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DOUBLY WARPED PRODUCT SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD OF QUASI-CONSTANT CURVATURE

BY

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Abstract. We establish a sharp inequality for a doubly warped product submanifold in a Riemannian manifold of quasi-constant curvature.

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1. Introduction

The notion of a Riemannian manifold (M, g) of *quasi-constant curvature* was introduced by Chen and Yano [6] and it defines a Riemannian manifold whose curvature tensor R satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) &= a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +b[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &\quad +g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)], \end{aligned} \tag{1}$$

where a and b are scalar functions and T is a 1-form given by $g(X,P) = T(X)$, with P a fixed unit vector field. It is easy to see that if R is of the form (1), then the manifold is conformally flat. If $b = 0$, then the manifold is called to be a *space of constant curvature*.

An n-dimensional $(n > 2)$ non-flat Riemannian manifold M is said to be a *quasi-Einstein manifold* (see [1]) if its Ricci tensor S satisfies the condition $S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y)$, where α and β are scalar functions such that $\beta \neq 0$ and A is a non-zero 1-form denoted by $g(X,U) = A(X)$, for any vector field X and where U is a fixed unit vector field . It is easy to see that any Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f_1, f_2 be positive, differentiable functions on M_1 and M_2, respectively. The *doubly warped product* $M ={}_{f_2} M_1 \times_{f_1} M_2$ (see [10]) is the product manifold $M_1 \times M_2$ equipped with the metric $g = f_2^2 g_1 + f_1^2 g_2.$ More precisely, if $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are canonical projections, then the metric g is defined by $g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2.$ The functions f_1 and f_2 are called *warping functions*. If either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we get a warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we obtain a Riemannian product manifold. If neither f_1 nor f_2 is constant, then we have a non-trivial doubly warped product (see [10]).

For a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$, let D_1 and D_2 denote the distributions obtained from the vectors tangent to leaves and fibres, respectively.

Assume that $x : {}_{f_2} M_1 \times_{f_1} M_2 \to N$ is an isometric immersion of a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into a Riemannian manifold N. We denote by σ the second fundamental form of x and by $H_i = \frac{1}{n_i} trace\sigma_i$ the partial mean curvatures, where $trace\sigma_i$ is the trace of σ restricted to M_i and $n_i = \dim M_i$ $(i = 1, 2)$. The immersion x is called *mixed totally geodesic* if $\sigma(X, Z) = 0$, for any vector fields X and Z tangent to D_1 and D_2, respectively.

In [5], Chen proved the following result for a warped product submanifold of a Riemannian manifold of constant sectional curvature:

Theorem 1.1. *Let $x : M_1 \times_f M_2 \to N(c)$ be an isometric immersion of an n-dimensional warped product $M_1 \times_f M_2$ into an m-dimensional Riemannian manifold $N(c)$ of constant sectional curvature c. Then we have*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 c, \tag{2}$$

where $n_i = \dim M_i$, $n = n_1 + n_2$, Δ is the Laplacian operator of M_1. The equality case of (2) *holds identically if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$ are the partial mean curvature vectors.*

As a generalization of Chen's result, in [8], Özgür and Murathan considered warped product submanifolds of a Riemannian manifold of quasi-constant curvature and obtained the following sharp inequality for a warped product isometrically immersed in a Riemannian manifold of quasi-constant curvature:

Theorem 1.2. *Let $x : M_1 \times_f M_2 \to N^m$ be an isometric immersion of an n-dimensional warped product $M_1 \times_f M_2$ into an m-dimensional Riemannian manifold N^m of quasi-constant curvature. Then we have:*

$$\begin{aligned} \frac{\Delta f}{f} \leq\ & \frac{n^2}{4n_2}\|H\|^2 + n_1 a - \frac{b}{n_2}\sum_{1\leq i\leq n_1}\sum_{n_1+1\leq s\leq n}(T(e_i)^2 + T(e_s)^2) \\ & + \frac{b}{n_2}(n-1)\left\|P^\top\right\|^2, \end{aligned} \tag{3}$$

where $n_i = \dim M_i$. The equality sign of (3) holds identically if and only if the immersion x is mixed totally geodesic with $tr\sigma_1 = tr\sigma_2$.

