!}{k!(\mathfrak{J}-k)!}$.

The DHM operator shares similar characteristics with the HM operator and offers a comparative evaluation based on these attributes.

2.3.1 *Weighted DHM operator*

Definition 17 [57] Consider a collection of non-negative real numbers denoted by $\digamma_\iota$ where ι takes values from 1 to $\mathfrak{J}$ and corresponding weight vectors $p^\gamma = (p_1^\gamma, p_2^\gamma, \ldots, p_{\mathfrak{J}}^\gamma)^T$, $p_\iota^\gamma \in [0, 1]$ and $\sum_{\iota=1}^{\mathfrak{J}} p_\iota^\gamma = 1$. Then,

$$\text{WDHM}^{(k)}(\digamma_1, \digamma_2, \ldots, \digamma_{\mathfrak{J}}) = \left(\prod_{1 \le \iota_1 < \ldots < \iota_k \le \mathfrak{J}} \left(\frac{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^\gamma \digamma_{\iota_{\mathfrak{c}}}}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^\gamma}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^k}}, \tag{13}$$

the weighted dual hamy mean operator denoted as $\text{WDHM}^{(k)}$ is defined by considering each k-tuple combination $(\iota_1, \iota_2, \ldots, \iota_k)$ from the set $(1, 2, 3, \ldots, \mathfrak{J})$. The binomial coefficient is expressed as $C_{\mathfrak{J}}^k = \frac{\mathfrak{J}!}{k!(\mathfrak{J}-k)!}$.

3. Linear diophantine fuzzy HM operator for LDFNs

This section, we investigate the combination of LDFNs and the HM operator presenting the LDFHM operator as well as the LDFWHM operator.

Definition 18 Consider a collection of LDFNs denoted by $\vartheta_{\mathfrak{c}} = ((\mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}})(\alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}))$, where $\mathfrak{c}$ takes values from 1 to $\mathfrak{J}$. The LDFHM operator is specified as follows:

$$\mathrm{HM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}}) = \frac{\bigoplus\limits_{1\leq\iota_1<\ldots<\iota_k\leq\mathfrak{J}}\left(\bigotimes\limits_{\mathfrak{c}=1}^{k}\vartheta_{\iota_{\mathfrak{c}}}\right)^{\frac{1}{k}}}{C_{\mathfrak{J}}^{k}}, \tag{14}$$

the hamy mean operator denoted as $\mathrm{HM}^{(k)}$ is defined by considering each k-tuple combination $(\iota_1,\ \iota_2,\ \ldots,\ \iota_k)$ from the set $(1,\ 2,\ 3,\ \ldots,\ \mathfrak{J})$. The binomial coefficient is expressed as $C_{\mathfrak{J}}^{k} = \dfrac{\mathfrak{J}!}{k!(\mathfrak{J}-k)!}$.

Theorem 1 If $\vartheta_{\mathfrak{c}} = (\langle\mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}}\rangle,\ \langle\alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}\rangle)$ for $\mathfrak{c} = 1,\ 2,\ \ldots,\ \mathfrak{J}$ is a collection of LDFNs, then the synthesis result of $\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}}$ applying the LDFHM operator is:

$$\mathrm{LDFHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}}) = \left\{\begin{array}{l}\left\langle 1-\left(\prod\limits_{\eth}\left(1-\left(\prod\limits_{\mathfrak{c}=1}^{k}\mu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod\limits_{\eth}\left(1-\left(\prod\limits_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle, \\ \left\langle 1-\left(\prod\limits_{\eth}\left(1-\left(\prod\limits_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod\limits_{\eth}\left(1-\left(\prod\limits_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle\end{array}\right\}, \tag{15}$$

for the subscript $(1\leq\iota_1<\ldots<\iota_k\leq\mathfrak{J})$, we utilize $\eth$ for simplicity.

Proof. Utilizing the basic LDFNs algorithms yields the following results:

$$\left(\bigotimes_{\mathfrak{c}=1}^{k}\vartheta_{\mathfrak{c}}\right) = \left\{\begin{array}{l}\left\langle\prod\limits_{\mathfrak{c}=1}^{k}(\mu_{\mathfrak{c}}),\ 1-\prod\limits_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})\right\rangle, \\ \left\langle\prod\limits_{\mathfrak{c}=1}^{k}(\alpha_{\mathfrak{c}}),\ 1-\prod\limits_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right\rangle\end{array}\right\},$$

and

$$\left(\bigotimes_{\mathfrak{c}=1}^{k}\vartheta_{\mathfrak{c}}\right)^{\frac{1}{k}} = \left\{\begin{array}{l}\left\langle\left(\prod\limits_{\mathfrak{c}=1}^{k}\mu_{\mathfrak{c}}\right)^{\frac{1}{k}},\ 1-\left(\prod\limits_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right\rangle, \\ \left\langle\left(\prod\limits_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}},\ 1-\left(\prod\limits_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}}\right\rangle\end{array}\right\},$$

also

$$\left(\bigoplus_{1\le\iota_1<\ldots<\iota_k\le\mathfrak{J}}\left(\bigotimes_{\mathfrak{c}=1}^{k}\vartheta_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)=\left\{\begin{array}{c}\left\langle 1-\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\mu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right),\ \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right\rangle\\ \left\langle 1-\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}}\right),\ \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right\rangle\end{array}\right\},$$

hence

$$\left(\frac{\bigoplus_{1\le\iota_1<\ldots<\iota_k\le\mathfrak{J}}\left(\bigotimes_{\mathfrak{c}=1}^{k}\vartheta_{\mathfrak{c}}\right)^{\frac{1}{k}}}{C_{\mathfrak{J}}^{k}}\right)=\left\{\begin{array}{c}\left\langle 1-\left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\mu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle,\\ \left\langle 1-\left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle\end{array}\right\}.$$

□

3.1 *Properties of LDFHM operator*

The LDFHM operator possesses the following properties.

Theorem 2 Consider $\vartheta_{\mathfrak{c}}=((\mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}})(\alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}))$, where $(\mathfrak{c}=1,\ 2,\ 3,\ \ldots,\ \mathfrak{J})$ be a collection of LDFNs. If $\vartheta_1=\vartheta_2=\ldots=\vartheta_{\mathfrak{J}}=\vartheta$, then

$$\mathrm{LDFHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}})=\vartheta. \tag{16}$$

Proof. We know that,

$$\mathrm{LDFHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}})=\left\{\begin{array}{c}\left\langle 1-\left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\mu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle,\\ \left\langle 1-\left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle\end{array}\right\}$$

Now if $\vartheta_1=\vartheta_2=\ldots=\vartheta_{\mathfrak{J}}=\vartheta$, then

$$\text{LDFHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}}) = \left\{ \begin{array}{l} \left\langle 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\mu\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\nu)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle, \\ \left\langle 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\langle 1-\left(\prod_{\mathfrak{d}}\left(1-(\mu^k)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\mathfrak{d}}\left(1-((1-\nu)^k)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle, \\ \left\langle 1-\left(\prod_{\mathfrak{d}}\left(1-(\alpha^k)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\mathfrak{d}}\left(1-((1-\beta)^k)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\langle 1-\left(\prod_{\mathfrak{d}}(1-(\mu))\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\mathfrak{d}}(1-(1-\nu))\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle, \\ \left\langle 1-\left(\prod_{\mathfrak{d}}(1-(\alpha)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\mathfrak{d}}(1-(1-\beta))\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\langle 1-\left(\prod_{\mathfrak{d}}(1-(\mu))\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\mathfrak{d}}\nu\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle, \\ \left\langle 1-\left(\prod_{\mathfrak{d}}(1-(\alpha))\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod_{\mathfrak{d}}(\beta)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\langle 1-\left((1-(\mu))^{C_{\mathfrak{J}}^{k}}\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left((\nu)^{C_{\mathfrak{J}}^{k}}\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle, \\ \left\langle 1-\left((1-(\alpha))^{C_{\mathfrak{J}}^{k}}\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left((\beta)^{C_{\mathfrak{J}}^{k}}\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle \end{array} \right\}$$

$$= (\langle\mu,\ \nu\rangle,\ \langle\alpha,\ \beta\rangle)$$

$$= \vartheta.$$

□

Theorem 3 (Monotonicity) Consider $\vartheta_{\mathfrak{c}} = ((\mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}})(\alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}))$, $\hat{\vartheta}_{\mathfrak{c}} = ((\hat{\mu}_{\mathfrak{c}},\ \hat{\nu}_{\mathfrak{c}})(\hat{\alpha}_{\mathfrak{c}},\ \hat{\beta}_{\mathfrak{c}}))$ where $(\mathfrak{c} = 1,\ 2,\ 3\ \ldots,\ \mathfrak{J})$ be collections of LDFNs. If $\vartheta_{\mathfrak{c}} = (\langle\mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}}\rangle,\ \langle\alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}\rangle) \leq \hat{\vartheta}_{\mathfrak{c}} = \left(\langle\hat{\mu}_{\mathfrak{c}},\ \hat{\nu}_{\mathfrak{c}}\rangle,\ \left\langle\hat{\alpha}_{\mathfrak{c}},\ \hat{\beta}_{\mathfrak{c}}\right\rangle\right)$, then

$$\mathrm{LDFHM}^{(k)}(\vartheta_{\mathfrak{c}}) \leq \mathrm{LDFHM}^{(k)}(\hat{\vartheta}_{\mathfrak{c}}), \quad \text{where} \quad \mathfrak{c} = 1,\ 2,\ 3,\ \ldots,\ \mathfrak{J}. \tag{17}$$

Proof. As $\vartheta_{\mathfrak{c}} \leq \hat{\vartheta}_{\mathfrak{c}}$, we get $\mu_{\mathfrak{c}} \leq \hat{\mu}_{\mathfrak{c}},\ \nu_{\mathfrak{c}} \geq \hat{\nu}_{\mathfrak{c}}$ and $\alpha_{\mathfrak{c}} \leq \hat{\alpha}_{\mathfrak{c}},\ \beta_{\mathfrak{c}} \geq \hat{\beta}_{\mathfrak{c}}$ for $(\mathfrak{c} = 1,\ 2,\ \ldots,\ \mathfrak{J})$, are LDFNs. Since $\mu_{\mathfrak{c}} \leq \hat{\mu}_{\mathfrak{c}}$, we have

$$\Rightarrow \prod_{\mathfrak{c}=1}^{k} \mu_{\mathfrak{c}} \leq \prod_{\mathfrak{c}=1}^{k} \hat{\mu}_{\mathfrak{c}} \Rightarrow \left(\prod_{\mathfrak{c}=1}^{k} \mu_{\mathfrak{c}}\right)^{\frac{1}{k}} \leq \left(\prod_{\mathfrak{c}=1}^{k} \hat{\mu}_{\mathfrak{c}}\right)^{\frac{1}{k}}$$

$$\Rightarrow 1 - \left(\prod_{\mathfrak{c}=1}^{k} \mu_{\mathfrak{c}}\right)^{\frac{1}{k}} \geq 1 - \left(\prod_{\mathfrak{c}=1}^{k} \hat{\mu}_{\mathfrak{c}}\right)^{\frac{1}{k}}$$

$$\Rightarrow \prod_{\mathfrak{d}} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \mu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right) \geq \prod_{\mathfrak{d}} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \hat{\mu}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)$$

$$\Rightarrow \left(\prod_{\mathfrak{d}} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \mu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \geq \left(\prod_{\mathfrak{d}} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \hat{\mu}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}$$

$$\Rightarrow 1 - \left(\prod_{\mathfrak{d}} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \mu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \leq 1 - \left(\prod_{\mathfrak{d}} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \hat{\mu}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}.$$

Also, since $\nu_{\mathfrak{c}} \leq \hat{\nu}_{\mathfrak{c}}$

$$\Rightarrow 1 - \nu_{\mathfrak{c}} \geq 1 - \hat{\nu}_{\mathfrak{c}} \Rightarrow \prod_{\mathfrak{c}=1}^{k} (1 - \nu_{\mathfrak{c}}) \geq \prod_{\mathfrak{c}=1}^{k} (1 - \hat{\nu}_{\mathfrak{c}})$$

$$\Rightarrow \left(\prod_{\mathfrak{c}=1}^{k} (1 - \nu_{\mathfrak{c}})\right)^{\frac{1}{k}} \geq \left(\prod_{\mathfrak{c}=1}^{k} (1 - \hat{\nu}_{\mathfrak{c}})\right)^{\frac{1}{k}}$$

$$\Rightarrow 1 - \left(\prod_{\mathfrak{c}=1}^{k} (1 - \nu_{\mathfrak{c}})\right)^{\frac{1}{k}} \leq 1 - \left(\prod_{\mathfrak{c}=1}^{k} (1 - \hat{\nu}_{\mathfrak{c}})\right)^{\frac{1}{k}}$$

$$\Rightarrow \prod_{\mathfrak{d}} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1 - \nu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right) \leq \prod_{\mathfrak{d}} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1 - \hat{\nu}_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)$$

$$\Rightarrow \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \leq \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\nu}_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}.$$

Similarly, we prove for reference parameters α and β.
For $\alpha_{\mathfrak{c}} \leq \hat{\alpha}_{\mathfrak{c}}$

$$\Rightarrow \prod_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}} \leq \prod_{\mathfrak{c}=1}^{k}\hat{\alpha}_{\mathfrak{c}} \Rightarrow \left(\prod_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}} \leq \left(\prod_{\mathfrak{c}=1}^{k}\hat{\alpha}_{\mathfrak{c}}\right)^{\frac{1}{k}}$$

$$\Rightarrow 1-\left(\prod_{\mathfrak{c}=1}^{k}\mu_{\mathfrak{c}}\right)^{\frac{1}{k}} \geq 1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\alpha}_{\mathfrak{c}}\right)^{\frac{1}{k}}$$

$$\Rightarrow \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}}\right) \geq \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\alpha}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)$$

$$\Rightarrow \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \geq \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\alpha}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}$$

$$\Rightarrow 1-\left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \leq 1-\left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\alpha}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}.$$

Also, since $\beta_{\mathfrak{c}} \leq \hat{\beta}_{\mathfrak{c}}$

$$\Rightarrow 1-\beta_{\mathfrak{c}} \geq 1-\hat{\beta}_{\mathfrak{c}} \Rightarrow \prod_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}}) \geq \prod_{\mathfrak{c}=1}^{k}(1-\hat{\beta}_{\mathfrak{c}})$$

$$\Rightarrow \left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}} \geq \left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\beta}_{\mathfrak{c}})\right)^{\frac{1}{k}}$$

$$\Rightarrow 1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}} \leq 1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\beta}_{\mathfrak{c}})\right)^{\frac{1}{k}}$$

$$\Rightarrow \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}}\right) \leq \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\beta}_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)$$

$$\Rightarrow \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \leq \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\beta}_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}.$$

Hence, the monotonicity property hold. □

Theorem 4 (Boundedness) Let $\vartheta_{\mathfrak{c}}(\mathfrak{c}=1, 2, ..., \mathfrak{J})$ be the collection of the LDFNs. If $\vartheta^{-}=\min(\vartheta_1, \vartheta_2, \vartheta_3, \ldots, \vartheta_{\mathfrak{J}})$ and $\vartheta^{+}=\max(\vartheta_1, \vartheta_2, \vartheta_3, \ldots, \vartheta_{\mathfrak{J}})$, then

$$\vartheta^{-} \leq \mathrm{LDFHM}^{(k)}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{\mathfrak{J}}) \leq \vartheta^{+}. \tag{18}$$

Proof. By using property 1 and 2, then:

$\mathrm{LDFHM}^{(k)}(\vartheta_1, \vartheta_2, ..., \vartheta_{\mathfrak{J}}) \leq \mathrm{LDFHM}^{(k)}(\vartheta_1^{+}, \vartheta_2^{+}, \ldots, \vartheta_{\mathfrak{J}}^{+})=\vartheta^{+}$.

