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Multiplicative modular metric spaces and some fixed point results

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Abstract

This study aims to introduce a novel type of metric, defined as the w-multiplicative modular metric space, which becomes an expanded idea that includes two metric spaces. We present and demonstrate some of this new space's topological properties. We also prove the Banach and Kannan contraction mapping theorems. Finally, the applicability of the obtained results is also examined.

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Keywords: Multiplicative metric spaces; Modular metric spaces; Fixed point

1 Introduction and preliminaries

The study of fixed points (FPs) of mappings with certain contractive properties is an effective method for answering several scientific challenges across different mathematical topics [1–3]. This technique is crucial for computational algorithms in physics and other disciplines associated with the natural sciences and engineering. Banach established the Banach [4] contraction (Cn.) principle in 1922. Due to its simple structure, it has been utilized to address existence problems across various areas of mathematical analysis.

A metric space is a collection of items that is endowed with a distance function that fulfills certain criteria, thereby allowing for the measurement of distances between each element. Metric spaces are fundamental in several mathematical fields, as they provide a structure for studying notions like compactness, continuity, and convergence. Mathematicians can enhance their comprehension of the topological and geometric characteristics of mathematical objects through the exploration of metric spaces. These spaces allow them to grasp the structure and attributes of sets that have a certain distance function. This research promotes the development of rigorous mathematical theories and their practical applications in various fields, including topology, geometry, and analysis.

Various generalizations of classical metric spaces may be found in the literature. Gahler [5] conducted research on the notion of 2-metric spaces. Dhage [6] expanded upon the findings in [5] and established the concept of D-metric spaces. The authors asserted that their findings extended the idea of metric spaces. Mustafa and Sims [7] detected topological errors in the characteristics of D-metric spaces initially proposed by Dhage [6].

Mustafa and Sims [8] proposed the concept of G-metric spaces as an extension of classical metric spaces to address the limitations of D-metric spaces. They also developed several FP theorems in this context.

In 2010, Chistyakov [9] presented the concept of modular metric spaces, known as parameterized metric spaces with a time λ parameter.

Bashirov et al. [10] examined the notion of multiplicative calculus and formulated a basic theorem pertaining to it. Moreover, they illustrated the applicability of multiplicative calculus through several compelling applications. Multiplicative calculus provides a straightforward and direct approach for determining the derivative of the product and quotient of two functions [10]. Multiplicative differential equations have been shown to be more suitable than ordinary differential equations for investigating certain issues in economics and finance [11]. Furthermore, they established the multiplicative distance between two non-negative real numbers and between two positive square matrices, grounded in the concept of the multiplicative absolute value function. This established the basis for multiplicative metric spaces. In recent years, several scientists have been drawn to it due to its practical simplicity and support of Newtonian calculus. Bashirov et al. [11] accomplished the mathematical modeling of problems related to multiplicative differential equations. Florack and van Assen [12] presented instances of the application of multiplicative calculus in biological image processing.

Ozavsar and Cevikel [13] formulated a similar version of the Banach Cn. principle in the context of multiplicative metric spaces. Furthermore, they acquired the topological properties of the multiplicative metric space.

In this paper, some fixed-point theorems are stated and proved in w-multiplicative modular metric spaces (w-MMMS). These spaces are generalized metric spaces in which the distance between points is multiplicative rather than additive. The concept of w-MMMS is new, and a significant study can be conducted on the subject. Potential applications of these spaces include modeling real-world phenomena with multiplicatively measured distances, such as in economics and finance, creating novel algorithms for optimization issues, and understanding the geometry of specific spaces.

Frechet [14] provided the following definition of metric spaces:

Definition 1 Consider a non-empty set χ and a function $\partial : \chi \times \chi \to [0,\infty)$ that meets the subsequent criteria for every $m, n, \gamma \in \chi$,

- $\partial(m,n) \geq 0$,
- $\partial(m,n) = 0$ if and only if $m = n$,
- $\partial(m,n) = \partial(n,m)$,
- $\partial(m,n) \leq \partial(m,\gamma) + \partial(\gamma,n)$.

A metric space is denoted by (χ,∂) and ∂ is referred to as a metric on χ.

The concept of a modular metric space was initially put forward by Chistyakov in [9].