Recently, in [7], Olteanu established the following general inequality for arbitrary isometric immersions of doubly warped product manifolds in arbitrary Riemannian manifolds:

Theorem 1.3. *Let x be an isometric immersion of an n-dimensional doubly warped product $M = {}_{f_2} M_1 \times_{f_1} M_2$ into an m-dimensional arbitrary Riemannian manifold $\widetilde{M}$. Then we have*

$$n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4}\|H\|^2 + n_1 n_2 \max\widetilde{K}, \tag{4}$$

where $n_i = \dim M_i$, $n = n_1 + n_2$, Δ_i is the Laplacian operator of M_i, $i = 1, 2$ and $\max\widetilde{K}(p)$ denotes the maximum of the sectional curvature function of $\widetilde{M}$ restricted to 2-plane sections of the tangent space T_pM of M at each point p in M. Moreover, the equality case of (4) holds if and only if the following two statements hold:

(1) *x is a mixed totally geodesic immersion satisfying $n_1H_1 = n_2H_2$, where H_i, $i = 1, 2$ are the partial mean curvature vectors of M_i.*

(2) *at each point $p = (p_1, p_2) \in M$, the sectional curvature function $\widetilde{K}$ of $\widetilde{M}$ satisfies $\widetilde{K}(u, v) = \max\widetilde{K}(p)$ for each unit vector $u \in T_{p_1}M_1$ and each unit vector $v \in T_{p_2}M_2$.*

Moreover, in [9], the present author and Özgür studied C-totally real doubly warped product submanifolds in (κ, μ)-contact space forms and non-Sasakian (κ, μ)-contact metric manifolds.

Motivated by the studies of the above mentioned authors, in the present paper, we establish a sharp inequality for a doubly warped product submanifold in a Riemannian manifold of quasi-constant curvature.

The paper is organized as follows: Section 2 is devoted to preliminaries. In section 3, we give a sharp inequality for a doubly warped product submanifold in a Riemannian manifold of quasi-constant curvature.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(u,v)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, where $\{u,v\}$ is an orthonormal basis of π. For any n-dimensional subspace $L \subseteq T_pM$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is denoted by $\tau(L) = \sum_{1\leq i<j\leq n} K(e_i \wedge e_j)$, where $\{e_1, ..., e_n\}$ is any orthonormal basis of L (see [4]). When $L = T_pM$, then the scalar curvature $\tau(L)$ is just the scalar curvature $\tau(p)$ of M at p.

For an n-dimensional submanifold M in a Riemannian m-manifold N, we denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and N, respectively. The Gauss and Weingarten formulas are given by $\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ and $\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X Y$, respectively, for vector fields X, Y tangent to M and ξ normal to M, where σ denotes the second fundamental form, $\nabla^\perp$ the normal connection and A the shape operator of M (see [2]).

Denote by R and $\widetilde{R}$ the Riemannian curvature tensors of M and N, respectively. Then the equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) + g(\sigma(Y,Z),\sigma(X,W)) \\ &\quad - g(\sigma(X,Z),\sigma(Y,W)), \end{aligned} \tag{5}$$

for all vector fields X, Y, Z, W tangent to M (see [2]).

For any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM, the mean curvature vector is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n}\sigma(e_i, e_i), \tag{6}$$

where $n = \dim M$.

We set $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r), i, j \in \{1, ..., n\}, r \in \{n+1, ..., m\}$, the coefficients of the second fundamental form σ with respect to $e_1, ..., e_n$,

$e_{n+1}, ..., e_m$, and

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)). \tag{7}$$

Let M be an n-dimensional Riemannian manifold and $\{e_1, ..., e_n\}$ be an orthonormal basis of M. For a differentiable function f on M, the Laplacian Δf of f is denoted by $\Delta f = \sum_{j=1}^{n}\{(\nabla_{e_j} e_j)f - e_j e_j f\}$.