$\mathrm{LDFHM}^{(k)}(\vartheta_1, \vartheta_2, ..., \vartheta_{\mathfrak{J}}) \geq \mathrm{LDFHM}^{(k)}(\vartheta_1^{-}, \vartheta_2^{-}, \ldots, \vartheta_{\mathfrak{J}}^{-})=\vartheta^{-}$.

Then, we obtained: $\vartheta^{-} \leq \mathrm{LDFHM}^{(k)}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{\mathfrak{J}}) \leq \vartheta^{+}$. □

In the context of different parameter values, the LDFHM operator exhibits distinct behaviors in the following special situations.

1. **Case 1:** When $k=1$, the LDFHM operator becomes the LDF arithmetic mean operator of LDFNs.

$$\mathrm{LDFHM}^{(1)}(\vartheta_1, \vartheta_2, ..., \vartheta_{\mathfrak{J}})=\frac{\bigoplus_{\mathfrak{c}=1}^{\mathfrak{J}}\vartheta_{\mathfrak{c}}}{\mathfrak{J}}=\left\{\begin{array}{c}\left\langle 1-\left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}}(1-\mu_{\mathfrak{c}})\right)^{\frac{1}{\mathfrak{J}}}, \left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}}\nu_{\mathfrak{c}}\right)^{\frac{1}{\mathfrak{J}}}\right\rangle, \\ \left\langle 1-\left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}}(1-\alpha_{\mathfrak{c}})\right)^{\frac{1}{\mathfrak{J}}}, \left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}}\beta_{\mathfrak{c}}\right)^{\frac{1}{\mathfrak{J}}}\right\rangle\end{array}\right\}. \tag{19}$$

2. **Case 2:** When k equals $\mathfrak{J}$ the LDFHM operator transforms into the LDF geometric operator for LDFNs.

$$\mathrm{LDFHM}^{(\mathfrak{J})}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{\mathfrak{J}})=\left(\bigotimes_{\mathfrak{c}=1}^{\mathfrak{J}}\vartheta_{\mathfrak{c}}\right)^{\frac{1}{\mathfrak{J}}}=\left\{\begin{array}{c}\left\langle\left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}}\mu_{\mathfrak{c}}\right)^{\frac{1}{\mathfrak{J}}}, 1-\left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}}(1-\nu_{\mathfrak{c}})\right)^{\frac{1}{\mathfrak{J}}}\right\rangle, \\ \left\langle\left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}}\alpha_{\mathfrak{c}}\right)^{\frac{1}{\mathfrak{J}}}, 1-\left(\prod_{\mathfrak{c}=1}^{\mathfrak{J}}(1-\beta_{\mathfrak{c}})\right)^{\frac{1}{\mathfrak{J}}}\right\rangle\end{array}\right\}. \tag{20}$$

3.2 *LDF weighted Hamy mean operator for LDFNs*

We will now describe the LDFWHM operator using the same basic actions as the HM operator. To solve MAGDM approaches, the DMk employs a weight value containing all attributes provided by the experts. In DM problems, the weights assigned to attributes and experts can play a significant role. Therefore, in this subsection, we examine the impact of weights on the HM operator and introduce the WHM operator as follows.

Definition 19 Assume a collection of LDFNs denoted by $\vartheta_{\mathfrak{c}}=(\langle\mu_{\mathfrak{c}}, \nu_{\mathfrak{c}}\rangle, \langle\alpha_{\mathfrak{c}}, \beta_{\mathfrak{c}}\rangle)$, where $\mathfrak{c}$ takes values from 1 to $\mathfrak{J}$ and corresponding weight vectors $p^{\gamma}=(p_1^{\gamma}, p_2^{\gamma}, ..., p_{\mathfrak{J}}^{\gamma})^{T}$, $p_{\mathfrak{c}}^{\gamma}\in[0, 1]$ and $\sum_{\mathfrak{c}=1}^{\mathfrak{J}}p_{\mathfrak{c}}^{\gamma}=1$. Let k range from 1 to $\mathfrak{J}$, and let $\iota_1, \iota_2, \ldots, \iota_k$ be elements chosen from the set $1, 2, \ldots, \mathfrak{J}$. A mapping: $\Lambda^{\mathfrak{J}}\longrightarrow\Lambda$ is referred to as the LDFWHM operator and is define as below:

$$\text{LDWHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ ...,\ \vartheta_{\mathfrak{J}}) = \frac{\bigoplus\limits_{1\le\iota_1<...<\iota_k\le\mathfrak{J}}\left(\bigotimes\limits_{\mathfrak{c}=1}^{k}\vartheta_{\iota_{\mathfrak{c}}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}}}{C_{\mathfrak{J}}^{k}}, \tag{21}$$

$C_{\mathfrak{J}}^{k} = \dfrac{\mathfrak{J}!}{k!(\mathfrak{J}-k)!}$ represents the binomial coefficient, while Λ refers to the family of LDFNs.

Theorem 5 If $\vartheta_{\mathfrak{c}} = (\langle\mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}}\rangle,\ \langle\alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}\rangle)$ for $\mathfrak{c} = 1,\ 2,\ \ldots,\ \mathfrak{J}$ is a collection of LDFNs. Then the aggregated value of $\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}}$ applying the LDFWHM operator is:

$$\text{LDFWHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ ...,\ \vartheta_{\mathfrak{J}})$$

$$= \left\{\begin{array}{l} \left\langle 1-\left(\prod\limits_{\eth}\left(1-\left(\prod\limits_{\mathfrak{c}=1}^{k}\mu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod\limits_{\eth}\left(1-\left(\prod\limits_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle, \\ \left\langle 1-\left(\prod\limits_{\eth}\left(1-\left(\prod\limits_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ \left(\prod\limits_{\eth}\left(1-\left(\prod\limits_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle \end{array}\right\}, \tag{22}$$

for the subscript $(1 \le \iota_1 < \ldots < \iota_k \le \mathfrak{J})$, we use $\eth$ for simplicity.

Proof. Utilizing the basic LDFNs algorithms yields the following results:

$$(\vartheta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}} = \left\{\begin{array}{l} \left\langle(\mu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}),\ 1-(1-\nu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right\rangle, \\ \left\langle(\alpha_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}),\ 1-(1-\beta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right\rangle \end{array}\right\},$$

$$\bigotimes_{\mathfrak{c}=1}^{\mathfrak{J}}(\vartheta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}} = \left\{\begin{array}{l} \left\langle\prod\limits_{\mathfrak{c}=1}^{k}(\mu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}),\ 1-\prod\limits_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right\rangle, \\ \left\langle\prod\limits_{\mathfrak{c}=1}^{k}(\alpha_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}),\ 1-\prod\limits_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right\rangle \end{array}\right\},$$

and

$$\left(\bigotimes_{\mathfrak{c}=1}^{\mathfrak{J}}(\vartheta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}} = \left\{\begin{array}{l} \left\langle\left(\prod\limits_{\mathfrak{c}=1}^{k}\mu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}},\ 1-\left(\prod\limits_{\mathfrak{c}=1}^{k}(1-\nu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}}\right\rangle, \\ \left\langle\left(\prod\limits_{\mathfrak{c}=1}^{k}\alpha_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}},\ 1-\left(\prod\limits_{\mathfrak{c}=1}^{k}(1-\beta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_{\mathfrak{c}}^{\gamma}}}\right\rangle \end{array}\right\},$$

also

$$\underset{1\leq\iota_1<\ldots<\iota_k\leq\mathfrak{J}}{\oplus}\left(\overset{\mathfrak{J}}{\underset{\mathfrak{c}=1}{\otimes}}(\vartheta_\mathfrak{c})^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}=\left\{\begin{array}{c}\left\langle 1-\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\mu_\mathfrak{c}^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}\right),\ \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\nu_\mathfrak{c})^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}\right)\right\rangle\\ \left\langle 1-\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha_\mathfrak{c}^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}\right),\ \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_\mathfrak{c})^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}\right)\right\rangle\end{array}\right\},$$

hence

$$\left(\frac{\underset{1\leq\iota_1<\ldots<\iota_k\leq\mathfrak{J}}{\oplus}\left(\overset{k}{\underset{\mathfrak{c}=1}{\otimes}}(\vartheta_\mathfrak{c})^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}}{C_\mathfrak{J}^k}\right)$$

$$=\left\{\begin{array}{c}\left\langle 1-\left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\mu_\mathfrak{c}^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}\right)\right)^{\frac{1}{C_\mathfrak{J}^k}},\ \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\nu_\mathfrak{c})^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}\right)\right)^{\frac{1}{C_\mathfrak{J}^k}}\right\rangle,\\ \left\langle 1-\left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\alpha_\mathfrak{c}^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}\right)\right)^{\frac{1}{C_\mathfrak{J}^k}},\ \left(\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\beta_\mathfrak{c})^{p_\mathfrak{c}^\gamma}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k}p_\mathfrak{c}^\gamma}}\right)\right)^{\frac{1}{C_\mathfrak{J}^k}}\right\rangle\end{array}\right\}. \quad (23)$$

□

Remark 1 The LDFWHM operator also satisfies the basic properties of aggregation as discussed in section 3.1.

3.3 *Linear diophantine fuzzy dual Hamy mean operator for LDFNs*

The combination of the LDFS and DHM operators gives rise to the creation of LDFDHM.

Definition 20 Assume $\vartheta_\mathfrak{c}=(\langle\mu_\mathfrak{c},\ \nu_\mathfrak{c}\rangle,\ \langle\alpha_\mathfrak{c},\ \beta_\mathfrak{c}\rangle)$ $(\mathfrak{c}=1,\ 2,\ \ldots,\ \mathfrak{J})$ be the collection of LDFNs and $k=1,\ 2,\ \ldots,\ \mathfrak{J},\ \iota_1,\ \iota_2,\ \ldots,\ \iota_k$, gets its values from the collection $\{1,\ 2,\ \ldots,\ \mathfrak{J}\}$. A map: $\Lambda^\mathfrak{J}\longrightarrow\Lambda$ is referred to as the LDFDHM operator and is define as follows:

$$\mathrm{LDFDHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_\mathfrak{J})=\left(\underset{1\leq\iota_1<\ldots<\iota_k\leq\mathfrak{J}}{\otimes}\left(\frac{\overset{k}{\underset{\mathfrak{c}=1}{\oplus}}\vartheta_{\iota_\mathfrak{c}}}{k}\right)\right)^{\frac{1}{C_\mathfrak{J}^k}}, \quad (24)$$

where $C_\mathfrak{J}^k=\dfrac{\mathfrak{J}!}{k!(\mathfrak{J}-k)!}$ represents the binomial coefficient and Λ is the collection of LDFNs.

Theorem 6 If $\vartheta_{\mathfrak{c}} = (\langle \mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}} \rangle,\ \langle \alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}} \rangle)$ represents a set of LDNFs where $(\mathfrak{c} = 1,\ 2,\ ...,\ \mathfrak{J})$, the result of synthesizing $\vartheta_1,\ \vartheta_2,\ ...,\ \vartheta_{\mathfrak{J}}$ using the LDFDHM operator is:

$$\text{LDFDHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ ...,\ \vartheta_{\mathfrak{J}}) = \left\{ \begin{array}{l} \left\langle \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\mu_{\mathfrak{c}}) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1 - \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \nu_{\mathfrak{c}} \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle, \\ \left\langle \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\alpha_{\mathfrak{c}}) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1 - \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \beta_{\mathfrak{c}} \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle \end{array} \right\}, \quad (25)$$

in the case of subscripts $(1 \leq \iota_1 < \ldots < \iota_k \leq \mathfrak{J})$, we use $\eth$ for simplicity.