Definition 2 [9] A function $w : (0,\infty) \times \chi \times \chi \to [0,\infty]$ is a (metric) modular on a set χ meeting the following three requirements for any $m, n, \gamma \in \chi$ and $\lambda, \mu > 0$:

1. $m = n$ if and only if $w(\lambda,m,n) = 0$,
2. $w(\lambda,m,n) = w(\lambda,n,m)$,
3. $w(\lambda+\mu,m,n) \leq w(\lambda,m,\gamma) + w(\mu,n,\gamma)$.

Then, (χ, w) is called a modular metric space.

For all $\lambda > 0$ and $m, n \in \chi$, the functions $w : (0, \infty) \times \chi \times \chi \to [0, \infty]$ will be expressed as $w(\lambda, m, n) = w_\lambda(m, n)$.

Lemma 3 [9] *Let (χ, w) be a modular metric space. The function $\lambda \to w_\lambda(m, n)$ is nonincreasing for each $m, n \in \chi$.*

The following is how multiplicative metric spaces were introduced by Bashirov et al. [10] in 2008.

Definition 4 [10] Consider a non-empty set χ and a function $\partial^* : \chi \times \chi \to [1, \infty)$ that meets the subsequent criteria for all $m, n, \gamma \in \chi$,

1. $\partial^*(m, n) \geq 1$,
2. $\partial^*(m, n) = 1$ if and only if $m = n$,
3. $\partial^*(m, n) = \partial^*(n, m)$,
4. $\partial^*(m, n) \leq \partial^*(m, \gamma).\partial^*(\gamma, n)$.

A multiplicative metric space is denoted as (χ, ∂^*) and ∂^* is referred to as a multiplicative metric on χ.

Remark 5 If the logarithm of Condition 4 of Definition 4 is taken, the multiplicative metric space is equivalent to the (standard) metric space. Indeed, this situation is easy to see. Since the logarithm function is a nondecreasing, we obtain

$$\begin{aligned}
\partial(m, n) &= \ln\left(\partial^*(m, n)\right) \\
&\leq \ln\left(\partial^*(m, \gamma)\,\partial^*(\gamma, n)\right) \\
&= \ln\left(\partial^*(m, \gamma)\right) + \ln\left(\partial^*(\gamma, n)\right) \\
&= \partial(m, \gamma) + \partial(\gamma, n).
\end{aligned}$$

2 Main results

The following definitions are introduced.

Definition 6 Let χ be a non-empty set and $w : (0, \infty) \times \chi \times \chi \to [1, \infty]$ be a function that satisfies the subsequent properties for all $m, n, \gamma \in \chi$ and $\lambda, \mu > 0$:

(i) $w_\lambda(m, n) = 1$ if and only if $m = n$,
(ii) $w_\lambda(m, n) = w_\lambda(n, m)$,
(iii) $w_{\lambda.\mu}(m, n) \leq w_\lambda(m, \gamma).w_\lambda(\gamma, n)$

Then, (χ, w) is referred to as a w-MMMS.

Remark 7 When the logarithm function is applied to Condition *(iii)* of the w-MMMS, the modular metric space is not obtained. Therefore, it makes our new structure more valuable.

Example 8 Let $\chi = [1,\infty)$ and $\lambda > 0$. Define the function $w : (0,\infty) \times \chi \times \chi \to [1,\infty)$ as follows:

$$w_\lambda(m,n) = \begin{cases} 1, & m = n \\ \frac{mn}{\lambda}, & m \neq n \end{cases}$$

Then, (χ, w) is a w-MMMS; however, it is not a modular metric space.

Example 9 Let (χ, ∂) be a metric space and the function $w : (0,\infty) \times \chi \times \chi \to [1,\infty)$ be defined as

$$w_\lambda(m,n) = e^{\frac{\partial(m,n)}{\lambda}}$$

for every $m, n \in \chi$ and $\lambda > 1$. Then, (χ, w) is a w-MMMS. The first two conditions of the w-MMMS properties can be easily observed. It will be sufficient to demonstrate only the triangle inequality condition. For $m, n, \gamma \in \chi$ and $\lambda, \mu > 1$, we obtain

$$\begin{aligned} w_{\lambda\mu}(m,n) &= e^{\frac{\partial(m,n)}{\lambda\mu}} \\ &\leq e^{\frac{\partial(m,\gamma)+\partial(\gamma,n)}{\lambda\mu}} \\ &\leq e^{\frac{\partial(m,\gamma)}{\lambda}} e^{\frac{\partial(\gamma,n)}{\mu}} \\ &= w_\lambda(m,\gamma)\, w_\mu(\gamma,n). \end{aligned}$$

It should be noted that this example is not a modular metric space.