We will need the following Chen's Lemma for later use:

Lemma 2.1 ([3]). *Let $n \geq 2$ and $a_1, a_2, ..., a_n, \lambda$ be real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + \lambda\right). \tag{8}$$

Then $2a_1a_2 \geq \lambda$, with equality holding if and only if $a_1 + a_2 = a_3 = ... = a_n$.

3. Doubly warped product submanifolds

In this section, we establish a sharp relationship between the warping functions f_1 and f_2 of a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ isometrically immersed in a Riemannian manifold of quasi-constant sectional curvature and the squared mean curvature $\|H\|^2$.

Decomposing the vector field P on M uniquely into its tangent and normal components $P^\top$ and $P^\perp$, respectively, we have

$$P = P^\top + P^\perp. \tag{9}$$

Now, let us begin with the following theorem:

Theorem 3.1. *Let $x :_{f_2} M_1 \times_{f_1} M_2 \to N$ be an isometric immersion of an n-dimensional doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into an m-dimensional Riemannian manifold N of quasi-constant curvature. Then we have:*

$$\begin{aligned} &n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4}\|H\|^2 + n_1 n_2 a \\ &- b\sum_{1\leq i\leq n_1}\sum_{n_1+1\leq s\leq n}(T(e_i)^2 + T(e_s)^2) + b(n-1)\|P^\top\|^2, \end{aligned} \tag{10}$$

where $n_i = \dim M_i$, $n = n_1 + n_2$ and Δ_i is the Laplacian of M_i, $i = 1, 2$. The equality case of (10) *holds identically if and only if the immersion x is mixed totally geodesic with $tr\sigma_1 = tr\sigma_2$.*

Proof. Let $M ={}_{f_2} M_1 \times_{f_1} M_2$ be a doubly warped product submanifold of a Riemannian manifold N of quasi-constant curvature. Since ${}_{f_2}M_1 \times_{f_1} M_2$ is a doubly warped product, we have

$$\nabla_X Y = \nabla^1_X Y - \frac{f_2^2}{f_1^2} g_1(X,Y)\nabla^2(\ln f_2), \tag{11}$$

$$\nabla_X Z = Z(\ln f_2)X + X(\ln f_1)Z, \tag{12}$$

for any vector fields X, Y on M_1 and Z on M_2, where ∇^1 and ∇^2 are Levi-Civita connections of the Riemannian metrics g_1 and g_2, respectively. Here, $\nabla^2(\ln f_2)$ denotes the gradient of $(\ln f_2)$ with respect to the metric g_2.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = \frac{1}{f_1}\{(\nabla^1_X X)f_1 - X^2 f_1\} + \frac{1}{f_2}\{(\nabla^1_Z Z)f_2 - Z^2 f_2\}. \tag{13}$$

If we choose a local orthonormal frame $\{e_1, ..., e_{n_1}, e_{n_1+1}, ..., e_n\}$ such that $e_1, ..., e_{n_1}$ are tangent to M_1, $e_{n_1+1}, ..., e_n$ are tangent to M_2 and e_{n+1} is parallel to the mean curvature vector H, we obtain

$$n_2 \frac{\Delta f_1}{f_1} + n_1 \frac{\Delta f_2}{f_2} = \sum_{1 \leq j \leq n_1 < s \leq n} K(e_j \wedge e_s), \tag{14}$$

for each $s \in \{n_1 + 1, ..., n\}$.

From the equation of Gauss, for $X = W = e_i$ and $Y = Z = e_j$ such that $i \neq j$, we have $2\tau = n^2\|H\|^2 - \|\sigma\|^2 + 2b(n-1)\|P^\top\|^2 + n(n-1)a$, where $\|\sigma\|^2$ is the squared norm of the second fundamental form σ of M in N and τ is the scalar curvature of $M ={}_{f_2} M_1 \times_{f_1} M_2$.