Proof. The following outcomes are derived using the fundamental LDFNs algorithms:

$$\overset{k}{\underset{\mathfrak{c}=1}{\oplus}} \vartheta_{\mathfrak{c}} = \left\{ \begin{array}{l} \left\langle 1 - \prod_{\mathfrak{c}=1}^{k} (1-\mu_{\mathfrak{c}}),\ \prod_{\mathfrak{c}=1}^{k} \nu_{\mathfrak{c}} \right\rangle, \\ \left\langle 1 - \prod_{\mathfrak{c}=1}^{k} (1-\alpha_{\mathfrak{c}}),\ \prod_{\mathfrak{c}=1}^{k} \beta_{\mathfrak{c}} \right\rangle \end{array} \right\},$$

and

$$\frac{\overset{k}{\underset{\mathfrak{c}=1}{\oplus}} \vartheta_{\mathfrak{c}}}{k} = \left\{ \begin{array}{l} \left\langle 1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\mu_{\mathfrak{c}}) \right)^{\frac{1}{k}},\ \left(\prod_{\mathfrak{c}=1}^{k} \nu_{\mathfrak{c}} \right)^{\frac{1}{k}} \right\rangle, \\ \left\langle 1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\alpha_{\mathfrak{c}}) \right)^{\frac{1}{k}},\ \left(\prod_{\mathfrak{c}=1}^{k} \beta_{\mathfrak{c}} \right)^{\frac{1}{k}} \right\rangle \end{array} \right\},$$

then

$$\underset{1 \leq \iota_1 < \ldots < \iota_{\mathfrak{c}} \leq \mathfrak{J}}{\otimes} \left(\frac{\overset{k}{\underset{\mathfrak{c}=1}{\oplus}} \vartheta_{\mathfrak{c}}}{k} \right) = \left\{ \begin{array}{l} \left\langle \prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\mu_{\mathfrak{c}}) \right)^{\frac{1}{k}} \right),\ 1 - \prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \nu_{\mathfrak{c}} \right)^{\frac{1}{k}} \right) \right\rangle, \\ \left\langle \prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\alpha_{\mathfrak{c}}) \right)^{\frac{1}{k}} \right),\ 1 - \prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \beta_{\mathfrak{c}} \right)^{\frac{1}{k}} \right) \right\rangle \end{array} \right\},$$

hence

$$\left(\underset{1\leq\iota_1<\ldots<\iota_{\mathfrak{c}}\leq\mathfrak{J}}{\otimes}\left(\frac{\overset{k}{\underset{\mathfrak{c}=1}{\oplus}}\vartheta_{\mathfrak{c}}}{k}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}=\left\{\begin{array}{l}\left\langle\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle,\\ \left\langle\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle\end{array}\right\}. \tag{26}$$

□

3.4 *Properties of LDFDHM operator*

The LDFDHM operator exhibits the following properties. Assume $\vartheta_{\mathfrak{c}}=(\langle\mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}}\rangle,\ \langle\alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}\rangle)$ and $\hat{\vartheta}_{\mathfrak{c}}=\left(\langle\hat{\mu}_{\mathfrak{c}},\ \hat{\nu}_{\mathfrak{c}}\rangle,\ \left\langle\hat{\alpha}_{\mathfrak{c}},\ \hat{\beta}_{\mathfrak{c}}\right\rangle\right)$ where $(\mathfrak{c}=1,\ 2,\ 3,\ \ldots,\ \mathfrak{J})$ are collections of LDFNs.

Theorem 7 (Idempotency) Assume $\vartheta_{\mathfrak{c}}=(\langle\mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}}\rangle,\ \langle\alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}\rangle)$ where $(\mathfrak{c}=1,\ 2,\ 3,\ \ldots,\ \mathfrak{J})$ is a collection of LDFNs. If $\vartheta_1=\vartheta_2=\ldots=\vartheta_{\mathfrak{J}}=\vartheta$, then,

$$\mathrm{LDFDHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}})=\vartheta. \tag{27}$$

Proof. We know that,

$$\mathrm{LDFDHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}})=\left\{\begin{array}{l}\left\langle\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle,\\ \left\langle\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle\end{array}\right\}.$$

Now if $\vartheta_1=\vartheta_2=\ldots=\vartheta_{\mathfrak{J}}=\vartheta$, then

$$=\left\{\begin{array}{l}\left\langle\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\mu)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\nu\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle,\\ \left\langle\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\beta\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}\right\rangle\end{array}\right\}$$

$$= \left\{ \begin{array}{l} \left\langle \left(\prod_{\mathfrak{d}}\left(1-\left(\left(1-\mu\right)^{k}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\nu^{k}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle, \\ \left\langle \left(\prod_{\mathfrak{d}}\left(1-\left(\left(1-\alpha\right)^{k}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\beta^{k}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\langle \left(\prod_{\mathfrak{d}}\left(1-\left(1-\mu\right)\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\nu\right)\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle, \\ \left\langle \left(\prod_{\mathfrak{d}}\left(1-\left(1-\alpha\right)\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\beta\right)\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\langle \left(\prod_{\mathfrak{d}}\left(\mu\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\nu\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle, \\ \left\langle \left(\prod_{\mathfrak{d}}\left(\alpha\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\prod_{\mathfrak{d}}\left(1-\beta\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\langle \left(\left(\mu\right)^{C_{\mathfrak{J}}^{k}}\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\left(1-\nu\right)^{C_{\mathfrak{J}}^{k}}\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle, \\ \left\langle \left(\left(\alpha\right)^{C_{\mathfrak{J}}^{k}}\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1-\left(\left(1-\beta\right)^{C_{\mathfrak{J}}^{k}}\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\langle \left(\mu\right),\ 1-\left(1-\nu\right) \right\rangle, \\ \left\langle \left(\alpha\right),\ 1-\left(1-\beta\right) \right\rangle \end{array} \right\}$$

$$= \left(\langle \mu,\ \nu\rangle,\ \langle \alpha,\ \beta\rangle\right) = \vartheta.$$

□

Theorem 8 (Monotonicity) Assume $\vartheta_{\mathfrak{c}} = (\langle \mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}}\rangle,\ \langle \alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}\rangle)$ and $\hat{\vartheta}_{\mathfrak{c}} = \left(\langle \hat{\mu}_{\mathfrak{c}},\ \hat{\nu}_{\mathfrak{c}}\rangle,\ \langle \hat{\alpha}_{\mathfrak{c}},\ \hat{\beta}_{\mathfrak{c}}\rangle\right)$ where ($\mathfrak{c} = 1,\ 2,\ 3,\ ...,\ \mathfrak{J}$) are collections of LDFNs. If $\vartheta_{\mathfrak{c}} = (\langle \mu_{\mathfrak{c}},\ \nu_{\mathfrak{c}}\rangle,\ \langle \alpha_{\mathfrak{c}},\ \beta_{\mathfrak{c}}\rangle) \leq \hat{\vartheta}_{\mathfrak{c}} = \left(\langle \hat{\mu}_{\mathfrak{c}},\ \hat{\nu}_{\mathfrak{c}}\rangle,\ \langle \hat{\alpha}_{\mathfrak{c}},\ \hat{\beta}_{\mathfrak{c}}\rangle\right)$, then

$$\text{LDFDHM}^{(k)}(\vartheta_{\mathfrak{c}}) \leq \text{LDFDHM}^{(k)}(\hat{\vartheta}_{\mathfrak{c}}) \quad \text{where} \quad (\mathfrak{c} = 1,\ 2,\ 3,\ ...,\ \mathfrak{J}). \tag{28}$$

Proof. As $\vartheta_{\mathfrak{c}} \leq \hat{\vartheta}_{\mathfrak{c}}$ we get that $\mu_{\mathfrak{c}} \leq \hat{\mu}_{\mathfrak{c}},\ \nu_{\mathfrak{c}} \geq \hat{\nu}_{\mathfrak{c}}$ and $\alpha_{\mathfrak{c}} \leq \hat{\alpha}_{\mathfrak{c}},\ \beta_{\mathfrak{c}} \geq \hat{\beta}_{\mathfrak{c}}$.
Since $\mu_{\mathfrak{c}} \leq \hat{\mu}_{\mathfrak{c}}$, we have

$$\Rightarrow\ 1-\mu_{\mathfrak{c}} \geq 1-\hat{\mu}_{\mathfrak{c}}\ \Rightarrow\ \prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}}) \geq \prod_{\mathfrak{c}=1}^{k}(1-\hat{\mu}_{\mathfrak{c}})$$

$$\Rightarrow\ \left(\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})\right)^{\frac{1}{k}} \geq \left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\mu}_{\mathfrak{c}})\right)^{\frac{1}{k}}$$

$$\Rightarrow\ 1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})\right)^{\frac{1}{k}} \leq 1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\mu}_{\mathfrak{c}})\right)^{\frac{1}{k}}$$

$$\Rightarrow \prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right) \leq \prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\mu}_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)$$

$$\Rightarrow\ \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \leq \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\mu}_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}.$$

Also, since $\nu_{\mathfrak{c}} \leq \hat{\nu}_{\mathfrak{c}}$

$$\Rightarrow\ \prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}} \leq \prod_{\mathfrak{c}=1}^{k}\hat{\nu}_{\mathfrak{c}}\ \Rightarrow\ \left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}\right)^{\frac{1}{k}} \leq \left(\prod_{\mathfrak{c}=1}^{k}\hat{\nu}_{\mathfrak{c}}\right)^{\frac{1}{k}}$$

$$\Rightarrow\ 1-\left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}\right)^{\frac{1}{k}} \geq 1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\nu}_{\mathfrak{c}}\right)^{\frac{1}{k}}$$

$$\Rightarrow \prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right) \geq \prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\nu}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)$$

$$\Rightarrow\ \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \geq \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\nu}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}$$

$$\Rightarrow\ 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \leq 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\nu}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}.$$

Similarly, we prove for reference parameters α and β.
For $\alpha_{\mathfrak{c}} \leq \hat{\alpha}_{\mathfrak{c}}$

$$\Rightarrow 1-\alpha_{\mathfrak{c}} \geq 1-\hat{\alpha}_{\mathfrak{c}} \Rightarrow \prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}}) \geq \prod_{\mathfrak{c}=1}^{k}(1-\hat{\alpha}_{\mathfrak{c}})$$

$$\Rightarrow \left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})\right)^{\frac{1}{k}} \geq \left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\alpha}_{\mathfrak{c}})\right)^{\frac{1}{k}}$$

$$\Rightarrow 1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})\right)^{\frac{1}{k}} \leq 1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\alpha}_{\mathfrak{c}})\right)^{\frac{1}{k}}$$

$$\Rightarrow \prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})\right)^{\frac{1}{k}}\right) \leq \prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\alpha}_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)$$

$$\Rightarrow \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \leq \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\hat{\alpha}_{\mathfrak{c}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}.$$

Also, since $\beta_{\mathfrak{c}} \leq \hat{\beta}_{\mathfrak{c}}$

$$\Rightarrow \prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}} \leq \prod_{\mathfrak{c}=1}^{k}\hat{\beta}_{\mathfrak{c}} \Rightarrow \left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}\right)^{\frac{1}{k}} \leq \left(\prod_{\mathfrak{c}=1}^{k}\hat{\beta}_{\mathfrak{c}}\right)^{\frac{1}{k}}$$

$$\Rightarrow 1-\left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}\right)^{\frac{1}{k}} \geq 1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\beta}_{\mathfrak{c}}\right)^{\frac{1}{k}}$$

$$\Rightarrow \prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}\right)^{\frac{1}{k}}\right) \geq \prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\beta}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)$$

$$\Rightarrow \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \geq \left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\beta}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}$$

$$\Rightarrow 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \leq 1-\left(\prod_{\mathfrak{d}}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\hat{\beta}_{\mathfrak{c}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_{\mathfrak{J}}^{k}}}.$$

Hence the monotonicity property hold. □

Theorem 9 (Boundedness) Assume $\vartheta_\mathfrak{c} = (\langle \mu_\mathfrak{c}, \nu_\mathfrak{c} \rangle, \langle \alpha_\mathfrak{c}, \beta_\mathfrak{c} \rangle)$ where $(\mathfrak{c} = 1, 2, 3, ..., \mathfrak{J})$ is a collection of LDFNs. If $\vartheta^- = \min(\vartheta_1, \vartheta_2, \vartheta_3, \ldots, \vartheta_\mathfrak{J})$ and $\vartheta^+ = \max(\vartheta_1, \vartheta_2, \vartheta_3, \ldots, \vartheta_\mathfrak{J})$, then

$$\vartheta^- \leq \mathrm{LDFDHM}^{(k)}(\vartheta_1, \vartheta_2, \ldots, \vartheta_\mathfrak{J}) \leq \vartheta^+. \tag{29}$$

Proof. The findings resulting from demonstrating the two aforementioned features of the proposed LDFDHM operator are as follows:

Suppose $\vartheta^- = \min_\mathfrak{c}\{\vartheta_\mathfrak{c}\}$ and $\vartheta^+ = \max_\mathfrak{c}\{\vartheta_\mathfrak{c}\}$.

For $\vartheta^- = \min_\mathfrak{c}\{\vartheta_\mathfrak{c}\} \leq \vartheta_\mathfrak{c}$, where $(\mathfrak{c} = 1, 2, ..., \mathfrak{J})$

$$\Rightarrow \mathrm{LDFDHM}^{(k)}(\vartheta^-, \vartheta^-, ..., \vartheta^-) = \vartheta^- \leq \mathrm{LDFDHM}^{(k)}(\vartheta_1, \vartheta_2, ..., \vartheta_\mathfrak{J})$$

$$\Rightarrow \vartheta^- \leq \mathrm{LDFDHM}^{(k)}(\vartheta_1, \vartheta_2, ..., \vartheta_\mathfrak{J}).$$

For $\vartheta_\mathfrak{c} \leq \vartheta^+ = \max_\mathfrak{c}\{\vartheta_\mathfrak{c}\}$, where $(\mathfrak{c} = 1, 2, ..., \mathfrak{J})$

$$\Rightarrow \mathrm{LDFDHM}^{(k)}(\vartheta_1, \vartheta_2, ..., \vartheta_\mathfrak{J}) \leq \mathrm{LDFDHM}^{(k)}(\vartheta^+, \vartheta^+, ..., \vartheta^+) = \vartheta^+$$

$$\Rightarrow \mathrm{LDFDHM}^{(k)}(\vartheta_1, \vartheta_2, ..., \vartheta_\mathfrak{J}) \leq \vartheta^+.$$

Hence, $\min_\mathfrak{c}\{\vartheta_\mathfrak{c}\} \leq \mathrm{LDFDHM}^{(k)}(\vartheta_1, \vartheta_2, ..., \vartheta_\mathfrak{J}) \leq \max_\mathfrak{c}\{\vartheta_\mathfrak{c}\}$.