Example 10 Let $\chi = \mathbb{N}$ and $\lambda > 1$. Define the function $w : (0,\infty) \times \chi \times \chi \to [1,\infty)$ as follows:

$$w_\lambda(m,n) = \begin{cases} 1, & m = n \\ e^{\frac{m+n}{\lambda}}, & m \neq n \end{cases}$$

Then, (χ, w) is a w-MMMS. However, it is not a modular metric space.

Lemma 11 *Let $\lambda > 0$ and (χ, w) be a w-MMMS. Then, for each $m, n \in \chi$, the function $\lambda \to w_\lambda(m,n)$ is nonincreasing.*

Proof Let $0 < \frac{\lambda}{\lambda_0} < \lambda$. For each $m, n \in \chi$, we have

$$w_\lambda(m,n) \leq w_{\lambda_0}(m,m)\, w_{\frac{\lambda}{\lambda_0}}(m,n) \leq w_{\frac{\lambda}{\lambda_0}}(m,n). \qquad \square$$

2.1 Topology of w-multiplicative modular metric space (w-MMMS)

Definition 12 Let $\lambda > 0$ and (χ, w) be a w-MMMS. A w-open ball $B_\lambda(\alpha, \epsilon)$ with radius $\epsilon > 1$ and center $\alpha \in \chi$ is defined as

$$B_\lambda(\alpha, \epsilon) = \{\beta \in \chi : w_\lambda(\alpha, \beta) < \epsilon\}.$$

Similarly, a w-closed ball can be defined as:

$$B_\lambda(\alpha, \epsilon) = \{\beta \in \chi : w_\lambda(\alpha, \beta) \leq \epsilon\}.$$

Definition 13 Let $\lambda > 0$ and (χ, w) be a w-MMMS. Take $A \subset \chi$. If there is $\epsilon > 1$ such that $B_\lambda(\alpha, \epsilon) \subset A$, then $\alpha \in A$ is considered a w-interior point of A. The collection of all w-interior points of A is known as the w-interior of A, and it is represented by $int(A)$.

Definition 14 When $\lambda > 0$ and $\epsilon > 1$ exist for each element $\alpha \in \chi$ such that $B_\lambda(\alpha, \epsilon) \subseteq A$, $A \subseteq \chi$ is referred to as a w-open set in χ.

Lemma 15 *Let $\lambda > 0$ and (χ, w) be a w-MMMS. Each w-open ball of χ is a w-open set.*

Proof Take a w-open ball $B_\lambda(\alpha, \epsilon)$. Then,

$$\beta \in B_\lambda(\alpha, \epsilon) \Longrightarrow w_\lambda(\alpha, \beta) < \epsilon.$$

Suppose that there is $\frac{\lambda}{\lambda_0} < \lambda$ such that $w_\lambda(\alpha, \beta) \leq w_{\frac{\lambda}{\lambda_0}}(\alpha, \beta) < \epsilon$. Now, consider the ball $B_{\frac{\lambda}{\lambda_0}}(\beta, \epsilon_0)$ such that $\epsilon_0 < \frac{\epsilon}{w_{\lambda_0}(\alpha,\beta)}$. We assert that $B_{\frac{\lambda}{\lambda_0}}(\beta, \epsilon_0) \subseteq B_\lambda(\alpha, \epsilon)$. If $\gamma \in B_{\frac{\lambda}{\lambda_0}}(\beta, \epsilon_0)$, then $w_{\frac{\lambda}{\lambda_0}}(\gamma, \beta) < \epsilon_0$. Therefore,

$$\begin{aligned}
w_\lambda(\alpha, \gamma) &\leq w_{\lambda_0}(\alpha, \beta) . w_{\frac{\lambda}{\lambda_0}}(\beta, \gamma) \\
&\leq w_{\lambda_0}(\alpha, \beta) . \epsilon_0 \\
&< w_{\lambda_0}(\alpha, \beta) . \frac{\epsilon}{w_{\lambda_0}(\alpha, \beta)} \\
&= \epsilon.
\end{aligned}$$