We set

$$\delta = 2\tau - \frac{n^2}{2}\|H\|^2 - 2b(n-1)\|P^\top\|^2 - n(n-1)a. \tag{15}$$

Then, we can write equation (15) as follows

$$n^2 \|H\|^2 = 2(\delta + \|\sigma\|^2). \tag{16}$$

For a chosen local orthonormal frame, the relation (16) takes the following form

$$\left(\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right)^2 = 2\left[\delta + \sum_{i=1}^{n}(\sigma_{ii}^{n+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2\right].$$

If we put $a_1 = \sigma_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} \sigma_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} \sigma_{tt}^{n+1}$, then the above equation turns into

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\Bigg[\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\le i\neq j\le n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n} (\sigma_{ij}^{r})^2 - \sum_{2\le j\neq k\le n_1} \sigma_{jj}^{n+1}\sigma_{kk}^{n+1} - \sum_{n_1+1\le s\neq t\le n} \sigma_{ss}^{n+1}\sigma_{tt}^{n+1}\Bigg].$$

Hence, a_1, a_2 and a_3 satisfy the Chen's Lemma (for $n = 3$), which implies that $(\sum_{i=1}^{3} a_i)^2 = 2(\lambda + \sum_{i=1}^{3} a_i^2)$ with

$$\lambda = \delta + \sum_{1\le i\neq j\le n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n} (\sigma_{ij}^{r})^2 - \sum_{2\le j\neq k\le n_1} \sigma_{jj}^{n+1}\sigma_{kk}^{n+1} - \sum_{n_1+1\le s\neq t\le n} \sigma_{ss}^{n+1}\sigma_{tt}^{n+1}.$$

Then we get $2a_1a_2 \ge \lambda$, with equality holding if and only if $a_1 + a_2 = a_3$. Equivalently, we have

$$\sum_{1\le j<k\le n_1} \sigma_{jj}^{n+1}\sigma_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} \sigma_{ss}^{n+1}\sigma_{tt}^{n+1} \ge \frac{\delta}{2} + \sum_{1\le \alpha<\beta\le n} (\sigma_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{\alpha,\beta=1}^{n} (\sigma_{\alpha\beta}^{r})^2. \tag{17}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} \sigma_{ii}^{n+1} = \sum_{t=n_1+1}^{n} \sigma_{tt}^{n+1}. \tag{18}$$

By making use of the Gauss equation again, we have

$$n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} = \tau - \sum_{1\le j<k\le n_1} K(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s\wedge e_t)$$

$$
\begin{aligned}
&= \tau - \frac{n_1(n_1-1)}{2}a - \sum_{r=n+1}^{m} \sum_{1\le j<k\le n_1} [\sigma^r_{jj}\sigma^r_{kk} - (\sigma^r_{jk})^2] \\
&- b\sum_{1\le j<k\le n} (T(e_k)^2 + T(e_j)^2) - \frac{n_2(n_2-1)}{2}a \qquad (19)\\
&- \sum_{r=n+1}^{m} \sum_{n_1+1\le s<t\le n_1} [\sigma^r_{ss}\sigma^r_{tt} - (\sigma^r_{st})^2] - b\sum_{n_1+1\le s<t\le n} (T(e_s)^2 + T(e_t)^2).
\end{aligned}
$$