Hence its hold. □

3.5 *LDFWDHM operator for LDFNs*

Definition 21 Consider a collection of LDFNs denoted by $\vartheta_\mathfrak{c} = (\langle \mu_\mathfrak{c}, \nu_\mathfrak{c} \rangle, \langle \alpha_\mathfrak{c}, \beta_\mathfrak{c} \rangle)$, where $\mathfrak{c}$ takes values from 1 to $\mathfrak{J}$ and corresponding weight vectors $p^\gamma = (p_1^\gamma, p_2^\gamma, ..., p_\mathfrak{J}^\gamma)^T$, $p_\mathfrak{c}^\gamma \in [0, 1]$ and $\sum_{\mathfrak{c}=1}^{\mathfrak{J}} p_\mathfrak{c}^\gamma = 1$. Let k range from 1 to $\mathfrak{J}$, and $\iota_1, \iota_2, \ldots, \iota_k$ be elements chosen from the set $1, 2, \ldots, \mathfrak{J}$. A mapping: $\Lambda^\mathfrak{J} \longrightarrow \Lambda$ is referred to as the LDFWDHM operator and is define as below:

$$\mathrm{LDFWDHM}^{(k)}(\vartheta_1, \vartheta_2, ..., \vartheta_\mathfrak{J}) = \left(\underset{1 \leq \iota_1 < \ldots < \iota_k \leq \mathfrak{J}}{\otimes} \left(\frac{\oplus_{\mathfrak{c}=1}^{k} p_{\iota_\mathfrak{c}}^\gamma \vartheta_{\iota_\mathfrak{c}}}{\sum_{\mathfrak{c}=1}^{k} p_{\iota_\mathfrak{c}}^\gamma} \right) \right)^{\frac{1}{C_\mathfrak{J}^k}}, \tag{30}$$

$C_\mathfrak{J}^k = \dfrac{\mathfrak{J}!}{k!(\mathfrak{J}-k)!}$ represents the binomial coefficient, while Λ refers to the family of LDFNs.

Theorem 10 If $\vartheta_\mathfrak{c} = (\langle \mu_\mathfrak{c}, \nu_\mathfrak{c} \rangle, \langle \alpha_\mathfrak{c}, \beta_\mathfrak{c} \rangle)$ indicates a collection of LDNFs, where $(\mathfrak{c} = 1, 2, ..., \mathfrak{J})$, the result of synthesizing $\vartheta_1, \vartheta_2, ..., \vartheta_\mathfrak{J}$ using the LDFWDHM operator is:

$$\mathrm{LDFWDHM}^{(k)}(\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{\mathfrak{J}})$$

$$= \left\{ \begin{array}{l} \left\langle \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\mu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}} \right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}} \right) \right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1 - \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \nu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}} \right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}} \right) \right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle, \\ \left\langle \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\alpha_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}} \right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}} \right) \right)^{\frac{1}{C_{\mathfrak{J}}^{k}}},\ 1 - \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \beta_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}} \right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}} \right) \right)^{\frac{1}{C_{\mathfrak{J}}^{k}}} \right\rangle \end{array} \right\}, \tag{31}$$

in the case of subscripts $(1 \leq \iota_1 < \ldots < \iota_k \leq \mathfrak{J})$, we use $\eth$ for simplicity.

Proof. The following outcomes are derived applying the fundamental LDFNs algorithms:

$$(\vartheta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}} = \left\{ \begin{array}{l} \left\langle 1-(1-\mu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}},\ \nu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}} \right\rangle, \\ \left\langle 1-(1-\alpha_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}},\ \beta_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}} \right\rangle \end{array} \right\},$$

$$\bigoplus_{\mathfrak{c}=1}^{k} (\vartheta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}} = \left\{ \begin{array}{l} \left\langle 1-\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}},\ \prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}} \right\rangle, \\ \left\langle 1-\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}},\ \prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}} \right\rangle \end{array} \right\},$$

and

$$\frac{\bigoplus_{\mathfrak{c}=1}^{k} (\vartheta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}}{k} = \left\{ \begin{array}{l} \left\langle 1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}},\ \left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}} \right\rangle, \\ \left\langle 1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}},\ \left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}} \right\rangle \end{array} \right\},$$

then

$$\bigotimes_{1\leq\iota_1<\ldots<\iota_{\mathfrak{c}}\leq\mathfrak{J}} \left(\frac{\bigoplus_{\mathfrak{c}=1}^{k} (\vartheta_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}}{k} \right) = \left\{ \begin{array}{l} \left\langle \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\mu_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}}\right),\ 1-\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\nu_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}}\right) \right\rangle, \\ \left\langle \prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}(1-\alpha_{\mathfrak{c}})^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}}\right),\ 1-\prod_{\eth}\left(1-\left(\prod_{\mathfrak{c}=1}^{k}\beta_{\mathfrak{c}}^{p_{\mathfrak{c}}^{\gamma}}\right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p_{\mathfrak{c}}^{\gamma}}}\right) \right\rangle \end{array} \right\},$$

hence

$$\left(\bigotimes_{1\le \iota_1<\ldots<\iota_{\mathfrak{c}}\le \mathfrak{J}} \left(\frac{\bigoplus_{\mathfrak{c}=1}^{k} (\vartheta_{\mathfrak{c}})^{p^{\gamma}_{\mathfrak{c}}}}{k} \right) \right)^{\frac{1}{C^{k}_{\mathfrak{J}}}}$$

$$= \left\{ \begin{array}{l} \left\langle \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\mu_{\mathfrak{c}})^{p^{\gamma}_{\mathfrak{c}}} \right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p^{\gamma}_{\mathfrak{c}}}} \right) \right)^{\frac{1}{C^{k}_{\mathfrak{J}}}}, 1 - \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \nu_{\mathfrak{c}}^{p^{\gamma}_{\mathfrak{c}}} \right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p^{\gamma}_{\mathfrak{c}}}} \right) \right)^{\frac{1}{C^{k}_{\mathfrak{J}}}} \right\rangle, \\ \left\langle \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} (1-\alpha_{\mathfrak{c}})^{p^{\gamma}_{\mathfrak{c}}} \right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p^{\gamma}_{\mathfrak{c}}}} \right) \right)^{\frac{1}{C^{k}_{\mathfrak{J}}}}, 1 - \left(\prod_{\eth} \left(1 - \left(\prod_{\mathfrak{c}=1}^{k} \beta_{\mathfrak{c}}^{p^{\gamma}_{\mathfrak{c}}} \right)^{\frac{1}{\sum_{\mathfrak{c}=1}^{k} p^{\gamma}_{\mathfrak{c}}}} \right) \right)^{\frac{1}{C^{k}_{\mathfrak{J}}}} \right\rangle \end{array} \right\}.$$

□

Remark 2 The LDFWDHM operator also satisfies the basic properties of aggregation as discussed in section 3.4.

4. Determine objective weights based on deviation maximization method

Li et al. [59] the deviation of $\mathfrak{v}^{\mathfrak{g}}{}_{i}$'s evaluation value from the evaluation values of all other alternatives under attribute $\mathfrak{x}^{\mathfrak{d}}{}_{j}$ is given by:

$$D_{ij}(p^{\gamma}_{j}) = \sum_{r=1}^{l} d(\hat{x}_{ij},\ \hat{x}_{rj}) p^{\gamma}_{j}, \tag{32}$$

where $\hat{x}_{ij}$ and $\hat{x}_{rj}$ represent the normalized evaluation values in the form of LDFNs, with $d(\hat{x}_{ij},\ \hat{x}_{rj})$ denoting the Hamming distance between these LDFNs, as calculated using Equation (9). The overall deviation across all alternatives for attribute $\mathfrak{x}^{\mathfrak{d}}{}_{j}$ is expressed as $D_j(p^{\gamma}_{j}) = \sum_{i=1}^{l} \sum_{r=1}^{l} d(\hat{x}_{ij},\ \hat{x}_{rj}) p^{\gamma}_{j}$. The total deviation across all alternatives and attributes is expressed as $D(p^{\gamma}_{j}) = \sum_{j=1}^{m} \sum_{i=1}^{l} \sum_{r=1}^{l} d(\hat{x}_{ij},\ \hat{x}_{rj}) p^{\gamma}_{j}$. Based on this, the following optimal model is developed:

$$\max D(p^{\gamma}_{j}) = \sum_{j=1}^{m} \sum_{i=1}^{l} \sum_{r=1}^{l} d(\hat{x}_{ij},\ \hat{x}_{rj}) p^{\gamma}_{j} \tag{33}$$

s.t.

$$\sum_{j=1}^{m} p^{\gamma 2}_{j} = 1 \text{ and } p^{\gamma}_{j} \ge 0,\ j = 1,\ 2,\ \ldots,\ m.$$

Formulate the Lagrange multiplier function:

$$L(p_j^{\gamma}, \eta) = \sum_{j=1}^{m}\sum_{i=1}^{l}\sum_{r=1}^{l} d(\hat{x}_{ij}, \hat{x}_{rj})p_j^{\gamma} + \eta\left(\sum_{j=1}^{m} p_j^{\gamma 2} - 1\right), \tag{34}$$

and we have

$$\begin{aligned} \frac{\partial L(p_j^{\gamma}, \eta)}{\partial p_j^{\gamma}} &= \sum_{i=1}^{l}\sum_{r=1}^{l} d(\hat{x}_{ij}, \hat{x}_{rj}) + 2\eta p_j^{\gamma} = 0 \\ \frac{\partial L(p_j^{\gamma}, \eta)}{\partial \eta} &= \sum_{j=1}^{m} p_j^{\gamma 2} - 1 = 0. \end{aligned} \tag{35}$$

Solving Equation (35), we get:

$$\begin{aligned} 2\eta &= \sqrt{\sum_{j=1}^{m}\sum_{i=1}^{l}\sum_{r=1}^{l} d(\hat{x}_{ij}, \hat{x}_{rj}))^2} \\ p_j^{\gamma} &= \frac{\sum_{i=1}^{l}\sum_{r=1}^{l} d(\hat{x}_{ij}, \hat{x}_{rj})}{\sqrt{\sum_{j=1}^{m}\sum_{i=1}^{l}\sum_{r=1}^{l} d(\hat{x}_{ij}, \hat{x}_{rj}))^2}}. \end{aligned} \tag{36}$$

Normalizing the vector $p^{\gamma} = (p_1^{\gamma}, p_2^{\gamma}, \ldots, p_m^{\gamma})$ results in the determination of the objective weight vector.

$$p_j^{\gamma} = \frac{p_j^{\gamma}}{\sum_{j=1}^{m} p_j^{\gamma}} = \frac{\sum_{i=1}^{l}\sum_{r=1}^{l} d(\hat{x}_{ij}, \hat{x}_{rj})}{\sum_{j=1}^{m}\sum_{i=1}^{l}\sum_{r=1}^{l} d(\hat{x}'_{ij}, \hat{x}_{rj})}. \tag{37}$$

5. Linear diophantine fuzzy MAGDM method

This part tackles real-world DM problem by addressing core MAGDM problems and introducing a novel MAGDM approach using the specified LDFWHM and LDFWDHM operators.

We will develop the LDFWHM and LDFWDHM operators to address MAGDM problems applying LDFN data. Let $\mathfrak{v}^{\mathfrak{g}} = \{\mathfrak{v}^{\mathfrak{g}}{}_1, \mathfrak{v}^{\mathfrak{g}}{}_2, \ldots, \mathfrak{v}^{\mathfrak{g}}{}_l\}$ denote a set of alternatives and $\mathfrak{x}^{\mathfrak{d}} = \{\mathfrak{x}^{\mathfrak{d}}{}_1, \mathfrak{x}^{\mathfrak{d}}{}_2, \ldots, \mathfrak{x}^{\mathfrak{d}}{}_m\}$ represent a set of attributes and the corresponding weight vector is $p^{\gamma} = (p_1^{\gamma}, p_2^{\gamma}, \ldots, p_m^{\gamma})$, where p_j^{γ} satisfies $p_j^{\gamma} \in [0, 1]$ and $\sum_{j=1}^{m} p_j^{\gamma} = 1$. The group of experts are represented as $E = \{E_1, E_2, \ldots, E_t\}$. Each expert E_s (where $s = 1, 2, \ldots, t$) provides their evaluation for each alternative $\mathfrak{v}^{\mathfrak{g}}{}_i$ (where $i = 1, 2, \ldots, l$) based on attribute $\mathfrak{x}^{\mathfrak{d}}{}_j$ (where $j = 1, 2, \ldots, m$), using LDFNs denoted by $\vartheta_{i_j}^{s} = \left(\left\langle \mu_{i_j}^{s}, \nu_{i_j}^{s} \right\rangle, \left\langle \alpha_{i_j}^{s}, \beta_{i_j}^{s} \right\rangle\right)$. The decision matrices are represented as $D^s = (\vartheta_{i_j}^{s})_{l\times m}$. To determine the optimal alternative, we propose a new MAGDM method utilizing the LDFWHM and LDFWDH operator, which involves the following steps and also represented by a flowchart in Figure 1.

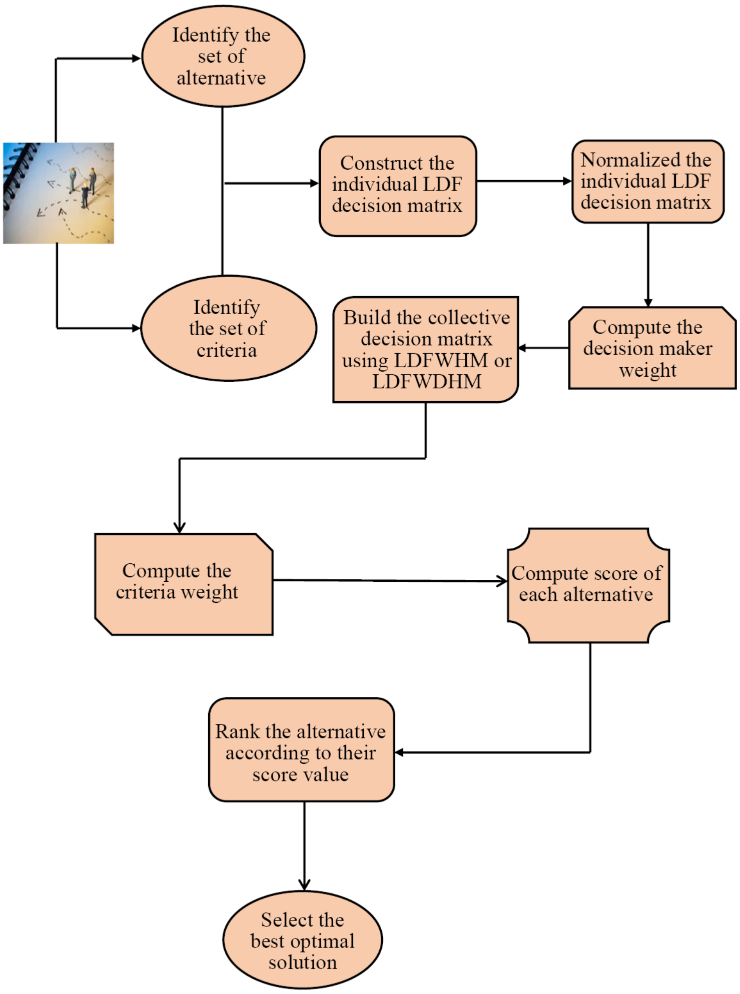

Figure 1. Flowchart of the proposed method

Step-1: Utilize the transformation method recommended by experts to develop the linear Diophantine fuzzy decision matrices.