Consequently, $\gamma \in B_\lambda(\alpha, \epsilon)$ and hence $B_{\frac{\lambda}{\lambda_0}}(\beta, \epsilon_0) \subseteq B_\lambda(\alpha, \epsilon)$. □

Definition 16 Let (χ, w) be a w-MMMS. A point $\alpha \in \chi$ is said to be w-limit point of $S \subset \chi$ if and only if $[B_\lambda(\alpha, \epsilon) \setminus \{\alpha\}] \cap S \neq \varnothing$ for every $\epsilon > 1$. The set of all w-limit points of the set S is denoted by S'.

Definition 17 Consider a w-MMMS (χ, w). A set $S \subset \chi$ is said to be w-closed in (χ, w) if S contains all of its w-limit points.

Definition 18 Let (χ, w_χ) and (Y, w_Y) be two w-MMMS and $T: \chi \to Y$ be a function. If T satisfies the criterion that, for every $\epsilon > 1$, there exists $\delta > 1$ such that

$$T(B_\lambda(\alpha, \delta)) \subset B_\lambda(T\alpha, \epsilon),$$

then we say that T w-continuous at $\alpha \in \chi$.

Definition 19 Consider a w-MMMS (χ, w) with a sequence $\{\alpha_n\}$ in χ and an element $\alpha \in \chi$. A sequence $\{\alpha_n\}$ is said to be w-convergent to α, represented by $\alpha_n \to \alpha$ $(n \to \infty)$, if for any w-open ball $B_\lambda(\alpha, \epsilon)$, there exists a natural number N such that $n \geq N$ implies $\alpha_n \in B_\lambda(\alpha, \epsilon)$.

Lemma 20 *Let $\{\alpha_n\}$ be a sequence within χ, and let (χ, w) be a w-MMMS. Then,* $\lim_{n\to\infty} w_\lambda(\alpha_n, \alpha) = 1$, *for all $\lambda > 0$ if and only if $\alpha_n \to \alpha$ as $n \to \infty$.*

Proof Assume that the $\lim_{n\to\infty} w_\lambda(\alpha_n,\alpha) = 1$, for all values of $\lambda \geq 0$. Let $\lambda > 0$ and $\epsilon > 1$. There is a natural number n_0 such that the limit of the function $w_\lambda(\alpha_n,\alpha) < \epsilon$ for all $n \geq n_0$. Therefore, it can be deduced that α_n belongs to the set $B_\lambda(\alpha,\epsilon)$. Therefore, the sequence $\{\alpha_n\}$ w-converges to α. On the other hand, if $\{\alpha_n\}$ w-converges to α, then for any positive λ and $\epsilon > 1$, there exists a natural number n_0 such that $\alpha_n \in B_\lambda(\alpha,\epsilon)$ for all $n \geq n_0$. This implies that $w_\lambda(\alpha_n,\alpha) < \epsilon$, for any $n \geq n_0$. Thus, $\lim_{n\to\infty} w_\lambda(\alpha_n,\alpha) = 1$. □

Lemma 21 *Let (χ,w) be a w-MMMS and $\{\alpha_n\}$ be a sequence in χ. If the sequence $\{\alpha_n\}$ is a w-convergent, then the limit point is unique.*

Proof Let α and β be the elements of χ such that $\alpha_n \to \alpha$ and $\alpha_n \to \beta$ as $n \to \infty$. In other words, for each $\epsilon > 1$, there is a N such that, for all $n \geq N$, $w_{\lambda\frac{1}{2}}(\alpha_n,\alpha) < \sqrt{\epsilon}$ and $w_{\lambda\frac{1}{2}}(\alpha_n,\beta) < \sqrt{\epsilon}$. Then, we have that

$$w_\lambda(\alpha,\beta) \leq w_{\lambda\frac{1}{2}}(\alpha,\alpha_n).w_{\lambda\frac{1}{2}}(\alpha_n,\beta) < \epsilon.$$

Since ϵ is arbitrary, $w_\lambda(\alpha,\beta) = 1$. The statement is that α is equal to β. □

Definition 22 Let $\{\alpha_n\}$ be a sequence within χ, and let (χ,w) be an w-MMMS. Then, $\{\alpha_n\}$ is said to be a w-Cauchy sequence if and only if $w_\lambda(\alpha_n,\alpha_m) \to 1$ as $n,m \to \infty$.