In view of the equations (14), (17) and (19) we obtain

$$
\begin{aligned}
&n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} \le \tau - \frac{n(n-1)}{2}a + n_1n_2a - \frac{\delta}{2} \\
&- \frac{1}{2}\sum_{r=n+2}^{m} \sum_{\alpha,\beta=1}^{n} (\sigma^r_{\alpha\beta})^2 + \sum_{r=n+2}^{m} \sum_{1\le j<k\le n_1} [(\sigma^r_{jk})^2 - \sigma^r_{jj}\sigma^r_{kk}] \\
&+ \sum_{r=n+2}^{m} \sum_{n_1+1\le s<t\le n_1} [(\sigma^r_{st})^2 - \sigma^r_{ss}\sigma^r_{tt}] - b\sum_{1\le j<k\le n} (T(e_k)^2 + T(e_j)^2) \\
&- b\sum_{n_1+1\le s<t\le n} (T(e_s)^2 + T(e_t)^2) \\
&= \tau - \frac{n(n-1)}{2}a + n_1n_2a - \frac{\delta}{2} - \sum_{r=n+1}^{m}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n} \left(\sigma^r_{jt}\right)^2 \qquad (20)\\
&- \frac{1}{2}\sum_{r=n+2}^{m} \left(\sum_{j=1}^{n_1}\sigma^r_{jj}\right)^2 - \frac{1}{2}\sum_{r=n+2}^{m} \left(\sum_{t=n_1+1}^{n}\sigma^r_{tt}\right)^2 \\
&- b\sum_{1\le j<k\le n} (T(e_k)^2 + T(e_j)^2) - b\sum_{n_1+1\le s<t\le n} (T(e_s)^2 + T(e_t)^2) \\
&\le \tau - \frac{n(n-1)}{2}a + n_1n_2a - \frac{\delta}{2} \\
&- b\sum_{1\le j<k\le n} (T(e_k)^2 + T(e_j)^2) - b\sum_{n_1+1\le s<t\le n} (T(e_s)^2 + T(e_t)^2) \\
&= \frac{n^2}{4}\|H\|^2 + n_1n_2a - b\sum_{1\le i\le n_1}\sum_{n_1+1\le s\le n} (T(e_i)^2 + T(e_s)^2) + b(n-1)\|P^\top\|^2,
\end{aligned}
$$

which gives us (10).

By using of (18) and (20), it can be easily seen that the equality sign of (10) holds if and only if

$$\sigma^r_{jt} = 0, \quad n+1 \leq r \leq m \tag{21}$$

and

$$\sum_{i=1}^{n_1} \sigma^r_{ii} = \sum_{t=n_1+1}^{n} \sigma^r_{tt} = 0, \tag{22}$$

for $1 \leq j \leq n_1$, $n_1 + 1 \leq t \leq n$ and $n + 2 \leq r \leq m$. The equation (21) means that the second fundamental form σ of ${}_{f_2}M_1 \times_{f_1} M_2$ in N is as follows $\sigma(D_1, D_2) = \{0\}$. Hence, the immersion x is mixed totally geodesic. From (18) and (22), we also get $\sum_{j=1}^{n_1} \sigma(e_j, e_j) = \sum_{s=n_1+1}^{n} \sigma(e_s, e_s)$, which implies that $tr\sigma_1 = tr\sigma_2$.

Conversely, assume that N is an m-dimensional Riemannian manifold of quasi-constant curvature and the immersion x is mixed totally geodesic with $tr\sigma_1 = tr\sigma_2$. Then, inequalities (17) and (20) reduce to equalities. Hence, we obtain the equality sign of (10) and finish the proof of the theorem. □

As a consequence of Theorem 3.1 we can give the following corollary:

Corollary 3.2. *Let $x :_{f_2} M_1 \times_{f_1} M_2 \to N$ be an isometric immersion of an n-dimensional doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into an m-dimensional Riemannian manifold N of quasi-constant curvature. If the vector field P is tangent to M, then we have:*

$$\begin{aligned} n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \quad \leq \quad & \frac{n^2}{4} \|H\|^2 + n_1 n_2 a + b(n-1) \\ & - b \sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq s \leq n} (T(e_i)^2 + T(e_s)^2). \end{aligned} \tag{23}$$

If the vector field P is normal to M, then we have:

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 a. \tag{24}$$

The equality case of (23) and (24) holds identically if and only if the immersion x is mixed totally geodesic with $tr\sigma_1 = tr\sigma_2$.

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