Step-2: Standardize the LDF data, as the expert matrix $E^{(s)}$ may include both benefit and cost attribute, defined as:

$$\vartheta_{i_j}^s = \begin{cases} \left(\langle \mu_{i_j}^s,\ \nu_{i_j}^s \rangle,\ \langle \alpha_{i_j}^s,\ \beta_{i_j}^s \rangle\right) & \text{for benefit attribute} \\ \left(\langle \nu_{i_j}^s,\ \mu_{i_j}^s \rangle,\ \langle \beta_{i_j}^s,\ \alpha_{i_j}^s \rangle\right) & \text{for cost attribute,} \end{cases} \tag{38}$$

where $s = 1,\ 2,\ \ldots,\ t$, $i = 1,\ 2,\ \ldots,\ l$, $j = 1,\ 2,\ \ldots,\ m$.

Step-3: Assign weights to each Expert Matrix to prioritize them, denoted by $p^\gamma = (p_1^\gamma,\ p_2^\gamma,\ \ldots,\ p_m^\gamma)^T$.

Step-4: Using the information provided by the expert matrices $E_s(s = 1,\ 2,\ 3)$, combine these into a unified LDF Expert Matrix by applying the LDFWHM and LDFWDHM operators, using the associated weights $p^\gamma = (p_1^\gamma,\ p_2^\gamma,\ \ldots,\ p_m^\gamma)^T$ for each attribute $\mathfrak{r}^{\mathfrak{b}}{}_j$.

Step-5: Calculate the collective aggregated values for each attributes $\mathfrak{r}^{\mathfrak{b}}{}_j$ using the weight vector $p^\gamma = (p_1^\gamma,\ p_2^\gamma,\ \ldots,\ p_m^\gamma)^T$. These weights were determined using Equation (37). The recalculations were performed by applying Equations (22) and (31) as defined in the LDFWHM and LDFWDHM aggregation operators.

Step-6: Calculate the scores for each alternative $\mathfrak{v}^{\mathfrak{g}}{}_i$ from the aggregated values by applying the defined expectation score functions using Equations (4) and (5).

Step-7: Utilize the definition 7 select the best optimal decision by ranking of all alternatives.

Step-8: Alternatives with higher scores receive the top rankings and should be selected for the final decision.

6. Numerical example

In this part, we provide a new MAGDM method that utilizes linear Diophantine fuzzy information within the LDFWHM and LDFWDHM operators. An approach for the numerical model is presented and by applying SF and AF, we obtain preventative and control rankings to make the final choice on emergency shelters. The results are then compared with those of previous operators to illustrate the practicality and effectiveness of the proposed approach.

6.1 *Materials for building emergency shelters*

Earthquakes pose a significant threat to human life, property, and public health. Unfortunately, scientists have yet to develop a dependable method for predicting earthquake events with accuracy. As a result, countries have developed a range of emergency response measures to be implemented after an earthquake occurs. These measures aim to ensure that those affected by the disaster are swiftly rescued, and their lives and livelihoods are restored. One crucial aspect of these efforts is the construction of emergency shelters [60]. Based on prior knowledge, five primary attribute are usually considered when constructing emergency shelters, specifically:

Availability ($\mathfrak{x}^{\mathfrak{d}}{}_1$), Convenience ($\mathfrak{x}^{\mathfrak{d}}{}_2$), Safety ($\mathfrak{x}^{\mathfrak{d}}{}_3$), Comprehensive support services ($\mathfrak{x}^{\mathfrak{d}}{}_4$), Sustainability ($\mathfrak{x}^{\mathfrak{d}}{}_5$).

Simultaneously, five types of materials (alternatives) are used in the construction of emergency shelters: Rubber ($\mathfrak{v}^{\mathfrak{g}}{}_1$), Bamboo ($\mathfrak{v}^{\mathfrak{g}}{}_2$), Plastic Sheeting ($\mathfrak{v}^{\mathfrak{g}}{}_3$), 3D-printed recycled materials ($\mathfrak{v}^{\mathfrak{g}}{}_4$) and Shipping Containers ($\mathfrak{v}^{\mathfrak{g}}{}_5$).

To quickly construct a safe and appropriate emergency shelter, the emergency command department assembled three expert panels (E_1, E_2, E_3) comprising government officials, emergency management experts, international rescue specialists, and local residents. These experts assessed five potential building materials (alternatives) based on five key attribute, with evaluation results presented in Tables 1-3. The fuzzy evaluation matrices provided by the experts are denoted as EMs. The following details outline five alternative emergency shelter options:

(1) Rubber ($\mathfrak{v}^{\mathfrak{g}}{}_1$): Rubber can be repurposed in various ways, including being transformed into bricks for building construction. A notable advantage of these eco-friendly bricks is their straightforward assembly and instant readiness for use. Their main advantage lies in their low density, making the blocks exceptionally lightweight. Additionally, the material is more durable than the waterproof fabrics typically used in emergency tent construction, allowing shelters built with this technology to remain useful even after the emergency has passed.

(2) Bamboo ($\mathfrak{v}^{\mathfrak{g}}{}_2$): Bamboo is an exceptionally adaptable material, renowned for its remarkable strength and resilience against pressure and torsion. Bamboo's abundance, particularly in tropical regions, and its swift growth rate are significant benefits, rendering it an excellent choice for temporary constructions, including scaffolding, partitions, and roofing. Additionally, it is quick to assemble and can even serve as permanent housing in emergency shelter situations. To qualify as a viable building material, bamboo must undergo chemical treatment to prevent deterioration and pest damage, thereby ensuring its durability and exceptional mechanical properties. It's also essential to protect bamboo components from exposure to sunlight and rainfall.

(3) Plastic sheeting ($\mathfrak{v}^{\mathfrak{g}}{}_3$): People affected by earthquakes often choose to remain in or close to their damaged homes rather than relocate to camps or communal centers. Plastic sheeting or tarps are commonly used for emergency shelter because of their adaptability. Since earthquake survivors can often salvage building materials from their homes, providing additional timber or bamboo poles is usually unnecessary.

(4) 3D Printed Recycled Materials ($\mathfrak{v}^{\mathfrak{g}}{}_4$): Although 3D printing comes with a relatively high price tag, it has emerged as a promising solution for building emergency shelters due to its growing popularity. A prime example is the "KamerMaker", a mobile 3D printer developed by DUS Architects that constructs entire buildings using recycled materials. Additionally, 3D

printers can be highly useful in emergencies for producing small connecting components. Designing joints to facilitate assembly and enable the connection of different materials can greatly simplify the installation process.

(5) Shipping Containers $(\mathfrak{v}^{\mathfrak{g}}{}_5)$: Shipping containers are engineered to be resilient in extreme weather conditions and withstand heavy usage.They are widely available, can be easily transported, and repurpose. Containers can be modified to provide basic amenities (e.g., windows, doors, insulation). Compared to traditional building materials, shipping containers can be relatively affordable. Containers can be quickly assembled and disassembled.

Table 1. Expert E_1 provided values for alternatives are

	$\mathfrak{x}^{\mathfrak{d}}{}_1$	$\mathfrak{x}^{\mathfrak{d}}{}_2$	$\mathfrak{x}^{\mathfrak{d}}{}_3$	$\mathfrak{x}^{\mathfrak{d}}{}_4$	$\mathfrak{x}^{\mathfrak{d}}{}_5$
$\mathfrak{v}^{\mathfrak{g}}{}_1$	$\begin{pmatrix}\langle 0.20, 0.90\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.30, 0.80\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.70\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.80\rangle \\ \langle 0.30, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.50\rangle \\ \langle 0.20, 0.60\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_2$	$\begin{pmatrix}\langle 0.30, 0.60\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.70\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.80\rangle \\ \langle 0.30, 0.40\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.50\rangle \\ \langle 0.20, 0.60\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.80\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_3$	$\begin{pmatrix}\langle 0.80, 0.40\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.80\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.50\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.60\rangle \\ \langle 0.30, 0.60\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.70\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_4$	$\begin{pmatrix}\langle 0.90, 0.50\rangle \\ \langle 0.60, 0.20\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.60\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.50\rangle \\ \langle 0.30, 0.60\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.70\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.80\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_5$	$\begin{pmatrix}\langle 0.70, 0.80\rangle \\ \langle 0.40, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.50\rangle \\ \langle 0.30, 0.60\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.70\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.80\rangle \\ \langle 0.30, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.70\rangle \\ \langle 0.20, 0.60\rangle\end{pmatrix}$

Table 2. Expert E_2 provided values for alternatives are

	$\mathfrak{x}^{\mathfrak{d}}{}_1$	$\mathfrak{x}^{\mathfrak{d}}{}_2$	$\mathfrak{x}^{\mathfrak{d}}{}_3$	$\mathfrak{x}^{\mathfrak{d}}{}_4$	$\mathfrak{x}^{\mathfrak{d}}{}_5$
$\mathfrak{v}^{\mathfrak{g}}{}_1$	$\begin{pmatrix}\langle 0.80, 0.40\rangle \\ \langle 0.60, 0.20\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.90\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.50\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, .70\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.80\rangle \\ \langle 0.30, 0.60\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_2$	$\begin{pmatrix}\langle 0.80, 0.70\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.30, 0.80\rangle \\ \langle 0.20, 0.60\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.20, 0.70\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.80\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.40\rangle \\ \langle 0.20, 0.70\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_3$	$\begin{pmatrix}\langle 0.50, 0.80\rangle \\ \langle 0.30, 0.60\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.20\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.40\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.30\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.80\rangle \\ \langle 0.70, 0.20\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_4$	$\begin{pmatrix}\langle 0.50, 0.70\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.70\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.20\rangle \\ \langle 0.70, 0.20\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.30\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.80\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_5$	$\begin{pmatrix}\langle 0.50, 0.30\rangle \\ \langle 0.70, 0.20\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.80\rangle \\ \langle 0.60, 0.20\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.50\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.50\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.80\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$

Table 3. Expert E_3 provided values for alternatives are

	$\mathfrak{x}^{\mathfrak{d}}{}_1$	$\mathfrak{x}^{\mathfrak{d}}{}_2$	$\mathfrak{x}^{\mathfrak{d}}{}_3$	$\mathfrak{x}^{\mathfrak{d}}{}_4$	$\mathfrak{x}^{\mathfrak{d}}{}_5$
$\mathfrak{v}^{\mathfrak{g}}{}_1$	$\begin{pmatrix}\langle 0.50, 0.30\rangle \\ \langle 0.70, 0.20\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.70\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.30, 0.60\rangle \\ \langle 0.40, 0.40\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.60\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.80\rangle \\ \langle 0.70, 0.20\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_2$	$\begin{pmatrix}\langle 0.70, 0.60\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.50\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.90, 0.40\rangle \\ \langle 0.20, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.30\rangle \\ \langle 0.20, 0.70\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.20\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_3$	$\begin{pmatrix}\langle 0.60, 0.80\rangle \\ \langle 0.30, 0.60\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.90\rangle \\ \langle 0.40, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.60\rangle \\ \langle 0.70, 0.20\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.90, 0.70\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.80\rangle \\ \langle 0.30, 0.60\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_4$	$\begin{pmatrix}\langle 0.80, 0.60\rangle \\ \langle 0.30, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.50\rangle \\ \langle 0.40, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.70\rangle \\ \langle 0.50, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.40, 0.80\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.30, 0.90\rangle \\ \langle 0.70, 0.20\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_5$	$\begin{pmatrix}\langle 0.40, 0.80\rangle \\ \langle 0.30, 0.50\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.50, 0.70\rangle \\ \langle 0.40, 0.30\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.60, 0.60\rangle \\ \langle 0.20, 0.60\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.70, 0.50\rangle \\ \langle 0.50, 0.40\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.80, 0.40\rangle \\ \langle 0.60, 0.30\rangle\end{pmatrix}$

Temporary measures and emergency structures provide vital support during episodes of extreme hardship and trauma. It is essential to prioritize fast and straightforward responses that address the needs and demands of those affected. Architects should take an active role in devising well-considered and effective solutions, exploring novel materials and alternatives, and leveraging local resources [61, 62]. The subsequent section outlines the solution derived from applying the aforementioned algorithm to a numerical table, focusing on the specific case of materials for emergency shelter construction. This will help us choose the most suitable alternative solution for the emergency situation.

6.2 *Implementation of the proposed decision making method*

In this subsection, we Implement the proposed method.

Step-1: Suppose that three experts are employed to evaluate the five material shelter alternatives ${\mathfrak{v}^{\mathfrak{g}}}_i$ $(i = 1, 2, 3, 4, 5)$ based on five attributes ${\mathfrak{x}^{\mathfrak{b}}}_j$ $(j = 1, 2, 3, 4, 5)$. The expert evaluation matrices $E_s(s = 1, 2, 3)$ are presented in Tables 1, 2, and 3. According to the experts, all attributes ${\mathfrak{x}^{\mathfrak{b}}}_1$, ${\mathfrak{x}^{\mathfrak{b}}}_2$, ${\mathfrak{x}^{\mathfrak{b}}}_3$, ${\mathfrak{x}^{\mathfrak{b}}}_4$ and ${\mathfrak{x}^{\mathfrak{b}}}_5$ are of the benefit type.