Definition 23 Let (χ,w) be a w-MMMS. If any Cauchy sequence in χ w-converges to the point of χ, then χ is called w-complete space.

Definition 24 Let (χ,w) be a w-MMMS. Define

$$\tau_w = \{A \subseteq \chi : \alpha \in A \Leftrightarrow \exists\lambda > 0, \epsilon > 1 \text{ such that } B_\lambda(\alpha,\epsilon) \subseteq A\}.$$

Then, (χ,τ_w) is a topological space.

Theorem 25 *Let (χ,w) be a w-MMMS. Then, τ_w is Hausdorff.*

Proof Consider α, β as two different points of χ and take for all $\lambda,\mu > 0$. We have $w_\lambda(\alpha,\beta) > 1$. For some $r > 1$, put $w_\lambda(\alpha,\beta) = r$. Furthermore, we obtain $B_{\frac{\lambda}{\mu}}(\alpha,r^2) \cap B_\mu(\beta,\frac{1}{r}) = \emptyset$ for $B_{\frac{\lambda}{\mu}}(\alpha,r^2)$ and $B_\mu(\beta,\frac{1}{r})$. In other words, we get to a contradiction if there is an element γ such that $\gamma \in B_{\frac{\lambda}{\mu}}(\alpha,r^2) \cap B_\mu(\beta,\frac{1}{r})$. This means that

$$r = w_\lambda(\alpha,\beta) \leq w_{\frac{\lambda}{\mu}}(\alpha,\gamma).w_\mu(\gamma,\beta) < r$$

Consequently, τ_w is Hausdorff. □

Definition 26 Let (χ,w) be a w-MMMS. If for any $\alpha,\beta \in \chi$ and $\lambda \in (0,\infty)$, there exists a real constant $k \in [0,1)$ such that

$$w_\lambda(T\alpha,T\beta) \leq [w_\lambda(\alpha,\beta)]^k,$$

then a mapping $T : \chi \to \chi$ is called w-multiplicative modular Cn.

We now propose the Banach Cn. principle for w-MMMS, which is based on the concept of multiplicative modular Cn.

Theorem 27 *Consider a complete w-MMMS (χ, w) with a w-multiplicative modular Cn. $T : \chi \to \chi$. Following that, T has a unique FP.*

Proof Let α_0 be a point in χ. We now define a sequence $\{\alpha_n\}$ in χ such that $\alpha_n = T\alpha_{n-1}$ for $n = 1, 2, 3...$ According to the w-multiplicative modular Cn. property of T, we have

$$w_\lambda(\alpha_n, \alpha_{n+1}) \leq [w_\lambda(\alpha_{n-1}, \alpha_n)]^k \leq \cdots \leq [w_\lambda(\alpha_0, \alpha_1)]^{k^n}.$$

If we assume that $m, n \in \mathbb{N}$ such that $m > n$, we obtain

$$\begin{aligned} w_\lambda(\alpha_n, \alpha_m) &\leq w_{\lambda\frac{1}{m-n}}(\alpha_n, \alpha_{n+1})\, w_{\lambda\frac{1}{m-n}}(\alpha_{n+1}, \alpha_{n+2}) \ldots w_{\lambda\frac{1}{m-n}}(\alpha_{m-1}, \alpha_m) \\ &\leq \left[w_{\lambda\frac{1}{m-n}}(\alpha_0, \alpha_1)\right]^{k^n} \left[w_{\lambda\frac{1}{m-n}}(\alpha_0, \alpha_1)\right]^{k^{n+1}} \ldots \left[w_{\lambda\frac{1}{m-n}}(\alpha_0, \alpha_1)\right]^{k^{m-1}} \\ &= \left[w_{\lambda\frac{1}{m-n}}(\alpha_0, \alpha_1)\right]^{k^n + k^{n+1} + \cdots + k^{m-1}} \\ &\leq \left[w_{\lambda\frac{1}{m-n}}(\alpha_0, \alpha_1)\right]^{\frac{k^n}{1-k}}. \end{aligned} \tag{1}$$