Step-2: Given that all attributes share the same unit of measurement, data normalization is unnecessary.

Step-3: For each expert's matrix, the corresponding weight vectors are $p^{\gamma} = \{0.325, 0.295, 0.38\}$.

Step-4: Next, we utilize the LDFWHM and LDFWDHM operators along with the associated weight vectors $p^{\gamma} = (0.325, 0.295, 0.38)^T$ to process the input data presented in Tables 1-3 into collective LDF information. Tables 4 and 5 display the expert matrix computed using the LDFWHM and LDFWDHM aggregation operators applying Equations (22) and (31).

Table 4. Integrate the decision matrix by using LDFWHM operator

	${\mathfrak{x}^{\mathfrak{b}}}_1$	${\mathfrak{x}^{\mathfrak{b}}}_2$	${\mathfrak{x}^{\mathfrak{b}}}_3$	${\mathfrak{x}^{\mathfrak{b}}}_4$	${\mathfrak{x}^{\mathfrak{b}}}_5$
${\mathfrak{v}^{\mathfrak{g}}}_1$	$\begin{pmatrix}\langle 0.4581, 0.5739\rangle \\ \langle 0.5661, 0.2988\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5569, 0.8061\rangle \\ \langle 0.4622, 0.4082\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4142, 0.6076\rangle \\ \langle 0.5235, 0.3369\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7071, 0.7033\rangle \\ \langle 0.4284, 0.3978\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4881, 0.7219\rangle \\ \langle 0.3807, 0.4685\rangle\end{pmatrix}$
${\mathfrak{v}^{\mathfrak{g}}}_2$	$\begin{pmatrix}\langle 0.5863, 0.6324\rangle \\ \langle 0.4659, 0.4049\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4926, 0.6715\rangle \\ \langle 0.4243, 0.3976\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5162, 0.6457\rangle \\ \langle 0.2866, 0.4681\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5918, 0.5444\rangle \\ \langle 0.2894, 0.5622\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5999, 0.4831\rangle \\ \langle 0.3917, 0.5022\rangle\end{pmatrix}$
${\mathfrak{v}^{\mathfrak{g}}}_3$	$\begin{pmatrix}\langle 0.6301, 0.7009\rangle \\ \langle 0.3585, 0.5119\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6325, 0.7353\rangle \\ \langle 0.4931, 0.4055\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7347, 0.5095\rangle \\ \langle 0.5660, 0.3304\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6375, 0.5622\rangle \\ \langle 0.4552, 0.4038\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5007, 0.7702\rangle \\ \langle 0.4747, 0.383\rangle\end{pmatrix}$
${\mathfrak{v}^{\mathfrak{g}}}_4$	$\begin{pmatrix}\langle 0.7413, 0.6028\rangle \\ \langle 0.4515, 0.3413\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7638, 0.5986\rangle \\ \langle 0.4622, 0.3309\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5276, 0.5005\rangle \\ \langle 0.4804, 0.3746\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5522, 0.6492\rangle \\ \langle 0.5306, 0.3676\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4488, 0.8418\rangle \\ \langle 0.5364, 0.3266\rangle\end{pmatrix}$
${\mathfrak{v}^{\mathfrak{g}}}_5$	$\begin{pmatrix}\langle 0.5226, 0.6934\rangle \\ \langle 0.4378, 0.3443\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6587, 0.6799\rangle \\ \langle 0.4173, 0.3746\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5664, 0.6076\rangle \\ \langle 0.4345, 0.4136\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6963, 0.6146\rangle \\ \langle 0.3992, 0.4641\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6338, 0.6439\rangle \\ \langle 0.4185, 0.4339\rangle\end{pmatrix}$

Table 5. Integrate the decision matrix by using LDFWDHM operator

	${\mathfrak{x}^{\mathfrak{b}}}_1$	${\mathfrak{x}^{\mathfrak{b}}}_2$	${\mathfrak{x}^{\mathfrak{b}}}_3$	${\mathfrak{x}^{\mathfrak{b}}}_4$	${\mathfrak{x}^{\mathfrak{b}}}_5$
${\mathfrak{v}^{\mathfrak{g}}}_1$	$\begin{pmatrix}\langle 0.5287, 0.4953\rangle \\ \langle 0.5806, 0.2781\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5919, 0.7951\rangle \\ \langle 0.4641, 0.4014\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4299, 0.5999\rangle \\ \langle 0.5312, 0.3348\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7291, 0.6959\rangle \\ \langle 0.4379, 0.3915\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5085, 0.7012\rangle \\ \langle 0.4208, 0.4345\rangle\end{pmatrix}$
${\mathfrak{v}^{\mathfrak{g}}}_2$	$\begin{pmatrix}\langle 0.6349, 0.6297\rangle \\ \langle 0.4682, 0.3972\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5370, 0.6557\rangle \\ \langle 0.4573, 0.3766\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6060, 0.6202\rangle \\ \langle 0.2944, 0.4659\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5986, 0.4995\rangle \\ \langle 0.3263, 0.5324\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6076, 0.4227\rangle \\ \langle 0.4166, 0.4804\rangle\end{pmatrix}$
${\mathfrak{v}^{\mathfrak{g}}}_3$	$\begin{pmatrix}\langle 0.6495, 0.6663\rangle \\ \langle 0.3676, 0.4915\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6415, 0.6413\rangle \\ \langle 0.4992, 0.3992\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7447, 0.5030\rangle \\ \langle 0.5806, 0.3149\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6805, 0.5324\rangle \\ \langle 0.4724, 0.3841\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5186, 0.7674\rangle \\ \langle 0.5011, 0.3561\rangle\end{pmatrix}$
${\mathfrak{v}^{\mathfrak{g}}}_4$	$\begin{pmatrix}\langle 0.7778, 0.5952\rangle \\ \langle 0.4655, 0.3265\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7663, 0.5918\rangle \\ \langle 0.4641, 0.3284\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5447, 0.4556\rangle \\ \langle 0.5103, 0.3433\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5668, 0.6018\rangle \\ \langle 0.5395, 0.3585\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4636, 0.8376\rangle \\ \langle 0.5512, 0.3099\rangle\end{pmatrix}$
${\mathfrak{v}^{\mathfrak{g}}}_5$	$\begin{pmatrix}\langle 0.5376, 0.6373\rangle \\ \langle 0.4676, 0.3285\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6735, 0.6617\rangle \\ \langle 0.4346, 0.3433\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5686, 0.5999\rangle \\ \langle 0.4685, 0.3964\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7051, 0.5913\rangle \\ \langle 0.4055, 0.4622\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6613, 0.6170\rangle \\ \langle 0.4490, 0.4194\rangle\end{pmatrix}$

Step-5.1: The collective aggregated matrix presented in Table 4 was recalculated for each attribute ${\mathfrak{x}^{\mathfrak{b}}}_j$ using the weight vector $p^{\gamma} = \{0.2055, 0.1577, 0.2189, 0.1909, 0.2270\}$. These weights were determined using Equation (37). The recalculations were performed by applying Equation (22) of the LDFWHM aggregation operators, respectively. The values obtained are shown in the Table 6.

Table 6. Information aggregated via proposed operators

Alternative	LDFWHM operator	LDFWDHM operator
$\mathfrak{v}^{\mathfrak{g}}{}_1$	$\begin{pmatrix} \langle 0.5184,\ 0.6817 \rangle \\ \langle 0.4710,\ 0.3815 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5561,\ 0.6513 \rangle \\ \langle 0.4882,\ 0.3656 \rangle \end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_2$	$\begin{pmatrix} \langle 0.5602,\ 0.5927 \rangle \\ \langle 0.3664,\ 0.4711 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6005,\ 0.5593 \rangle \\ \langle 0.3893,\ 0.4529 \rangle \end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_3$	$\begin{pmatrix} \langle 0.6266,\ 0.6588 \rangle \\ \langle 0.4686,\ 0.4056 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6483,\ 0.6244 \rangle \\ \langle 0.4857,\ 0.3857 \rangle \end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_4$	$\begin{pmatrix} \langle 0.5959,\ 0.6498 \rangle \\ \langle 0.4938,\ 0.3487 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6223,\ 0.6222 \rangle \\ \langle 0.5091,\ 0.3329 \rangle \end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}{}_5$	$\begin{pmatrix} \langle 0.6127,\ 0.6467 \rangle \\ \langle 0.4218,\ 0.4079 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6286,\ 0.6193 \rangle \\ \langle 0.4462,\ 0.3911 \rangle \end{pmatrix}$

Step-5.2: The collective aggregated matrix presented in Table 5 was recalculated for each attribute $\mathfrak{x}^{\mathfrak{d}}{}_j$ using the weight vector $p^{\gamma} = \{0.2044,\ 0.1389,\ 0.2251,\ 0.1929,\ 0.2387\}$. These weights were determined using Equation (37). The recalculations were performed by applying Equation (31) of the LDFWDHM aggregation operators, respectively. The calculated values are displayed in Table 6.

Step-6: Table 7 presents the overall score function values for the LDFWHM, and LDFWDHM aggregation operators.

Table 7. Scores of $\mathfrak{v}^{\mathfrak{g}}{}_i$ $(i = 1,\ \dots,\ 5)$ are as follows

Alternatives	LDFWHM operator	LDFDWHM operator
$\mathfrak{v}^{\mathfrak{g}}{}_1$	$S(\mathfrak{v}^{\mathfrak{g}}{}_1) = -0.0375$	$S(\mathfrak{v}^{\mathfrak{g}}{}_1) = 0.0137$
$\mathfrak{v}^{\mathfrak{g}}{}_2$	$S(\mathfrak{v}^{\mathfrak{g}}{}_2) = -0.0688$	$S(\mathfrak{v}^{\mathfrak{g}}{}_2) = -0.0112$
$\mathfrak{v}^{\mathfrak{g}}{}_3$	$S(\mathfrak{v}^{\mathfrak{g}}{}_3) = 0.0154$	$S(\mathfrak{v}^{\mathfrak{g}}{}_3) = 0.0619$
$\mathfrak{v}^{\mathfrak{g}}{}_4$	$S(\mathfrak{v}^{\mathfrak{g}}{}_4) = 0.0456$	$S(\mathfrak{v}^{\mathfrak{g}}{}_4) = 0.0882$
$\mathfrak{v}^{\mathfrak{g}}{}_5$	$S(\mathfrak{v}^{\mathfrak{g}}{}_4) = -0.0098$	$S(\mathfrak{v}^{\mathfrak{g}}{}_5) = 0.0322$

Step-7: The rankings of the suggested AOs are displayed in Table 8 and shown by Figure 2.

Step-8: Based on Table 8, we conclude that $\mathfrak{v}^{\mathfrak{g}}{}_4$, which represents 3D printed recycled materials, is the preferred alternative for addressing emergencies and earthquakes.

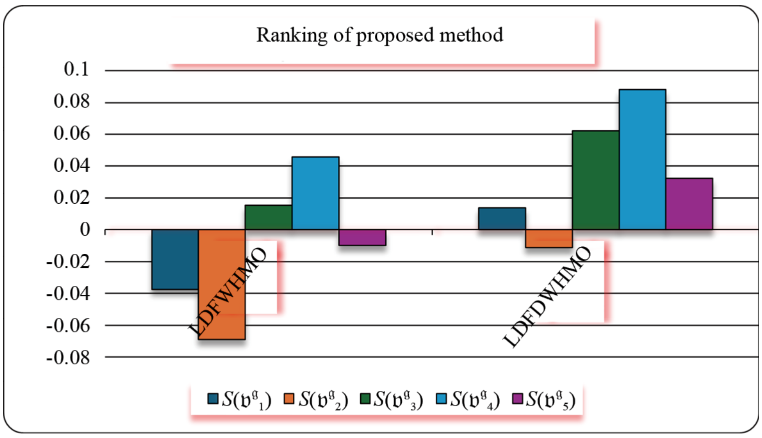

Figure 2. Ranking of alternatives using proposed method

Table 8. The alternatives are ranked in the following order

Operators	Order relation of $\mathfrak{vg}_i$ $(i = 1, 2, ..., 5)$	Most optimal
LDFWHM operator	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$
LDFWDHM operator	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$

6.3 *Sensitivity analysis*

The parameter holds significant importance in integrating information for deriving the final ranking outcomes. Since the LDFWHM and LDFWDHM operators include a modifiable parameter, varying its values reflects differing levels of risk tolerance among decision-makers in tackling MAGDM problems. Hence, examining the impact of parameter variations on the ranking outcomes becomes essential. To explore this, various values of the parameter k are applied to the LDFWHM and LDFWDHM operators, and the resulting rankings are presented in Tables 9 and 10 and shown by Figure 3. Based on these tables, the influence of k on decision outcomes can be summarized as follows:

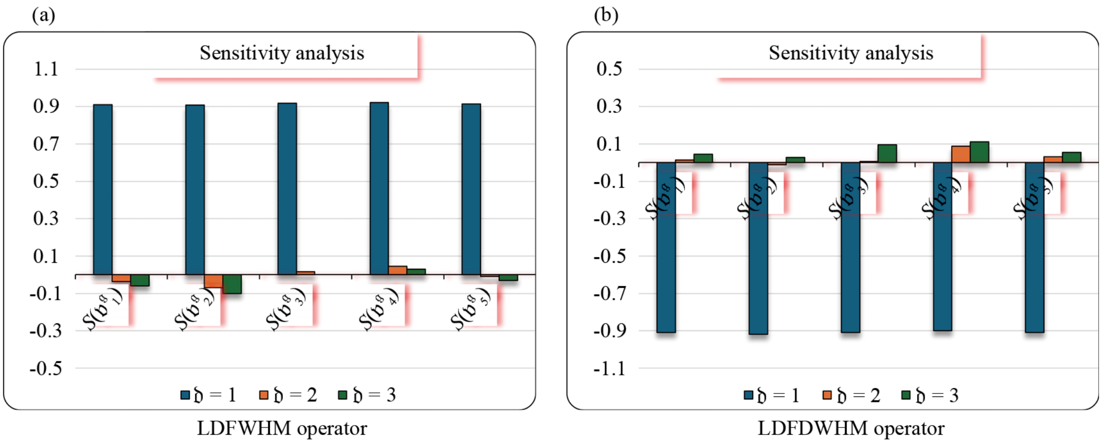

Figure 3. Sensitivity analysis of parameter k

Table 9. Ranking results using different values of k in the LDFWHM operator

Value of a parameter	$S(\mathfrak{vg}_1)$	$S(\mathfrak{vg}_2)$	$S(\mathfrak{vg}_3)$	$S(\mathfrak{vg}_4)$	$S(\mathfrak{vg}_5)$	Ranking order	Most optimal
$k = 1$	0.9110	0.9080	0.9190	0.9220	0.9150	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$
$k = 2$	-0.0375	-0.0688	0.0154	0.0456	-0.0098	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$
$k = 3$	-0.0600	-0.1000	0.0000	0.0300	-0.03000	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$

Table 10. Ranking results using different k values in the LDFDWHM operator

Value of a parameter	$S(\mathfrak{vg}_1)$	$S(\mathfrak{vg}_2)$	$S(\mathfrak{vg}_3)$	$S(\mathfrak{vg}_4)$	$S(\mathfrak{vg}_5)$	Ranking order	Most optimal
$k = 1$	-0.9100	-0.9200	-0.9100	-0.9000	-0.9100	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$
$k = 2$	0.0137	-0.0112	0.0619	0.0882	0.0322	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$
$k = 3$	0.0460	0.0280	0.0970	0.1120	0.0550	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$

Assigning different values to k leads to varying score values, but the rankings remain unchanged, with alternative $\mathfrak{v}^{\mathfrak{g}}_4$ consistently emerges as the best option. This variation stems from the different values of the parameter, which influence how attribute correlations are considered. For instance, setting $k = 2$ accounts for correlations between two attributes, while $k = 3$ reflects the interrelation among three attributes. This illustrates the HM operator's strong robustness in handling information fusion tasks. By adjusting the parameter k, the HM operator can adapt and resemble the arithmetic or geometric mean operators. This flexibility allows decision-makers to evaluate generalized results in specific practical cases.

Different parameter values yield different scores and ranking outcomes, demonstrating that k effectively regulates the decision-making process. It can be interpreted as an expression of evaluators' risk preferences, enabling them to select a suitable value for k based on their individual attitudes toward risk.

7. Comparison

This section presents a comparative analysis of our proposed method with established approaches, specifically the Linear Diophantine Fuzzy Weighted Average (LDFWA) aggregation operator, the LDFWG operator and q-ROF Hamy Mean (q-ROFHM) operator.

7.1 *Comparison with LDFWA aggregation operator*

This subsection presents a comparison between our proposed method and the LDFWA operator [63]. The LDFWA aggregation operators were utilized to process our input data, as presented in Tables 1-3, the expert matrices obtained through the existing LDFWA aggregation operators are shown in Table 11.

Table 11. Use the LDFWA operator to integrate the decision matrix

	$\mathfrak{x}^{\mathfrak{d}}_1$	$\mathfrak{x}^{\mathfrak{d}}_2$	$\mathfrak{x}^{\mathfrak{d}}_3$	$\mathfrak{x}^{\mathfrak{d}}_4$	$\mathfrak{x}^{\mathfrak{d}}_5$
$\mathfrak{v}^{\mathfrak{g}}_1$	(⟨0.5555, 0.4667⟩, ⟨0.5909, 0.2694⟩)	(⟨0.6049, 0.7873⟩, ⟨0.4641, 0.4000⟩)	(⟨0.4355, 0.5978⟩, ⟨0.5334, 0.3347⟩)	(⟨0.7379, 0.6895⟩, ⟨0.4422, 0.3855⟩)	(⟨0.5210, 0.6867⟩, ⟨0.4702, 0.3952⟩)
$\mathfrak{v}^{\mathfrak{g}}_2$	(⟨0.6494, 0.6279⟩, ⟨0.4695, 0.3951⟩)	(⟨0.5864, 0.6407⟩, ⟨0.4723, 0.3681⟩)	(⟨0.6884, 0.5910⟩, ⟨0.2963, 0.4650⟩)	(⟨0.6000, 0.4730⟩, ⟨0.3479, 0.5185⟩)	(⟨0.6109, 0.3850⟩, ⟨0.4256, 0.4718⟩)
$\mathfrak{v}^{\mathfrak{g}}_3$	(⟨0.6590, 0.6386⟩, ⟨0.3725, 0.4790⟩)	(⟨0.6458, 0.5558⟩, ⟨0.5016, 0.3965⟩)	(⟨0.7495, 0.5017⟩, ⟨0.5909, 0.3036⟩)	(⟨0.7288, 0.5185⟩, ⟨0.4778, 0.3758⟩)	(⟨0.5389, 0.7660⟩, ⟨0.5113, 0.3464⟩)
$\mathfrak{v}^{\mathfrak{g}}_4$	(⟨0.7908, 0.5918⟩, ⟨0.4716, 0.3193⟩)	(⟨0.7667, 0.5859⟩, ⟨0.4641, 0.3266⟩)	(⟨0.5531, 0.4336⟩, ⟨0.5202, 0.3334⟩)	(⟨0.5713, 0.5736⟩, ⟨0.5437, 0.3542⟩)	(⟨0.4680, 0.8366⟩, ⟨0.5655, 0.2990⟩)
$\mathfrak{v}^{\mathfrak{g}}_5$	(⟨0.5461, 0.5990⟩, ⟨0.4815, 0.3232⟩)	(⟨0.6807, 0.6527⟩, ⟨0.4403, 0.3334⟩)	(⟨0.5699, 0.5978⟩, ⟨0.4795, 0.3904⟩)	(⟨0.7077, 0.5825⟩, ⟨0.4114, 0.4594⟩)	(⟨0.6779, 0.5886⟩, ⟨0.4648, 0.4091⟩)

We recalculated the collective aggregated matrix from Table 11, for each attribute $\mathfrak{x}^{\mathfrak{d}}_j$, using the weight vector $p^{\gamma} = \{0.200,\ 0.136,\ 0.231,\ 0.191,\ 0.242\}$. The LDFWA aggregation operators where applied in this process. The outcomes of these calculation are shown in Table 12. Table 13 presents the overall score function values for the LDFWA aggregation operators. Table 14 and Figure 4 shows the ranking results for the existing aggregation operators.

Table 12. Information aggregated via the LDFWA operator

Alternatives	LDFWA operator
v^{Ξ}_1	(⟨0.5745, 0.6277⟩ ⟨0.5058, 0.3512⟩)
v^{Ξ}_2	(⟨0.6330, 0.5225⟩ ⟨0.4000, 0.4467⟩)
v^{Ξ}_3	(⟨0.6712, 0.5953⟩ ⟨0.4993, 0.3709⟩)
v^{Ξ}_4	(⟨0.6363, 0.5947⟩ ⟨0.5198, 0.3247⟩)
v^{Ξ}_5	(⟨0.6385, 0.6000⟩ ⟨0.4586, 0.3839⟩)

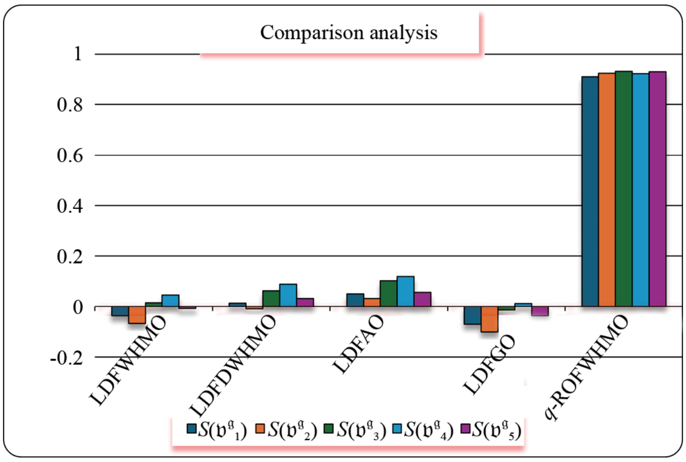

Figure 4. Comparison of presented technique with previous methods

Table 13. The score index of alternatives $\mathfrak{vg}_i$ $(i = 1, 2, ..., 5)$

Alternatives	LDFWA operator	LDFWG operator
$\mathfrak{vg}_1$	$S(\mathfrak{vg}_1) = 0.0507$	$S(\mathfrak{vg}_1) = -0.0720$
$\mathfrak{vg}_2$	$S(\mathfrak{vg}_2) = 0.0319$	$S(\mathfrak{vg}_2) = -0.1030$
$\mathfrak{vg}_3$	$S(\mathfrak{vg}_3) = 0.1021$	$S(\mathfrak{vg}_3) = -0.0150$
$\mathfrak{vg}_4$	$S(\mathfrak{vg}_4) = 0.1183$	$S(\mathfrak{vg}_4) = 0.0116$
$\mathfrak{vg}_5$	$S(\mathfrak{vg}_4) = 0.0566$	$S(\mathfrak{vg}_5) = -0.0380$

Table 14. The ranking order of all alternatives

Operators	Ranking order of $\mathfrak{vg}_i$ $(i = 1, 2, ..., 5)$	Best option
LDFWA Operator	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$
LDFWG Operator	$\mathfrak{vg}_4 > \mathfrak{vg}_3 > \mathfrak{vg}_5 > \mathfrak{vg}_1 > \mathfrak{vg}_2$	$\mathfrak{vg}_4$

7.2 *Comparison with LDFWG aggregation operator*

In this subsection, we compare our proposed work with the LDFWG operator [63]. The LDFWG aggregation operators were utilized to process the input data, as illustrated in Tables 1-3, the expert matrices obtained through the existing LDFWG aggregation operators are shown in Table 15.

We recalculated the collective aggregated matrix from Table 15, for each attribute $\mathfrak{r}^{\mathfrak{d}}{}_j$, using the weight vector $p^{\gamma} = \{0.202, 0.166, 0.225, 0.191, 0.217\}$. The LDFWG aggregation operators where applied in this process. The outcomes obtained from these calculations are displayed in Table 16. Table 13 presents the overall score function values for the LDFWG aggregation operators.

The ranking results for the existing AOs are presented in Table 14 and its shown in Figure 4. Table 13 displays the comprehensive ranking results for the proposed and existing methods. Based on these results, it is concluded that $\mathfrak{vg}_4$, representing 3D printed recycled materials, is the optimal choice for emergency situations and earthquakes. The results presented in Table 17 indicate that the proposed method achieves the same ranking order as the existing approaches, highlighting its flexibility and dependability.

Table 15. Use the LDFWG operator to integrate the Decision Matrix

	$\mathfrak{x}^{\mathfrak{b}}_1$	$\mathfrak{x}^{\mathfrak{b}}_2$	$\mathfrak{x}^{\mathfrak{b}}_3$	$\mathfrak{x}^{\mathfrak{b}}_4$	$\mathfrak{x}^{\mathfrak{b}}_5$
$\mathfrak{v}^{\mathfrak{g}}_1$	$\begin{pmatrix}\langle 0.4264, 0.6446\rangle \\ \langle 0.5576, 0.3133\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5315, 0.8098\rangle \\ \langle 0.4594, 0.4141\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4041, 0.6109\rangle \\ \langle 0.5143, 0.3398\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6964, 0.7067\rangle \\ \langle 0.4235, 0.4004\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4798, 0.7306\rangle \\ \langle 0.3628, 0.4795\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}_2$	$\begin{pmatrix}\langle 0.5529, 0.6325\rangle \\ \langle 0.4650, 0.4082\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4782, 0.6768\rangle \\ \langle 0.4089, 0.4065\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.2799, 0.4695\rangle \\ \langle 0.6578, 0.4695\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5859, 0.5664\rangle \\ \langle 0.2766, 0.5771\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5978, 0.5317\rangle \\ \langle 0.3816, 0.5110\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}_3$	$\begin{pmatrix}\langle 0.6243, 0.7142\rangle \\ \langle 0.3542, 0.5202\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6275, 0.7687\rangle \\ \langle 0.4874, 0.4114\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7286, 0.5153\rangle \\ \langle 0.5576, 0.3398\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6251, 0.5771\rangle \\ \langle 0.4469, 0.4164\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4948, 0.7718\rangle \\ \langle 0.4548, 0.4114\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}_4$	$\begin{pmatrix}\langle 0.7236, 0.6049\rangle \\ \langle 0.4369, 0.3567\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.7604, 0.6000\rangle \\ \langle 0.4594, 0.3311\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5222, 0.5270\rangle \\ \langle 0.4677, 0.3930\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5383, 0.6698\rangle \\ \langle 0.5259, 0.3725\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.4345, 0.8463\rangle \\ \langle 0.5320, 0.3332\rangle\end{pmatrix}$
$\mathfrak{v}^{\mathfrak{g}}_5$	$\begin{pmatrix}\langle 0.5124, 0.7106\rangle \\ \langle 0.4229, 0.3593\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6433, 0.6858\rangle \\ \langle 0.4106, 0.3930\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.5655, 0.6109\rangle \\ \langle 0.3952, 0.4341\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6925, 0.6288\rangle \\ \langle 0.3965, 0.4641\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.6140, 0.6536\rangle \\ \langle 0.3979, 0.4424\rangle\end{pmatrix}$

Table 16. Information aggregated via the LDFWG operator

Alternatives	LDFWG operator
$\mathfrak{v}^{\mathfrak{g}}_1$	$(\langle 0.4924, 0.7032\rangle\ \langle 0.4584, 0.3917\rangle)$
$\mathfrak{v}^{\mathfrak{g}}_2$	$(\langle 0.5370, 0.6149\rangle\ \langle 0.3523, 0.4802\rangle)$
$\mathfrak{v}^{\mathfrak{g}}_3$	$(\langle 0.6153, 0.6811\rangle\ \langle 0.4564, 0.4214\rangle)$
$\mathfrak{v}^{\mathfrak{g}}_4$	$(\langle 0.5738, 0.6754\rangle\ \langle 0.4837, 0.3590\rangle)$
$\mathfrak{v}^{\mathfrak{g}}_5$	$(\langle 0.5993, 0.6581\rangle\ \langle 0.4040, 0.4209\rangle)$