Letting $n, m \to \infty$ in (1), we have

$$\lim_{n,m\to\infty} w_\lambda(\alpha_n, \alpha_m) = 1.$$

Thus, $\{\alpha_n\}$ is a w-Cauchy sequence in χ. By the completeness of χ, there is $\alpha \in \chi$ such that $\alpha_n \to \alpha$ for $n \to \infty$. Since

$$\begin{aligned} w_\lambda(T\alpha, \alpha) &\leq w_{\lambda\frac{1}{2}}(T\alpha, \alpha_n) . w_{\lambda\frac{1}{2}}(\alpha_n, \alpha) \\ &= w_{\lambda\frac{1}{2}}(T\alpha, T\alpha_{n-1}) . w_{\lambda\frac{1}{2}}(\alpha_n, \alpha) \\ &\leq \left[w_{\lambda\frac{1}{2}}(\alpha, \alpha_{n-1})\right]^k . w_{\lambda\frac{1}{2}}(\alpha_n, \alpha), \end{aligned}$$

we obtain $w_\lambda(T\alpha, \alpha) = 1$ for $n \to \infty$. That is, $T\alpha = \alpha$ and so this says that α is a FP of T.

In the final step, we prove that α is the unique FP of T. Assume that β is another FP of T. Then, we have $T\beta = \beta$. Since

$$w_\lambda(\alpha, \beta) = w_\lambda(T\alpha, T\beta) \leq [w_\lambda(\alpha, \beta)]^k,$$

we obtain a contradiction. This implies that α is the unique FP of T. □

We now provide w-MMMS proof of Kannan [15] type Cn., which is generalization of the Banach Cn. mapping.

Theorem 28 *Consider a complete w-MMMS (χ, w) and assume that the mapping $T : \chi \to \chi$ satisfies the Cn. condition*

$$w_\lambda(T\alpha, T\beta) \leq [w_\lambda(T\alpha,\alpha).w_\lambda(T\beta,\beta)]^k, \tag{2}$$

where $k \in [0, \frac{1}{2})$ is a constant. Following that, T has a unique FP.

Proof Choose $\alpha_0 \in \chi$. For a sequence $\{\alpha_n\}$ in χ such that

$$\alpha_n = T\alpha_{n-1} = T^n\alpha_0, n = 1, 2, 3, \dots$$

with (2), we have

$$\begin{aligned} w_\lambda(\alpha_n,\alpha_{n+1}) &= w_\lambda(T\alpha_{n-1}, T\alpha_n) \\ &\leq [w_\lambda(\alpha_{n-1},\alpha_n).w_\lambda(\alpha_{n+1},\alpha_n)]^k \end{aligned}$$

and

$$w_\lambda(\alpha_n,\alpha_{n+1}) \leq w_\lambda(\alpha_{n-1},\alpha_n)^{\frac{k}{1-k}} = w_\lambda(\alpha_{n-1},\alpha_n)^h,$$

where $h = \frac{k}{1-k}$.

For $m > n$,

$$\begin{aligned} w_\lambda(\alpha_n,\alpha_m) &\leq w_{\lambda\frac{1}{m-n}}(\alpha_n,\alpha_{n+1}).w_{\lambda\frac{1}{m-n}}(\alpha_{n+1},\alpha_{n+2})\dots w_{\lambda\frac{1}{m-n}}(\alpha_0,\alpha_1) \\ &\leq \left[w_{\lambda\frac{1}{m-n}}(\alpha_0,\alpha_1)\right]^{h^n+h^{n+1}+\dots+h^m} \\ &\leq \left[w_{\lambda\frac{1}{m-n}}(\alpha_0,\alpha_1)\right]^{\frac{h^n}{1-h}}. \end{aligned}$$

This means that $w_\lambda(\alpha_n,\alpha_m) \to 1$ as $n, m \to \infty$. Hence, $\{\alpha_n\}$ is a Cauchy sequence. The completeness of χ implies that there is a $\alpha \in \chi$ such that $\alpha_n \to \alpha$ as $n \to \infty$. Since

$$w_\lambda(\alpha, T\alpha) \leq w_{\lambda\frac{1}{2}}(\alpha,\alpha_n).w_{\lambda\frac{1}{2}}(\alpha_n, T\alpha),$$

letting $n \to \infty$, we get

$$w_\lambda(\alpha, T\alpha) \leq 1. \tag{3}$$

If $w_\lambda(\alpha, T\alpha) \neq 1$, then the inequality (3) is false. So, we obtain $\alpha = T\alpha$.