Table 17. Comparison of proposed methods with existing methods

Aggregation operators	The score value of alternatives	Alternatives ranking
LDFA operator proposed in [63]	$S(\mathfrak{v}^{\mathfrak{g}}_1) = 0.0507$, $S(\mathfrak{v}^{\mathfrak{g}}_2) = 0.0319$, $S(\mathfrak{v}^{\mathfrak{g}}_3) = 0.1021$, $S(\mathfrak{v}^{\mathfrak{g}}_4) = 0.1183$, $S(\mathfrak{v}^{\mathfrak{g}}_5) = 0.0566$	$\mathfrak{v}^{\mathfrak{g}}_4 > \mathfrak{v}^{\mathfrak{g}}_3 > \mathfrak{v}^{\mathfrak{g}}_5 > \mathfrak{v}^{\mathfrak{g}}_1 > \mathfrak{v}^{\mathfrak{g}}_2$
LDFG operator proposed in [63]	$S(\mathfrak{v}^{\mathfrak{g}}_1) = -0.0720$, $S(\mathfrak{v}^{\mathfrak{g}}_2) = -0.1030$, $S(\mathfrak{v}^{\mathfrak{g}}_3) = -0.0150$, $S(\mathfrak{v}^{\mathfrak{g}}_4) = 0.0116$, $S(\mathfrak{v}^{\mathfrak{g}}_5) = -0.0380$	$\mathfrak{v}^{\mathfrak{g}}_4 > \mathfrak{v}^{\mathfrak{g}}_3 > \mathfrak{v}^{\mathfrak{g}}_5 > \mathfrak{v}^{\mathfrak{g}}_1 > \mathfrak{v}^{\mathfrak{g}}_2$
q-Rung Orthopair Fuzzy Weighted Hamy Mean (ROFWHM) operator proposed in [47]	$S(\mathfrak{v}^{\mathfrak{g}}_1) = 0.9111$, $S(\mathfrak{v}^{\mathfrak{g}}_2) = 0.9248$, $S(\mathfrak{v}^{\mathfrak{g}}_3) = 0.9327$, $S(\mathfrak{v}^{\mathfrak{g}}_4) = 0.9235$, $S(\mathfrak{v}^{\mathfrak{g}}_5) = 0.9317$	$\mathfrak{v}^{\mathfrak{g}}_3 > \mathfrak{v}^{\mathfrak{g}}_5 > \mathfrak{v}^{\mathfrak{g}}_4 > \mathfrak{v}^{\mathfrak{g}}_2 > \mathfrak{v}^{\mathfrak{g}}_1$
LDFWHM operator presented in this work ($k = 2$)	$S(\mathfrak{v}^{\mathfrak{g}}_1) = -0.0375$, $S(\mathfrak{v}^{\mathfrak{g}}_2) = -0.0688$, $S(\mathfrak{v}^{\mathfrak{g}}_3) = 0.0144$, $S(\mathfrak{v}^{\mathfrak{g}}_4) = 0.0476$, $S(\mathfrak{v}^{\mathfrak{g}}_5) = -0.0098$	$\mathfrak{v}^{\mathfrak{g}}_4 > \mathfrak{v}^{\mathfrak{g}}_3 > \mathfrak{v}^{\mathfrak{g}}_5 > \mathfrak{v}^{\mathfrak{g}}_1 > \mathfrak{v}^{\mathfrak{g}}_2$
LDFWDHMO presented in this work ($k = 2$)	$S(\mathfrak{v}^{\mathfrak{g}}_1) = 0.0137$, $S(\mathfrak{v}^{\mathfrak{g}}_2) = -0.0112$, $S(\mathfrak{v}^{\mathfrak{g}}_3) = 0.0619$, $S(\mathfrak{v}^{\mathfrak{g}}_4) = 0.0882$, $S(\mathfrak{v}^{\mathfrak{g}}_5) = 0.0322$	$\mathfrak{v}^{\mathfrak{g}}_4 > \mathfrak{v}^{\mathfrak{g}}_3 > \mathfrak{v}^{\mathfrak{g}}_5 > \mathfrak{v}^{\mathfrak{g}}_1 > \mathfrak{v}^{\mathfrak{g}}_2$

7.3 *Comparison with q-ROFHM operator based method*

Our method's effectiveness is confirmed by a comparison with Wang et al. [47] q-ROFHM-based method, showcasing its superior performance. To adapt the Wang et al. [47] method to our specific use case, we removed the reference parameters from Tables 1-3, as they were not relevant to our application. For a consistent comparison, we applied the same expert and criteria weights from our case study to the method of Wang et al. [47] and considered $q = 4$. Then aggregated these tables by using q-ROFWHM operator as detailed in the Equation (39) and the aggregated result shown in Table 18.

$$q\text{-ROFWHM}^{(\kappa)}(\eth_1, \eth_2, \ldots, \eth_{\mathfrak{n}}) = \left(\begin{array}{c} \left(1 - \prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \left(\prod_{j=1}^{k} \mu_{i_j}^{w_{ij}}\right)^{\frac{q}{k}}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{q}}, \\ \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \left(\prod_{j=1}^{k} (1 - \nu_{i_j}^{q})^{w_{ij}}\right)^{\frac{1}{k}}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{q}} \end{array} \right), \tag{39}$$

where $\eth_i$, $1 \le i \le n$ represent q-ROFNs.

By employing the q-ROFWHM operator, the overall evaluation results for the alternatives $\mathfrak{vg}_i$ $(i = 1 \text{ to } 5)$ are calculated as follows: $\mathfrak{vg}_1 = (0.9575,\ 0.3691)$, $\mathfrak{vg}_2 = (0.9630,\ 0.3211)$, $\mathfrak{vg}_3 = (0.9694,\ 0.3653)$, $\mathfrak{vg}_4 = (0.9644,\ 0.3666)$, $\mathfrak{vg}_5 = (0.9681,\ 0.3512)$.

Table 18. Collective matrix of q-ROFWHMO

	$\mathfrak{r}^{\mathfrak{b}}{}_1$	$\mathfrak{r}^{\mathfrak{b}}{}_2$	$\mathfrak{r}^{\mathfrak{b}}{}_3$	$\mathfrak{r}^{\mathfrak{b}}{}_4$	$\mathfrak{r}^{\mathfrak{b}}{}_5$
$\mathfrak{vg}_1$	(0.7735, 0.4773)	(0.8239, 0.6457)	(0.7477, 0.4735)	(0.8910, 0.5526)	(0.7884, 0.5797)
$\mathfrak{vg}_2$	(0.83846, 0.4893)	(0.7897, 0.5348)	(0.8029, 0.5266)	(0.8536, 0.4127)	(0.8436, 0.4187)
$\mathfrak{vg}_3$	((0.8577, 0.5736)	(0.8586, 0.6283)	(0.9025, 0.3969)	(0.8607, 0.4591)	(0.7939, 0.6075)
$\mathfrak{vg}_4$	(0.8710, 0.5447)	(0.9143, 0.4669)	(0.8085, 0.4232)	(0.8220, 0.5374)	(0.7681, 0.6768)
$\mathfrak{vg}_5$	(0.8068, 0.5768)	(0.8710, 0.54071)	(0.8276, 0.4735)	(0.8866, 0.4879)	(0.8596, 0.5253)

q-ROFN score function is applied to determine score values for $\mathfrak{vg}_i$ $(i = 1 \text{ to } 5)$: $S(\mathfrak{vg}_1) = 0.9111$, $S(\mathfrak{vg}_2) = 0.9248$, $S(\mathfrak{vg}_3) = 0.9327$, $S(\mathfrak{vg}_4) = 0.9235$, $S(\mathfrak{vg}_5) = 0.9317$ and its shown in Figure 4. Alternatives are ranked based on their score values: $\mathfrak{vg}_3 \succ \mathfrak{vg}_5 \succ \mathfrak{vg}_4 \succ \mathfrak{vg}_2 \succ \mathfrak{vg}_1$. From above ranking, it is clear that ranking results from q-ROFWHM and our proposed method are totally different. This change in the ranking is due to neglecting reference parameter.

7.4 *Analytical comparison with other aggregation operators*

The introduction of the LDFHM, LDFWHM, LDFDHM and LDFWDHM operators enriches the toolbox for LDF information aggregation. Their unique characteristics and advantages become clear when compared to other operators in the same LDF structure. The following Table 19 provides a succinct comparison.

Table 19. Comparison of proposed operators with other existing aggregation operators

Feature/Operator	LDFWA/LDFWG	LDFOWA/LDFOWG	LDFHM/LDFWHM and LDFDHM/ LDFWDHM (Proposed)
Core Function	Basic averaging/ geometric mean.	Averages/geometric mean based on ordered positions.	Captures correlations between multiple (k) arguments.
Consider Weights?	LDFWA/WG: Yes (on arguments).	Yes (on ordered positions).	LDFWHM/LDFWDHM: Yes (on arguments in combinations).
Interrelationship Capture	None. Treats all arguments as independent.	None. Considers only the order, not relationships between values.	Yes, explicitly. Models the synergy between any k input arguments.
Parameter (s)	None.	None (besides the weight vector).	Parameter k. Control the granularity of interaction.
Flexibility	Low. A fixed, fundamental operation.	Medium. Allows emphasis on specific ordered positions (e.g., “most important”).	High. By varying k, it can model different scenarios from pessimistic to optimistic.
When to Use	When attributes are considered independent and no specific synergy is expected.	When the decision-maker’s attitude (e.g., optimism, pessimism) is more important than raw data.	When complex interdependencies among attributes are suspected or known to exist.

7.5 *Limitations and shortcomings of the proposed method*

While the proposed Linear Diophantine Fuzzy Hamy Mean operators demonstrate significant advantages in handling uncertainty and modeling interrelationships among criteria, it is important to acknowledge their limitations. This helps to delineate the scope of applicability of the framework and to motivate future research. In particular, the LDFHM-based approach may not perform optimally under the following conditions.

1. Computational Complexity with Large-Scale Data

• Limitation: The LDFHM operator, by its mathematical nature, requires the evaluation of combinations of the input arguments. For a problem with n criteria and a parameter k (the number of interrelated criteria considered), the number of required computations grows combinatorially, i.e., on the order of $\binom{n}{k}$. While this is manageable for moderate-sized problems (e.g., $n = 5$-10 criteria), it may lead to significant computational overhead for problems with a very large number of criteria (e.g., $n > 20$) or alternatives.

• Context of suboptimal performance: The method can become computationally intensive and less practical in real-time decision-making systems, big data analytics, or complex systems engineering applications involving hundreds of interdependent factors.

2. Sensitivity to Parameter Selection

• Limitation: The performance and behavior of the LDFHM operator are highly dependent on the choice of the parameter k (where $1 \leq k \leq n$), which determines the degree of interrelationship captured. If $k = 1$, the operator degenerates to a simple weighted averaging operator and fails to model interrelationships. If $k = n$, it considers all criteria simultaneously, which may over-smooth the outcomes and dilute the influence of strong individual performances.

• Context of suboptimal performance: In situations where the decision-maker has no prior knowledge or intuition regarding the level of interdependence among criteria, an inappropriate choice of k may lead to misleading aggregation results. Thus, the method requires careful tuning of k, which can introduce a degree of subjectivity.

3. Requirement for Numerical Input

• Limitation: The proposed formulation is developed for numerical LDF data and does not directly handle purely qualitative, textual, or categorical information.

• Context of suboptimal performance: In comprehensive decision problems where some crucial criteria are best expressed in narrative or linguistic form, a pre-processing step is needed to transform such qualitative judgments into LDF numbers. This conversion may be lossy and can introduce additional bias into the evaluation process.

4. Assumption of Homogeneous Interrelationship

• Limitation: The underlying Hamy Mean operator assumes a homogeneous and symmetric interrelationship among the selected k criteria. It does not distinguish between different strengths or types of interactions.

• Context of suboptimal performance: In complex systems where criteria exhibit heterogeneous and asymmetric dependencies, the LDFHM may be less suitable than more flexible approaches, such as those based on the Choquet integral or fuzzy measures, which can explicitly model varying interaction strengths.

These limitations do not diminish the value of the proposed framework; rather, they delineate clear avenues for future research. Promising directions include the development of fast approximation algorithms for large-scale problems, data-driven or theoretical guidelines for selecting the optimal parameter k, extensions to accommodate heterogeneous interrelationships, and hybrid models capable of integrating qualitative information directly within the LDF-based aggregation process.

8. Conclusion

LDFS had proven to be an effective tool for expressing DMk' evaluation values in MAGDM processes. To better capture decision-makers' evaluation information in complex MAGDM scenarios, this paper proposed a novel MAGDM approach based on new aggregation operators under LDFS. While intuitionistic fuzzy sets, PyFSs, and q-Rung orthopair FSs, were fundamental concepts in computational intelligence, they had limitations in modeling uncertainty and MCDM due to constraints on MD and NMD. LDFS offered a new approach to uncertainty, providing more flexibility than existing theories by incorporating reference parameters. This enabled decision experts to comprehensively address vague and uncertain information. Under these environments, we introduce several HMOs named as LDFHM, LDFWHM, LDFDHM, and LDFWDH operators, specifically designed to address MAGDM problems. Our focus was on creating tools that facilitate the selection of optimal alternatives when dealing with complex and uncertain DM scenarios. We explored the properties of these operators, including idempotency, monotonicity, and boundedness, to validate their reliability and applicability. To demonstrate the practical utility of these operators, we provided illustrative examples and conducted a case study focused on material selection for emergency shelters. This example showcased how the proposed operators enhance DM accuracy in real-world applications. Through this work, we concluded that the LDFWHM and LDFWDHM operators significantly improve the efficiency and reliability of decision-making processes, making them valuable tools for addressing MAGDM problems.

Future work

Future research will extend the proposed operators to Large-Scale Group Decision-Making (LSGDM) by integrating clustering algorithms to manage expert subgroups. We will also explore hybrid models combining LDFHM with TOPSIS/VIKOR for dynamic MAGDM problems. Additionally, optimizing computational efficiency for high-dimensional data and developing adaptive parameter-tuning methods for real-time applications remain key priorities. Finally, applications in emerging domains like sustainable supply chains and Artificial Intelligence (AI)-assisted healthcare diagnostics will be investigated.

Conflict of interest

The authors declare no competing financial interest.

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