To demonstrate that α is a unique FP of T, we suppose that β is also an FP of T. Then, we have

$$\begin{aligned} w_\lambda(\alpha,\beta) &= w_\lambda(T\alpha, T\beta) \\ &\leq [w_\lambda(\alpha, T\alpha).w_\lambda(\beta, T\beta)]^k \\ &= [w_\lambda(\alpha,\beta)]^{2k}. \end{aligned}$$

This is a contradiction. So, we obtain $\alpha = \beta$. $\square$

3 Application

The aim of this section is to determine solutions to a system of linear equations using Theorem 27. Hence, we present the following theorem.

Theorem 29 *Let $\chi = \mathbb{R}^n$ be a w-MMMS with*

$$w_\lambda(\alpha, \beta) = e^{\frac{\partial(\alpha,\beta)}{\lambda}},$$

where $\alpha, \beta \in \chi$, $\lambda > 1$ and

$$\partial(\alpha, \beta) = \sum_{i=1}^{n} |\alpha_i - \beta_i|.$$

If

$$\sum_{i=1}^{n} \left|\delta_{ij}\right| \le k < 1, \text{ for all } j = 1, 2, \ldots, n,$$

then the system of n-linear equations

$$\begin{aligned}
\delta_{11}\alpha_1 + \delta_{12}\alpha_2 + \cdots + \delta_{1n}\alpha_n &= \beta_1 \\
\delta_{12}\alpha_2 + \delta_{22}\alpha_2 + \cdots + \delta_{2n}\alpha_n &= \beta_2 \\
&\vdots \\
\delta_{n1}\alpha_n + \delta_{n2}\alpha_2 + \cdots + \delta_{nn}\alpha_n &= \beta_n
\end{aligned} \tag{4}$$

has a unique solution.

Proof Since $\chi = \mathbb{R}^n$ with w-multiplicative modular metric is w-complete, we have to demonstrate that $T : \chi \to \chi$ defined by

$$T\alpha = A\alpha + B,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, $B = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, and

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is a w-multiplicative modular Cn. Since

$$\begin{aligned}
w_\lambda(T\alpha, T\beta) &= e^{\frac{1}{\lambda}\sum_{i=1}^{n}\left|\sum_{j=1}^{n}\alpha_{ij}(\alpha_j-\beta_j)\right|} \\
&\le e^{\frac{1}{\lambda}\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_{ij}||\alpha_i-\beta_j|} \\
&= \left(e^{\frac{1}{\lambda}\sum_{j=1}^{n}|\alpha_i-\beta_j|}\right)^{\sum_{i=1}^{n}|\alpha_{ij}|}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(e^{\frac{1}{\lambda}\sum_{j=1}^{n}|\alpha_i-\beta_j|}\right)^k \\
&= \left(e^{\frac{\partial(\alpha,\beta)}{\lambda}}\right)^k \\
&= [w_\lambda(\alpha,\beta)]^k,
\end{aligned}
$$

we can deduce that T is a w-Cn. mapping. According to Theorem 27, the linear equation system (4) has a unique solution. □

4 Conclusions

One of the most important research areas of recent years has been the study of FP theorems by obtaining various metric spaces. In this research, a new generalized metric space structure, called w-multiplicative modular metric space, is constructed. The topological properties of this structure are also defined. It is important to note that when the logarithmic function is applied to multiplicative metric spaces [10] and multiplicative S-metric spaces [16], standard metric and S-metric spaces are obtained, respectively. It is important to note that this is not the case for our space. As a result, the findings produced will not overlap with modular metric spaces. Many different FP theorems can be derived in this new space, and their application areas can be investigated.

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Author contributions

E.K., N.T., S.H. and N.M. wrote the main manuscript. All authors reviewed the manuscript.

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Ethical approval

Not applicable.

Competing interests

The authors declare no competing interests.

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